Bruce H. Edwards

Complete Solutions Guide

Volume I



Calculus

Seventh Edition

Larson • Hostetler • Edwards

For use with

Calculus with Analytic Geometry, Seventh Edition

Calculus of a Single Variable, Seventh Edition


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PART I

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CHAPTER P
Preparation for Calculus

Section P.1 Graphs and Models

Solutions to Odd-Numbered Exercises

1. $y = -\frac{1}{2}x + 2$

x-intercept: $(4, 0)$

y-intercept: $(0, 2)$

Matches graph (b)

3. $y = 4 - x^2$

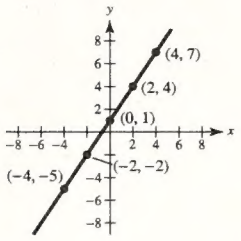
x-intercepts: $(2, 0), (-2, 0)$

y-intercept: $(0, 4)$

Matches graph (a)

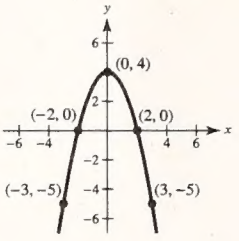
5. $y = \frac{3}{2}x + 1$

x	-4	-2	0	2	4
y	-5	-2	1	4	7



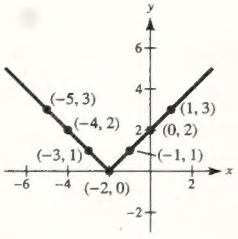
7. $y = 4 - x^2$

x	-3	-2	0	2	3
y	-5	0	4	0	-5



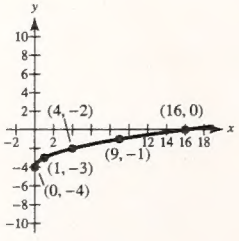
9. $y = |x + 2|$

x	-5	-4	-3	-2	-1	0	1
y	3	2	1	0	1	2	3



11. $y = \sqrt{x} - 4$

x	0	1	4	9	16
y	-4	-3	-2	-1	0

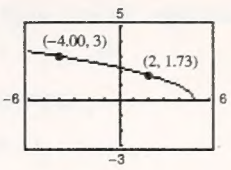


13.

```
Xmin = -3
Xmax = 5
Xscl = 1
Ymin = -3
Ymax = 5
Yscl = 1
```

Note that $y = 4$ when $x = 0$.

15.



(a) $(2, y) = (2, 1.73)$ $\left(y = \sqrt{5 - 2} = \sqrt{3} \approx 1.73\right)$

(b) $(x, 3) = (-4, 3)$ $\left(3 = \sqrt{5 - (-4)}\right)$

17. $y = x^2 + x - 2$

y-intercept: $y = 0^2 + 0 - 2$

$y = -2; (0, -2)$

x-intercepts: $0 = x^2 + x - 2$

$0 = (x + 2)(x - 1)$

$x = -2, 1; (-2, 0), (1, 0)$

19. $y = x^2\sqrt{25 - x^2}$

y-intercept: $y = 0^2\sqrt{25 - 0^2}$

$y = 0; (0, 0)$

x-intercepts: $0 = x^2\sqrt{25 - x^2}$

$0 = x^2\sqrt{(5 - x)(5 + x)}$

$x = 0, \pm 5; (0, 0); (\pm 5, 0)$

21. $y = \dfrac{3\left(2 - \sqrt{x}\right)}{x}$

y-intercept: None. x cannot equal 0.

x-intercepts: $0 = \dfrac{3\left(2 - \sqrt{x}\right)}{x}$

$0 = 2 - \sqrt{x}$

$x = 4; (4, 0)$

23. $x^2y - x^2 + 4y = 0$

y-intercept:

$0^2(y) - 0^2 + 4y = 0$

$y = 0; (0, 0)$

x-intercept:

$x^2(0) - x^2 + 4(0) = 0$

$x = 0; (0, 0)$

25. Symmetric with respect to the y-axis since

$y = (-x)^2 - 2 = x^2 - 2.$

27. Symmetric with respect to the x-axis since

$(-y)^2 = y^2 = x^3 - 4x.$

29. Symmetric with respect to the origin since

$(-x)(-y) = xy = 4.$

31. $y = 4 - \sqrt{x + 3}$

No symmetry with respect to either axis or the origin.

33. Symmetric with respect to the origin since

$$-y = \frac{-x}{(-x)^2 + 1}$$

$$y = \frac{x}{x^2 + 1}.$$

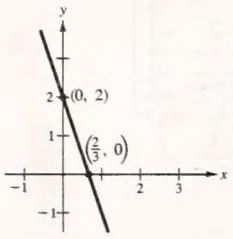
35. $y = |x^3 + x|$ is symmetric with respect to the y-axis since $y = |(-x)^3 + (-x)| = |-(x^3 + x)| = |x^3 + x|$.

37. $y = -3x + 2$

Intercepts:

$\left(\frac{2}{3}, 0\right), (0, 2)$

Symmetry: none

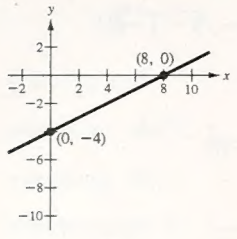


39. $y = \frac{1}{2}x - 4$

Intercepts:

$(8, 0), (0, -4)$

Symmetry: none

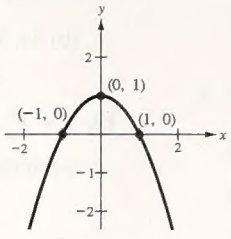


41. $y = 1 - x^2$

Intercepts:

$(1, 0), (-1, 0), (0, 1)$

Symmetry: y-axis

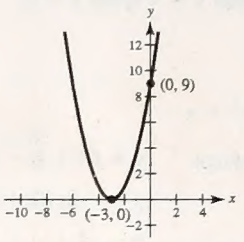


43. $y = (x + 3)^2$

Intercepts:

$(-3, 0), (0, 9)$

Symmetry: none

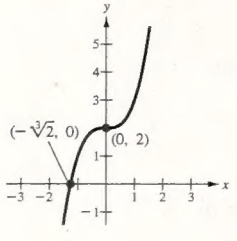


45. $y = x^3 + 2$

Intercepts:

$\left(-\sqrt[3]{2}, 0\right), (0, 2)$

Symmetry: none



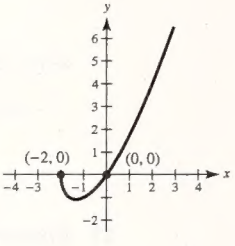
47. $y = x\sqrt{x + 2}$

Intercepts:

$(0, 0), (-2, 0)$

Symmetry: none

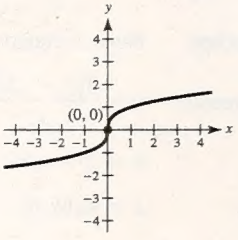
Domain: $x \geq -2$



49. $x = y^3$

Intercepts: $(0, 0)$

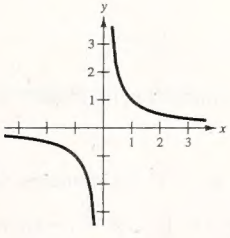
Symmetry: origin



51. $y = \frac{1}{x}$

Intercepts: none

Symmetry: origin

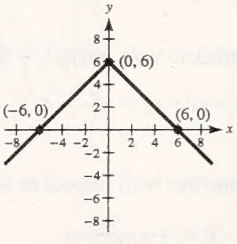


53. $y = 6 - |x|$

Intercepts:

$(0, 6), (-6, 0), (6, 0)$

Symmetry: y-axis



55. $y^2 - x = 9$

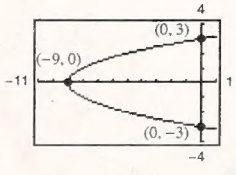
$y^2 = x + 9$

$y = \pm\sqrt{x + 9}$

Intercepts:

$(0, 3), (0, -3), (-9, 0)$

Symmetry: x-axis



57. $x + 3y^2 = 6$

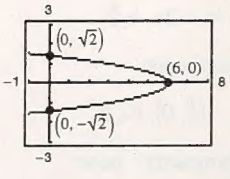
$3y^2 = 6 - x$

$y = \pm\sqrt{2 - \frac{x}{3}}$

Intercepts:

$(6, 0), \left(0, \sqrt{2}\right), \left(0, -\sqrt{2}\right)$

Symmetry: x-axis



59. $y = (x + 2)(x - 4)(x - 6)$ (other answers possible)

61. Some possible equations:

$y = x$

$y = x^3$

$y = 3x^3 - x$

$y = \sqrt[3]{x}$

63. $x + y = 2 \Rightarrow y = 2 - x$

$2x - y = 1 \Rightarrow y = 2x - 1$

$2 - x = 2x - 1$

$3 = 3x$

$1 = x$

The corresponding y-value is $y = 1$.

Point of intersection: $(1, 1)$

65. $x + y = 7 \Rightarrow y = 7 - x$

$3x - 2y = 11 \Rightarrow y = \dfrac{3x - 11}{2}$

$7 - x = \dfrac{3x - 11}{2}$

$14 - 2x = 3x - 11$

$-5x = -25$

$x = 5$

The corresponding y-value is $y = 2$.

Point of intersection: $(5, 2)$

67. $x^2 + y = 6 \Rightarrow y = 6 - x^2$

$x + y = 4 \Rightarrow y = 4 - x$

$6 - x^2 = 4 - x$

$0 = x^2 - x - 2$

$0 = (x - 2)(x + 1)$

$x = 2, -1$

The corresponding y-values are $y = 2$ (for $x = 2$) and $y = 5$ (for $x = -1$).

Points of intersection: $(2, 2), (-1, 5)$

69. $x^2 + y^2 = 5 \Rightarrow y^2 = 5 - x^2$

$x - y = 1 \Rightarrow y = x - 1$

$5 - x^2 = (x - 1)^2$

$5 - x^2 = x^2 - 2x + 1$

$0 = 2x^2 - 2x - 4 = 2(x + 1)(x - 2)$

$x = -1$ or $x = 2$

The corresponding y-values are $y = -2$ and $y = 1$.

Points of intersection: $(-1, -2), (2, 1)$

71. $y = x^3$

$y = x$

$x^3 = x$

$x^3 - x = 0$

$x(x + 1)(x - 1) = 0$

$x = 0, x = -1$, or $x = 1$

The corresponding y-values are $y = 0$, $y = -1$, and $y = 1$.

Points of intersection: $(0, 0), (-1, -1), (1, 1)$

73. $y = x^3 - 2x^2 + x - 1$

$y = -x^2 + 3x - 1$

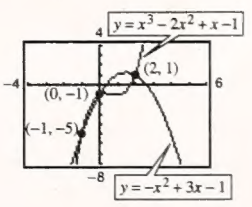
$x^3 - 2x^2 + x - 1 = -x^2 + 3x - 1$

$x^3 - x^2 - 2x = 0$

$x(x - 2)(x + 1) = 0$

$x = -1, 0, 2$

$(-1, -5), (0, -1), (2, 1)$



75. $5.5\sqrt{x} + 10{,}000 = 3.29x$

$$(5.5\sqrt{x})^2 = (3.29x - 10{,}000)^2$$

$$30.25x = 10.8241x^2 - 65{,}800x + 100{,}000{,}000$$

$$0 = 10.8241x^2 - 65{,}830.25x + 100{,}000{,}000 \quad \text{Use the Quadratic Formula.}$$

$$x \approx 3133 \text{ units}$$

The other root, $x \approx 2949$, does not satisfy the equation $R = C$.

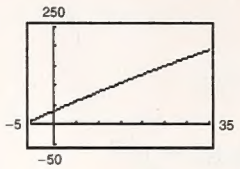
This problem can also be solved by using a graphing utility and finding the intersection of the graphs of C and R.

77. (a) Using a graphing utility, you obtain

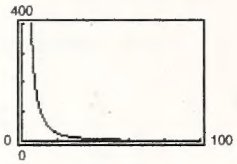
$$y = -0.0153t^2 + 4.9971t + 34.9405$$

(c) For the year 2004, $t = 34$ and $y \approx 187.2$ CPI.

(b)



79.



If the diameter is doubled, the resistance is changed by approximately a factor of $(1/4)$. For instance, $y(20) \approx 26.555$ and $y(40) \approx 6.36125$.

81. False; x-axis symmetry means that if $(1, -2)$ is on the graph, then $(1, 2)$ is also on the graph.

83. True; the x-intercepts are

$$\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, 0\right).$$

85. Distance to the origin $= K \times$ Distance to $(2, 0)$

$$\sqrt{x^2 + y^2} = K\sqrt{(x-2)^2 + y^2}, \; K \neq 1$$

$$x^2 + y^2 = K^2(x^2 - 4x + 4 + y^2)$$

$$(1 - K^2)x^2 + (1 - K^2)y^2 + 4K^2x - 4K^2 = 0$$

Note: This is the equation of a circle!

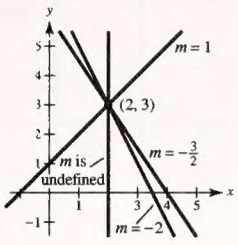
Section P.2 Linear Models and Rates of Change

1. $m = 1$

3. $m = 0$

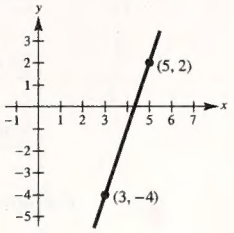
5. $m = -12$

7.



9. $m = \dfrac{2-(-4)}{5-3}$

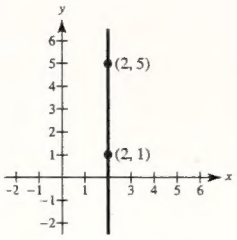
$= \dfrac{6}{2} = 3$



11. $m = \dfrac{5-1}{2-2}$

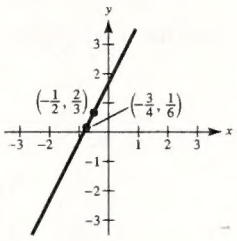
$= \dfrac{4}{0}$

undefined



13. $m = \dfrac{2/3 - 1/6}{-1/2 - (-3/4)}$

$= \dfrac{1/2}{1/4} = 2$



15. Since the slope is 0, the line is horizontal and its equation is $y = 1$. Therefore, three additional points are (0, 1), (1, 1), and (3, 1).

17. The equation of this line is

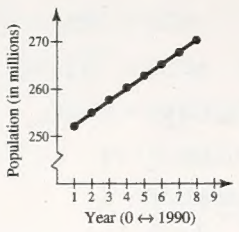
$$y - 7 = -3(x - 1)$$

$$y = -3x + 10.$$

Therefore, three additional points are (0, 10), (2, 4), and (3, 1).

19. Given a line L, you can use any two distinct points to calculate its slope. Since a line is straight, the ratio of the change in y-values to the change in x-values will always be the same. See Section P.2 Exercise 93 for a proof.

21. (a)



(b) The slopes of the line segments are

$$\frac{255.0 - 252.1}{2 - 1} = 2.9$$

$$\frac{257.7 - 255.0}{3 - 2} = 2.7$$

$$\frac{260.3 - 257.7}{4 - 3} = 2.6$$

$$\frac{262.8 - 260.3}{5 - 4} = 2.5$$

$$\frac{265.2 - 262.8}{6 - 5} = 2.4$$

$$\frac{267.7 - 265.2}{7 - 6} = 2.5$$

$$\frac{270.3 - 267.7}{8 - 7} = 2.6$$

The population increased most rapidly from 1991 to 1992.

($m = 2.9$)

23. $x + 5y = 20$

$y = -\frac{1}{5}x + 4$

Therefore, the slope is $m = -\frac{1}{5}$ and the y-intercept is $(0, 4)$.

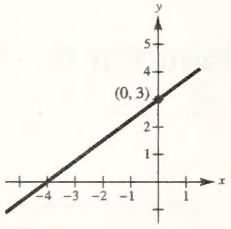
25. $x = 4$

The line is vertical. Therefore, the slope is undefined and there is no y-intercept.

27. $y = \frac{3}{4}x + 3$

$4y = 3x + 12$

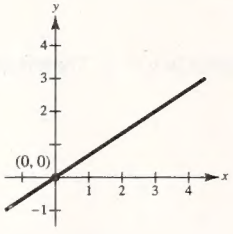
$0 = 3x - 4y + 12$



29. $y = \frac{2}{3}x$

$3y = 2x$

$2x - 3y = 0$

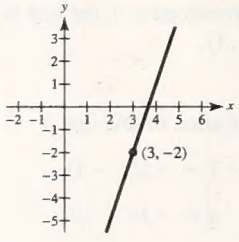


31. $y + 2 = 3(x - 3)$

$y + 2 = 3x - 9$

$y = 3x - 11$

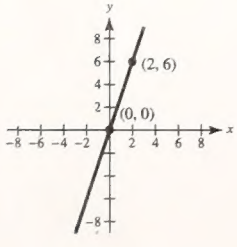
$y - 3x + 11 = 0$



33. $m = \frac{6 - 0}{2 - 0} = 3$

$y - 0 = 3(x - 0)$

$y = 3x$

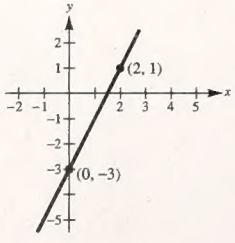


35. $m = \frac{1 - (-3)}{2 - 0} = 2$

$y - 1 = 2(x - 2)$

$y - 1 = 2x - 4$

$0 = 2x - y - 3$

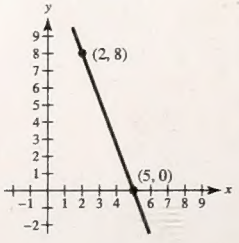


37. $m = \frac{8 - 0}{2 - 5} = -\frac{8}{3}$

$y - 0 = -\frac{8}{3}(x - 5)$

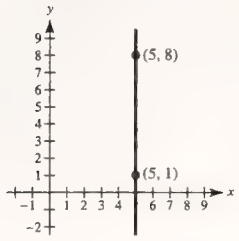
$y = -\frac{8}{3}x + \frac{40}{3}$

$3y + 8x - 40 = 0$



39. $m = \dfrac{8-1}{5-5}$ Undefined.

Vertical line $x = 5$

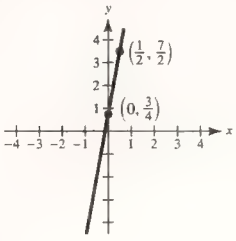


41. $m = \dfrac{7/2 - 3/4}{1/2 - 0} = \dfrac{11/4}{1/2} = \dfrac{11}{2}$

$$y - \frac{3}{4} = \frac{11}{2}(x - 0)$$

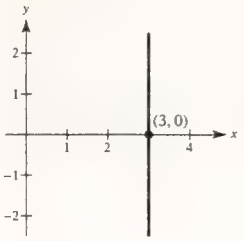
$$y = \frac{11}{2}x + \frac{3}{4}$$

$$22x - 4y + 3 = 0$$



43. $x = 3$

$$x - 3 = 0$$



45. $\dfrac{x}{2} + \dfrac{y}{3} = 1$

$$3x + 2y - 6 = 0$$

47. $\dfrac{x}{a} + \dfrac{y}{a} = 1$

$$\frac{1}{a} + \frac{2}{a} = 1$$

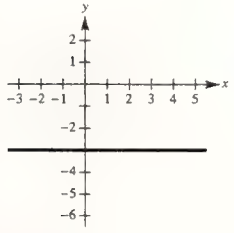
$$\frac{3}{a} = 1$$

$$a = 3 \Rightarrow x + y = 3$$

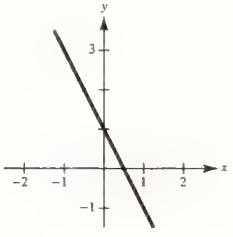
$$x + y - 3 = 0$$

49. $y = -3$

$$y + 3 = 0$$



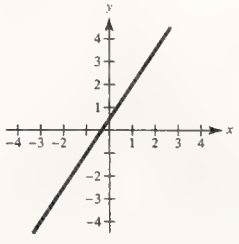
51. $y = -2x + 1$



53. $y - 2 = \frac{3}{2}(x - 1)$

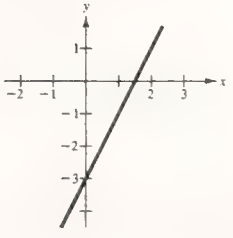
$$y = \tfrac{3}{2}x + \tfrac{1}{2}$$

$$2y - 3x - 1 = 0$$

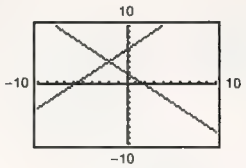


55. $2x - y - 3 = 0$

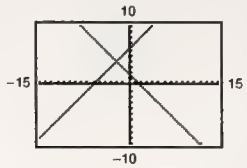
$$y = 2x - 3$$



57.



The lines do not appear perpendicular.



The lines appear perpendicular.

The lines are perpendicular because their slopes 1 and -1 are negative reciprocals of each other. You must use a square setting in order for perpendicular lines to appear perpendicular.

59. $4x - 2y = 3$

$y = 2x - \frac{3}{2}$

$m = 2$

(a) $y - 1 = 2(x - 2)$

$y - 1 = 2x - 4$

$2x - y - 3 = 0$

(b) $y - 1 = -\frac{1}{2}(x - 2)$

$2y - 2 = -x + 2$

$x + 2y - 4 = 0$

61. $5x - 3y = 0$

$y = \frac{5}{3}x$

$m = \frac{5}{3}$

(a) $y - \frac{7}{8} = \frac{5}{3}\left(x - \frac{3}{4}\right)$

$24y - 21 = 40x - 30$

$24y - 40x + 9 = 0$

(b) $y - \frac{7}{8} = -\frac{3}{5}\left(x - \frac{3}{4}\right)$

$40y - 35 = -24x + 18$

$40y + 24x - 53 = 0$

63. (a) $x = 2 \Rightarrow x - 2 = 0$

(b) $y = 5 \Rightarrow y - 5 = 0$

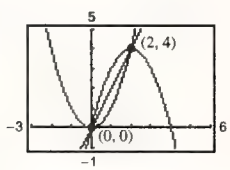
65. The slope is 125. Hence, $V = 125(t - 1) + 2540$

$= 125t + 2415$

67. The slope is -2000. Hence, $V = -2000(t - 1) + 20{,}400$

$= -2000t + 22{,}400$

69.



You can use the graphing utility to determine that the points of intersection are (0, 0) and (2, 4). Analytically,

$x^2 = 4x - x^2$

$2x^2 - 4x = 0$

$2x(x - 2) = 0$

$x = 0 \Rightarrow y = 0 \Rightarrow (0, 0)$

$x = 2 \Rightarrow y = 4 \Rightarrow (2, 4).$

The slope of the line joining (0, 0) and (2, 4) is $m = (4 - 0)/(2 - 0) = 2$. Hence, an equation of the line is

$y - 0 = 2(x - 0)$

$y = 2x.$

71. $m_1 = \dfrac{1-0}{-2-(-1)} = -1$

$m_2 = \dfrac{-2-0}{2-(-1)} = -\dfrac{2}{3}$

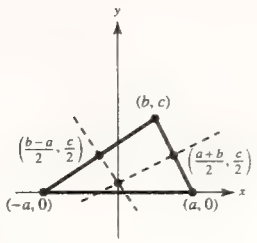
$m_1 \neq m_2$

The points are not collinear.

73. Equations of perpendicular bisectors:

$$y - \frac{c}{2} = \frac{a-b}{c}\left(x - \frac{a+b}{2}\right)$$

$$y - \frac{c}{2} = \frac{a+b}{-c}\left(x - \frac{b-a}{2}\right)$$



Letting $x = 0$ in either equation gives the point of intersection:

$$\left(0, \frac{-a^2 + b^2 + c^2}{2c}\right).$$

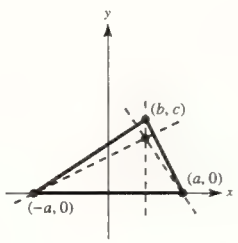
This point lies on the third perpendicular bisector, $x = 0$.

75. Equations of altitudes:

$$y = \frac{a-b}{c}(x + a)$$

$$x = b$$

$$y = -\frac{a+b}{c}(x - a)$$



Solving simultaneously, the point of intersection is

$$\left(b, \frac{a^2 - b^2}{c}\right).$$

77. Find the equation of the line through the points (0, 32) and (100, 212).

$$m = \tfrac{180}{100} = \tfrac{9}{5}$$

$$F - 32 = \tfrac{9}{5}(C - 0)$$

$$F = \tfrac{9}{5}C + 32$$

$$5F - 9C - 160 = 0$$

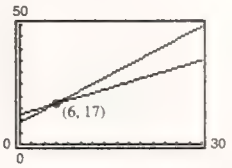
For $F = 72°$, $C \approx 22.2°$.

79. (a) $W_1 = 0.75x + 12.50$

$W_2 = 1.30x + 9.20$

(c) Both jobs pay \$17 per hour if 6 units are produced. For someone who can produce more than 6 units per hour, the second offer would pay more. For a worker who produces less than 6 units per hour, the first offer pays more.

(b)



Using a graphing utility, the point of intersection is approximately (6, 17). Analytically,

$$0.75x + 12.50 = 1.30x + 9.20$$

$$3.3 = 0.55x \Rightarrow x = 6$$

$$y = 0.75(6) + 12.50 = 17.$$

81. (a) Two points are (50, 580) and (47, 625). The slope is

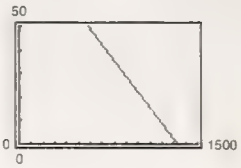
$$m = \frac{625 - 580}{47 - 50} = -15.$$

$$p - 580 = -15(x - 50)$$

$$p = -15x + 750 + 580 = -15x + 1330$$

$$\text{or } x = \tfrac{1}{15}(1330 - p)$$

(b)



If $p = 655$, $x = \frac{1}{15}(1330 - 655) = 45$ units.

(c) If $p = 595$, $x = \frac{1}{15}(1330 - 595) = 49$ units.

83. $4x + 3y - 10 = 0 \Rightarrow d = \dfrac{|4(0) + 3(0) - 10|}{\sqrt{4^2 + 3^2}} = \dfrac{10}{5} = 2$

85. $x - y - 2 = 0 \Rightarrow d = \dfrac{|1(-2) + (-1)(1) - 2|}{\sqrt{1^2 + 1^2}} = \dfrac{5}{\sqrt{2}} = \dfrac{5\sqrt{2}}{2}$

87. A point on the line $x + y = 1$ is (0, 1). The distance from the point (0, 1) to $x + y - 5 = 0$ is

$$d = \frac{|1(0) + 1(1) - 5|}{\sqrt{1^2 + 1^2}} = \frac{|1 - 5|}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

89. If $A = 0$, then $By + C = 0$ is the horizontal line $y = -C/B$. The distance to (x_1, y_1) is

$$d = \left|y_1 - \left(\frac{-C}{B}\right)\right| = \frac{|By_1 + C|}{|B|} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

If $B = 0$, then $Ax + C = 0$ is the vertical line $x = -C/A$. The distance to (x_1, y_1) is

$$d = \left|x_1 - \left(\frac{-C}{A}\right)\right| = \frac{|Ax_1 + C|}{|A|} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

(Note that A and B cannot both be zero.)

The slope of the line $Ax + By + C = 0$ is $-A/B$. The equation of the line through (x_1, y_1) perpendicular to $Ax + By + C = 0$ is:

$$y - y_1 = \frac{B}{A}(x - x_1)$$

$$Ay - Ay_1 = Bx - Bx_1$$

$$Bx_1 - Ay_1 = Bx - Ay$$

The point of intersection of these two lines is:

$$Ax + By = -C \quad \Rightarrow \quad A^2x + ABy = -AC \qquad (1)$$

$$Bx - Ay = Bx_1 - Ay_1 \Rightarrow \quad \underline{B^2x - ABy} = \underline{B^2x_1 - ABy_1} \quad (2)$$

$$(A^2 + B^2)x = -AC + B^2x_1 - ABy_1 \quad \text{(By adding equations (1) and (2))}$$

$$x = \frac{-AC + B^2x_1 - ABy_1}{A^2 + B^2}$$

$$Ax - By = -C \quad \Rightarrow \quad ABx + B^2y = -BC \qquad (3)$$

$$Bx - Ay = Bx_1 - Ay_1 \Rightarrow \quad \underline{-ABx + A^2y} = \underline{-ABx_1 + A^2y_1} \quad (4)$$

$$(A^2 + B^2)y = -BC - ABx_1 + A^2y_1 \quad \text{(By adding equations (3) and (4))}$$

$$y = \frac{-BC - ABx_1 + A^2y_1}{A^2 + B^2}$$

—CONTINUED—

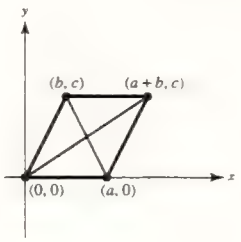
89. —CONTINUED—

$$\left(\frac{-AC + B^2x_1 - ABy_1}{A^2 + B^2}, \frac{-BC - ABx_1 + A^2y_1}{A^2 + B^2}\right) \text{ point of intersection}$$

The distance between (x_1, y_1) and this point gives us the distance between (x_1, y_1) and the line $Ax + By + C = 0$.

$$\begin{aligned} d &= \sqrt{\left[\frac{-AC + B^2x_1 - ABy_1}{A^2 + B^2} - x_1\right]^2 + \left[\frac{-BC - ABx_1 + A^2y_1}{A^2 + B^2} - y_1\right]^2} \\ &= \sqrt{\left[\frac{-AC - ABy_1 - A^2x_1}{A^2 + B^2}\right]^2 + \left[\frac{-BC - ABx_1 - B^2y_1}{A^2 + B^2}\right]^2} \\ &= \sqrt{\left[\frac{-A(C + By_1 + Ax_1)}{A^2 + B^2}\right]^2 + \left[\frac{-B(C + Ax_1 + By_1)}{A^2 + B^2}\right]^2} \\ &= \sqrt{\frac{(A^2 + B^2)(C + Ax_1 + By_1)^2}{(A^2 + B^2)^2}} \\ &= \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \end{aligned}$$

91. For simplicity, let the vertices of the rhombus be $(0, 0)$, $(a, 0)$, (b, c), and $(a + b, c)$, as shown in the figure. The slopes of the diagonals are then



$$m_1 = \frac{c}{a + b} \text{ and } m_2 = \frac{c}{b - a}.$$

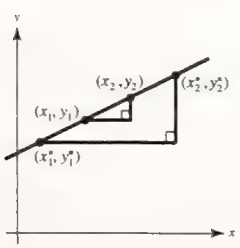
Since the sides of the Rhombus are equal, $a^2 = b^2 + c^2$, and we have

$$m_1m_2 = \frac{c}{a + b} \cdot \frac{c}{b - a} = \frac{c^2}{b^2 - a^2} = \frac{c^2}{-c^2} = -1.$$

Therefore, the diagonals are perpendicular.

93. Consider the figure below in which the four points are collinear. Since the triangles are similar, the result immediately follows.

$$\frac{y_2^* - y_1^*}{x_2^* - x_1^*} = \frac{y_2 - y_1}{x_2 - x_1}$$



95. True.

$$ax + by = c_1 \Rightarrow y = -\frac{a}{b}x + \frac{c_1}{b} \Rightarrow m_1 = -\frac{a}{b}$$

$$bx - ay = c_2 \Rightarrow y = \frac{b}{a}x - \frac{c_2}{a} \Rightarrow m_2 = \frac{b}{a}$$

$$m_2 = -\frac{1}{m_1}$$

Section P.3 Functions and Their Graphs

1. (a) $f(0) = 2(0) - 3 = -3$

(b) $f(-3) = 2(-3) - 3 = -9$

(c) $f(b) = 2b - 3$

(d) $f(x - 1) = 2(x - 1) - 3 = 2x - 5$

3. (a) $g(0) = 3 - 0^2 = 3$

(b) $g(\sqrt{3}) = 3 - (\sqrt{3})^2 = 3 - 3 = 0$

(c) $g(-2) = 3 - (-2)^2 = 3 - 4 = -1$

(d) $g(t - 1) = 3 - (t - 1)^2 = -t^2 + 2t + 2$

5. (a) $f(0) = \cos(2(0)) = \cos 0 = 1$

(b) $f\left(-\frac{\pi}{4}\right) = \cos\left(2\left(-\frac{\pi}{4}\right)\right) = \cos\left(-\frac{\pi}{2}\right) = 0$

(c) $f\left(\frac{\pi}{3}\right) = \cos\left(2\left(\frac{\pi}{3}\right)\right) = \cos\frac{2\pi}{3} = -\frac{1}{2}$

7. $$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{(x + \Delta x)^3 - x^3}{\Delta x} = \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2, \Delta x \neq 0$$

9. $$\frac{f(x) - f(2)}{x - 2} = \frac{(1/\sqrt{x - 1} - 1)}{x - 2}$$

$$= \frac{1 - \sqrt{x - 1}}{(x - 2)\sqrt{x - 1}} \cdot \frac{1 + \sqrt{x - 1}}{1 + \sqrt{x - 1}} = \frac{2 - x}{(x - 2)\sqrt{x - 1}(1 + \sqrt{x - 1})} = \frac{-1}{\sqrt{x - 1}(1 + \sqrt{x - 1})}, x \neq 2$$

11. $h(x) = -\sqrt{x + 3}$

Domain: $x + 3 \geq 0 \Rightarrow [-3, \infty)$

Range: $(-\infty, 0]$

13. $f(t) = \sec\frac{\pi t}{4}$

$\frac{\pi t}{4} \neq \frac{(2k + 1)\pi}{2} \Rightarrow t \neq 4k + 2$

Domain: all $t \neq 4k + 2$, k an integer

Range: $(-\infty, -1], [1, \infty)$

15. $f(x) = \frac{1}{x}$

Domain: $(-\infty, 0), (0, \infty)$

Range: $(-\infty, 0), (0, \infty)$

17. $f(x) = \begin{cases} 2x + 1, x < 0 \\ 2x + 2, x \geq 0 \end{cases}$

(a) $f(-1) = 2(-1) + 1 = -1$

(b) $f(0) = 2(0) + 2 = 2$

(c) $f(2) = 2(2) + 2 = 6$

(d) $f(t^2 + 1) = 2(t^2 + 1) = 2t^2 + 4$

(Note: $t^2 + 1 \geq 0$ for all t)

Domain: $(-\infty, \infty)$

Range: $(-\infty, 1), [2, \infty)$

19. $f(x) = \begin{cases} |x| + 1, x < 1 \\ -x + 1, x \geq 1 \end{cases}$

(a) $f(-3) = |-3| + 1 = 4$

(b) $f(1) = -1 + 1 = 0$

(c) $f(3) = -3 + 1 = -2$

(d) $f(b^2 + 1) = -(b^2 + 1) + 1 = -b^2$

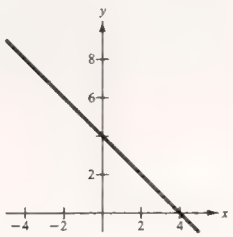
Domain: $(-\infty, \infty)$

Range: $(-\infty, 0] \cup [1, \infty)$

21. $f(x) = 4 - x$

Domain: $(-\infty, \infty)$

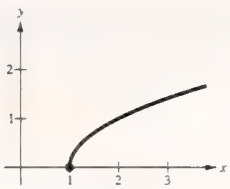
Range: $(-\infty, \infty)$



23. $h(x) = \sqrt{x - 1}$

Domain: $[1, \infty)$

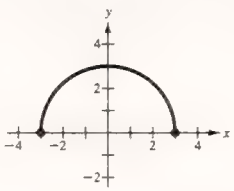
Range: $[0, \infty)$



25. $f(x) = \sqrt{9 - x^2}$

Domain: $[-3, 3]$

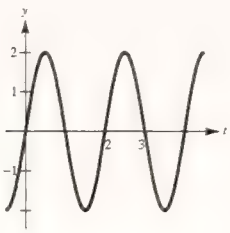
Range: $[0, 3]$



27. $g(t) = 2 \sin \pi t$

Domain: $(-\infty, \infty)$

Range: $[-2, 2]$



29. $x - y^2 = 0 \Rightarrow y = \pm\sqrt{x}$

y is not a function of x. Some vertical lines intersect the graph twice.

31. y is a function of x. Vertical lines intersect the graph at most once.

33. $x^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4 - x^2}$

y is not a function of x since there are two values of y for some x.

35. $y^2 = x^2 - 1 \Rightarrow y = \pm\sqrt{x^2 - 1}$

y is not a function of x since there are two values of y for some x.

37. $f(x) = |x| + |x - 2|$

If $x < 0$, then $f(x) = -x - (x - 2) = -2x + 2 = 2(1 - x)$.

If $0 \le x < 2$, then $f(x) = x - (x - 2) = 2$.

If $x \ge 2$, then $f(x) = x + (x - 2) = 2x - 2 = 2(x - 1)$.

Thus,

$$f(x) = \begin{cases} 2(1 - x), & x < 0 \\ 2, & 0 \le x < 2. \\ 2(x - 1), & x \ge 2. \end{cases}$$

39. The function is $g(x) = cx^2$. Since $(1, -2)$ satisfies the equation, $c = -2$. Thus, $g(x) = -2x^2$.

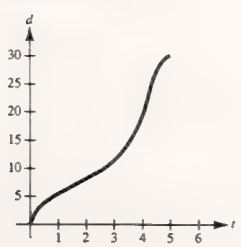
41. The function is $r(x) = c/x$, since it must be undefined at $x = 0$. Since $(1, 32)$ satisfies the equation, $c = 32$. Thus, $r(x) = 32/x$.

43. (a) For each time t, there corresponds a depth d.

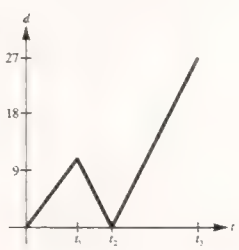
(b) Domain: $0 \le t \le 5$

Range: $0 \le d \le 30$

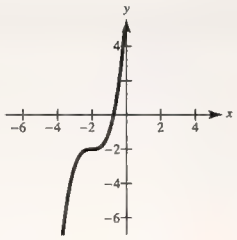
(c)



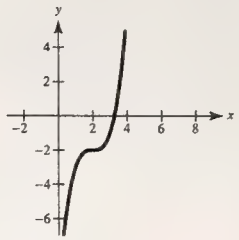
45.



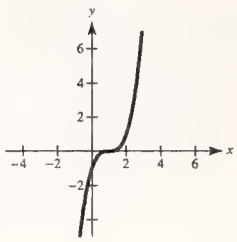
47. (a) The graph is shifted 3 units to the left.



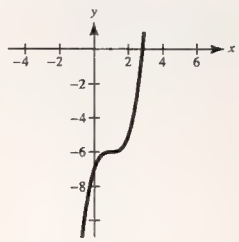
(b) The graph is shifted 1 unit to the right.



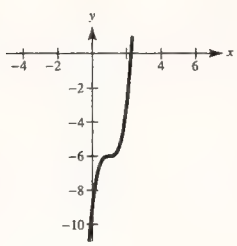
(c) The graph is shifted 2 units upward.



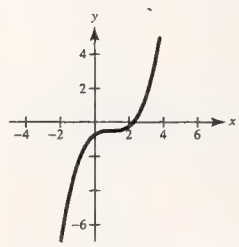
(d) The graph is shifted 4 units downward.



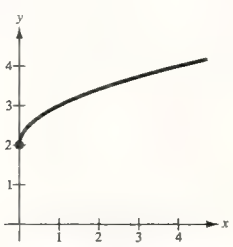
(e) The graph is stretched vertically by a factor of 3.



(f) The graph is stretched vertically by a factor of $\frac{1}{4}$.

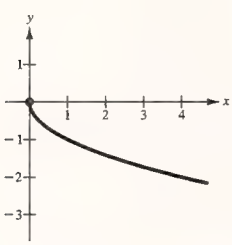


49. (a) $y = \sqrt{x} + 2$



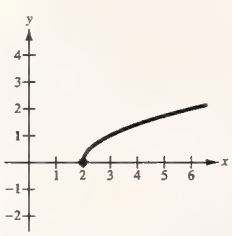
Vertical shift 2 units upward

(b) $y = -\sqrt{x}$



Reflection about the x-axis

(c) $y = \sqrt{x - 2}$



Horizontal shift 2 units to the right

51. (a) $T(4) = 16°$, $T(15) \approx 23°$

(b) If $H(t) = T(t - 1)$, then the program would turn on (and off) one hour later.

(c) If $H(t) = T(t) - 1$, then the overall temperature would be reduced 1 degree.

53. $f(x) = x^2$, $g(x) = \sqrt{x}$

$(f \circ g)(x) = f(g(x)) = f\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^2 = x, \quad x \geq 0$

Domain: $[0, \infty)$

$(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = |x|$

Domain: $(-\infty, \infty)$

No. Their domains are different. $(f \circ g) = (g \circ f)$ for $x \geq 0$.

55. $f(x) = \dfrac{3}{x}$, $g(x) = x^2 - 1$

$(f \circ g)(x) = f(g(x)) = f(x^2 - 1) = \dfrac{3}{x^2 - 1}$

Domain: all $x \neq \pm 1$

$(g \circ f)(x) = g(f(x)) = g\left(\dfrac{3}{x}\right) = \left(\dfrac{3}{x}\right)^2 - 1 = \dfrac{9}{x^2} - 1 = \dfrac{9 - x^2}{x^2}$

Domain: all $x \neq 0$

No, $f \circ g \neq g \circ f$.

57. $(A \circ r)(t) = A(r(t)) = A(0.6t) = \pi(0.6t)^2 = 0.36\pi t^2$

$(A \circ r)(t)$ represents the area of the circle at time t.

59. $f(-x) = (-x)^2(4 - (-x)^2) = x^2(4 - x^2) = f(x)$

Even

61. $f(-x) = (-x)\cos(-x) = -x\cos x = -f(x)$

Odd

63. (a) If f is even, then $\left(\frac{3}{2}, 4\right)$ is on the graph.

(b) If f is odd, then $\left(\frac{3}{2}, -4\right)$ is on the graph.

65. $f(-x) = a_{2n+1}(-x)^{2n+1} + \cdots + a_3(-x)^3 + a_1(-x)$

$= -[a_{2n+1}x^{2n+1} + \cdots + a_3x^3 + a_1x]$

$= -f(x)$

Odd

67. Let $F(x) = f(x)g(x)$ where f and g are even. Then

$F(-x) = f(-x)g(-x) = f(x)g(x) = F(x)$.

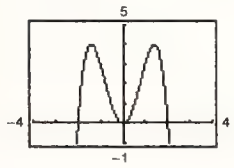
Thus, $F(x)$ is even. Let $F(x) = f(x)g(x)$ where f and g are odd. Then

$F(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = F(x)$.

Thus, $F(x)$ is even.

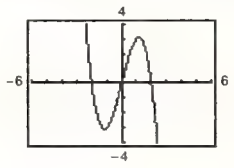
69. $f(x) = x^2 + 1$ and $g(x) = x^4$ are even.

$f(x)g(x) = (x^2 + 1)(x^4) = x^6 + x^4$ is even.



$f(x) = x^3 - x$ is odd and $g(x) = x^2$ is even.

$f(x)g(x) = (x^3 - x)(x^2) = x^5 - x^3$ is odd.



71. (a)

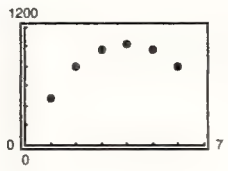
x	*length and width*	*volume V*
1	$24 - 2(1)$	484
2	$24 - 2(2)$	800
3	$24 - 2(3)$	972
4	$24 - 2(4)$	1024
5	$24 - 2(5)$	980
6	$24 - 2(6)$	864

The maximum volume appears to be 1024 cm^3.

(c) $V = x(24 - 2x)^2 = 4x(12 - x)^2$

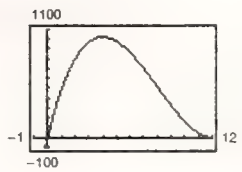
Domain: $0 < x < 12$

(b)



Yes, V is a function of x.

(d)



Maximum volume is $V = 1024$ cm^3 for box having dimensions $4 \times 16 \times 16$ cm.

73. False; let $f(x) = x^2$.

Then $f(-3) = f(3) = 9$, but $-3 \neq 3$.

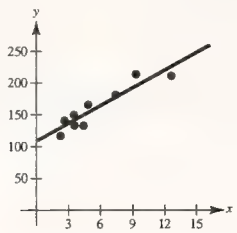
75. True, the function is even.

Section P.4 Fitting Models to Data

1. Quadratic function

3. Linear function

5. (a), (b)

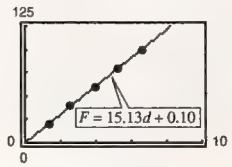


Yes. The cancer mortality increases linearly with increased exposure to the carcinogenic substance.

(c) If $x = 3$, then $y \approx 136$.

7. (a) $d = 0.066F$ or $F = 15.1d + 0.1$

(b)



The model fits well.

(c) If $F = 55$, then $d \approx 0.066(55) = 3.63$ cm.

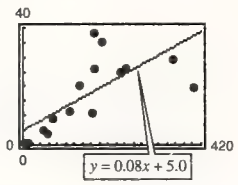
9. (a) Let x = per capita energy usage (in millions of Btu)

y = per capita gross national product (in thousands)

$y = 0.0764x + 4.9985 \approx 0.08x + 5.0$

$r = 0.7052$

(b)



(c) Denmark, Japan, and Canada

(d) Deleting the data for the three countries above,

$y = 0.0959x + 1.0539$

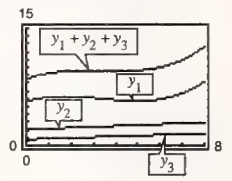
($r = 0.9202$ is much closer to 1.)

11. (a) $y_1 = 0.0343t^3 - 0.3451t^2 + 0.8837t + 5.6061$

$y_2 = 0.1095t + 2.0667$

$y_3 = 0.0917t + 0.7917$

(b)

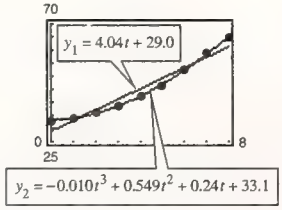


For 2002, $t = 12$ and $y_1 + y_2 + y_3 \approx 31.06$ cents/mile

13. (a) $y_1 = 4.0367t + 28.9644$

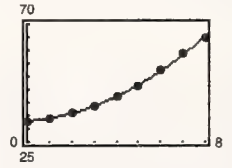
$y_2 = -0.0099t^3 + 0.5488t^2 + 0.2399t + 33.1414$

(b)



(c) The cubic model is better.

(d) $y_3 = 0.4297t^2 + 0.5994t + 32.9745$

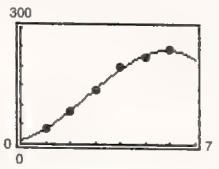


(e) The slope represents the average increase per year in the number of people (in millions) in HMOs.

(f) For 2000, $t = 10$, and $y_1 \approx 69.3$ million. (linear)

$y_2 \approx 80.5$ million (cubic)

15. (a) $y = -1.81x^3 + 14.58x^2 + 16.39x + 10$

(b) 

(c) If $x = 4.5$, $y \approx 214$ horsepower.

17. (a) Yes, y is a function of t. At each time t, there is one and only one displacement y.

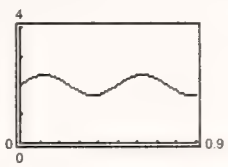
(b) The amplitude is approximately

$$(2.35 - 1.65)/2 = 0.35.$$

The period is approximately

$$2(0.375 - 0.125) = 0.5.$$

(c) One model is $y = 0.35 \sin(4\pi t) + 2$.

(d) 

19. Answers will vary.

Review Exercises for Chapter P

1. $y = 2x - 3$

$x = 0 \Rightarrow y = 2(0) - 3 = -3 \Rightarrow (0, -3)$ y-intercept

$y = 0 \Rightarrow 0 = 2x - 3 \Rightarrow x = \frac{3}{2} \Rightarrow \left(\frac{3}{2}, 0\right)$ x-intercept

3. $y = \dfrac{x - 1}{x - 2}$

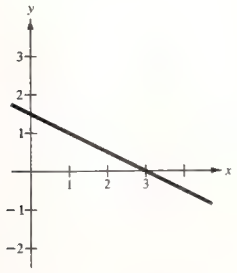
$x = 0 \Rightarrow y = \dfrac{0 - 1}{0 - 2} = \dfrac{1}{2} \Rightarrow \left(0, \dfrac{1}{2}\right)$ y-intercept

$y = 0 \Rightarrow 0 = \dfrac{x - 1}{x - 2} \Rightarrow x = 1 \Rightarrow (1, 0)$ x-intercept

5. Symmetric with respect to y-axis since

$$(-x)^2y - (-x)^2 + 4y = 0$$
$$x^2y - x^2 + 4y = 0.$$

7. $y = -\frac{1}{2}x + \frac{3}{2}$



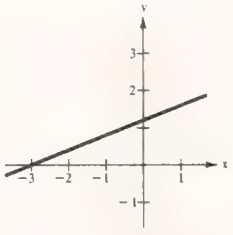
9. $-\frac{1}{3}x + \frac{5}{6}y = 1$

$-\frac{2}{5}x + y = \frac{6}{5}$

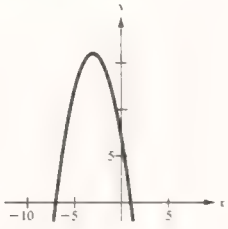
$y = \frac{2}{5}x + \frac{6}{5}$

Slope: $\frac{2}{5}$

y-intercept: $\frac{6}{5}$

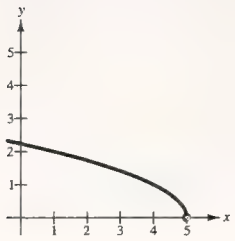


11. $y = 7 - 6x - x^2$



13. $y = \sqrt{5 - x}$

Domain: $(-\infty, 5]$



15. $y = 4x^2 - 25$

Xmin = -5
Xmax = 5
Xscl = 1
Ymin = -30
Ymax = 10
Yscl = 5

17. $3x - 4y = 8$

$4x + 4y = 20$

$7x \quad = 28$

$x = 4$

$y = 1$

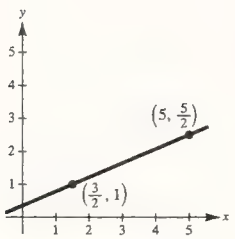
Point: (4, 1)

19. You need factors $(x + 2)$ and $(x - 2)$. Multiply by x to obtain origin symmetry

$y = x(x + 2)(x - 2)$

$= x^3 - 4x.$

21.



$$\text{Slope} = \frac{(5/2) - 1}{5 - (3/2)} = \frac{3/2}{7/2} = \frac{3}{7}$$

23. $\dfrac{1 - t}{1 - 0} = \dfrac{1 - 5}{1 - (-2)}$

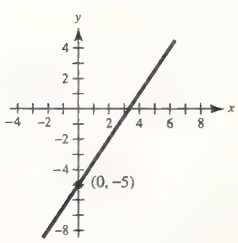
$$1 - t = -\frac{4}{3}$$

$$t = \frac{7}{3}$$

25. $y - (-5) = \frac{3}{2}(x - 0)$

$y = \frac{3}{2}x - 5$

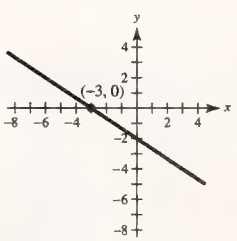
$2y - 3x + 10 = 0$



27. $y - 0 = -\frac{2}{3}(x - (-3))$

$y = -\frac{2}{3}x - 2$

$3y + 2x + 6 = 0$



29. (a) $y - 4 = \frac{7}{16}(x + 2)$

$16y - 64 = 7x + 14$

$0 = 7x - 16y + 78$

(b) Slope of line is $\frac{5}{3}$.

$y - 4 = \frac{5}{3}(x + 2)$

$3y - 12 = 5x + 10$

$0 = 5x - 3y + 22$

(c) $m = \frac{4 - 0}{-2 - 0} = -2$

$y = -2x$

$2x + y = 0$

(d) $x = -2$

$x + 2 = 0$

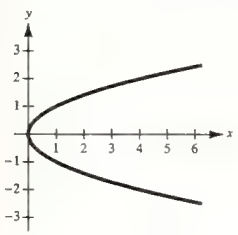
31. The slope is -850. $V = -850t + 12{,}500$.

$V(3) = -850(3) + 12{,}500 = \$9950$

33. $x - y^2 = 0$

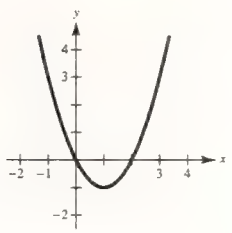
$y = \pm\sqrt{x}$

Not a function of x since there are two values of y for some x.



35. $y = x^2 - 2x$

Function of x since there is one value of y for each x.



37. $f(x) = \frac{1}{x}$

(a) $f(0)$ does not exist.

(b) $\frac{f(1 + \Delta x) - f(1)}{\Delta x} = \frac{\frac{1}{1 + \Delta x} - \frac{1}{1}}{\Delta x} = \frac{1 - 1 - \Delta x}{(1 + \Delta x)\Delta x}$

$= \frac{-1}{1 + \Delta x}, \Delta x \neq -1, 0$

39. (a) Domain: $36 - x^2 \geq 0 \Rightarrow -6 \leq x \leq 6$ or $[-6, 6]$

Range: $[0, 6]$

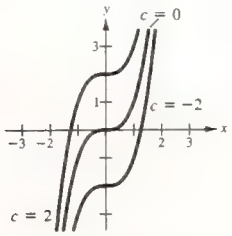
(b) Domain: all $x \neq 5$ or $(-\infty, 5), (5, \infty)$

Range: all $y \neq 0$ or $(-\infty, 0), (0, \infty)$

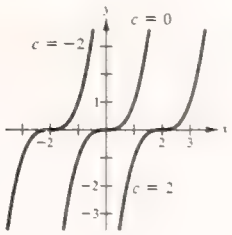
(c) Domain: all x or $(-\infty, \infty)$

Range: all y or $(-\infty, \infty)$

41. (a) $f(x) = x^3 + c, c = -2, 0, 2$



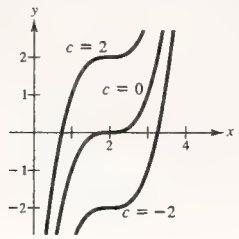
(b) $f(x) = (x - c)^3, c = -2, 0, 2$



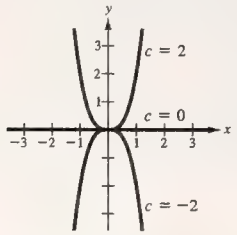
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41. —CONTINUED—

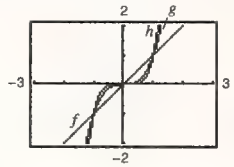
(c) $f(x) = (x - 2)^3 + c, c = -2, 0, 2$



(d) $f(x) = cx^3, c = -2, 0, 2$

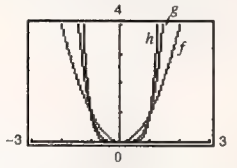


43. (a) Odd powers: $f(x) = x, g(x) = x^3, h(x) = x^5$



The graphs of f, g, and h all rise to the right and fall to the left. As the degree increases, the graph rises and falls more steeply. All three graphs pass through the points $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

Even powers: $f(x) = x^2, g(x) = x^4, h(x) = x^6$

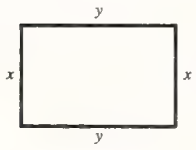


The graphs of f, g, and h all rise to the left and to the right. As the degree increases, the graph rises more steeply. All three graphs pass through the points $(0, 0)$, $(1, 1)$, and $(-1, 1)$.

(b) $y = x^7$ will look like $h(x) = x^5$, but rise and fall even more steeply.

$y = x^8$ will look like $h(x) = x^6$, but rise even more steeply.

45. (a)

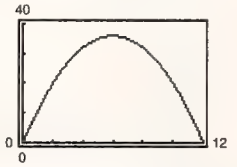


$2x + 2y = 24$

$y = 12 - x$

$A = xy = x(12 - x) = 12x - x^2$

(b) Domain: $0 < x < 12$



(c) Maximum area is $A = 36$. In general, the maximum area is attained when the rectangle is a square. In this case, $x = 6$.

47. (a) 3 (cubic), negative leading coefficient

(b) 4 (quartic), positive leading coefficient

(c) 2 (quadratic), negative leading coefficient

(d) 5, positive leading coefficient

49. (a) Yes, y is a function of t. At each time t, there is one and only one displacement y.

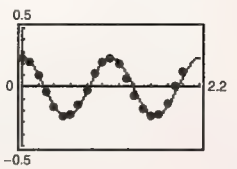
(b) The amplitude is approximately

$$(0.25 - (-0.25))/2 = 0.25.$$

The period is approximately 1.1.

(c) One model is $y = \frac{1}{4}\cos\left(\frac{2\pi}{1.1}t\right) \approx \frac{1}{4}\cos(5.7t)$

(d)



Problem Solving for Chapter P

1. (a) $x^2 - 6x + y^2 - 8y = 0$

$(x^2 - 6x + 9) + (y^2 - 8y + 16) = 9 + 16$

$(x - 3)^2 + (y - 4)^2 = 25$

Center: $(3, 4)$ Radius: 5

(c) Slope of line from $(6, 0)$ to $(3, 4)$ is $\dfrac{4 - 0}{3 - 6} = -\dfrac{4}{3}$.

Slope of tangent line is $\dfrac{3}{4}$. Hence,

$y - 0 = \dfrac{3}{4}(x - 6) \Rightarrow y = \dfrac{3}{4}x - \dfrac{9}{2}$ Tangent line

(b) Slope of line from $(0, 0)$ to $(3, 4)$ is $\dfrac{4}{3}$ Slope of tangent line is $-\dfrac{3}{4}$ Hence,

$y - 0 = -\dfrac{3}{4}(x - 0) \Rightarrow y = -\dfrac{3}{4}x$ Tangent line

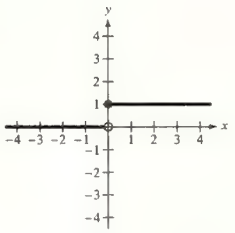
(d) $-\dfrac{3}{4}x = \dfrac{3}{4}x - \dfrac{9}{2}$

$\dfrac{3}{2}x = \dfrac{9}{2}$

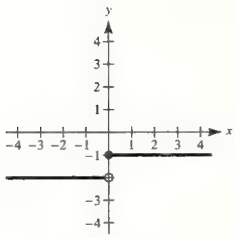
$x = 3$

Intersection: $\left(3, -\dfrac{9}{4}\right)$

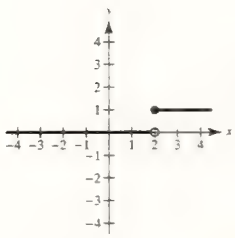
3. $H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$



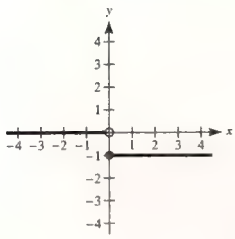
(a) $H(x) - 2$



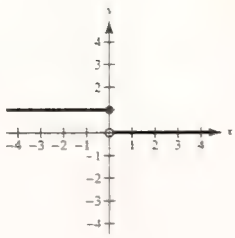
(b) $H(x - 2)$



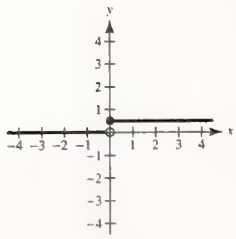
(c) $-H(x)$



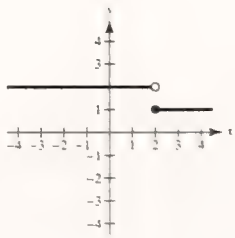
(d) $H(-x)$



(e) $\frac{1}{2}H(x)$



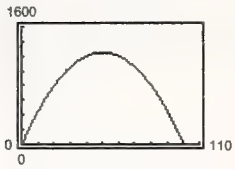
(f) $-H(x - 2) + 2$



5. (a) $x + 2y = 100 \Rightarrow y = \dfrac{100 - x}{2}$

$$A(x) = xy = x\left(\frac{100 - x}{2}\right) = -\frac{x^2}{2} + 50x$$

Domain: $0 < x < 100$

(b) 

Maximum of 1250 m^2 at $x = 50$ m, $y = 25$ m.

(c) $A(x) = -\dfrac{1}{2}(x^2 - 100x)$

$$= -\frac{1}{2}(x^2 - 100x + 2500) + 1250$$

$$= -\frac{1}{2}(x - 50)^2 + 1250$$

$A(50) = 1250$ m^2 is the maximum. $x = 50$ m, $y = 25$ m.

7. The length of the trip in the water is $\sqrt{2^2 + x^2}$, and the length of the trip over land is $\sqrt{1 + (3 - x)^2}$. Hence, the total time is

$$T = \frac{\sqrt{4 + x^2}}{2} + \frac{\sqrt{1 + (3 - x)^2}}{4} \text{ hours.}$$

9. (a) Slope $= \dfrac{9 - 4}{3 - 2} = 5$. Slope of tangent line is less than 5.

(b) Slope $= \dfrac{4 - 1}{2 - 1} = 3$. Slope of tangent line is greater than 3.

(c) Slope $= \dfrac{4.41 - 4}{2.1 - 2} = 4.1$. Slope of tangent line is less than 4.1.

(d) Slope $= \dfrac{f(2 + h) - f(2)}{(2 + h) - 2}$

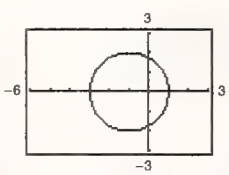
$$= \frac{(2 + h)^2 - 4}{h}$$

$$= \frac{4h + h^2}{h}$$

$$= 4 + h, h \neq 0$$

(e) Letting h get closer and closer to 0, the slope approaches 4. Hence, the slope at $(2, 4)$ is 4.

11. (a) At $x = 1$ and $x = -3$ the sounds are equal.



(b) $$\frac{I}{\sqrt{x^2 + y^2}} = \frac{2I}{\sqrt{(x - 3)^2 + y^2}}$$

$$(x - 3)^2 + y^2 = 4(x^2 + y^2)$$

$$3x^2 + 3y^2 + 6x = 9$$

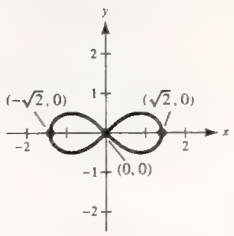
$$x^2 + 2x + y^2 = 3$$

$$(x + 1)^2 + y^2 = 4$$

Circle of radius 2 centered at $(-1, 0)$

13.

$$\begin{aligned} d_1d_2 &= 1 \\ [(x+1)^2 + y^2][(x-1)^2 + y^2] &= 1 \\ (x+1)^2(x-1)^2 + y^2[(x+1)^2 + (x-1)^2] + y^4 &= 1 \\ (x^2-1)^2 + y^2[2x^2+2] + y^4 &= 1 \\ x^4 - 2x^2 + 1 + 2x^2y^2 + 2y^2 + y^4 &= 1 \\ (x^4 + 2x^2y^2 + y^4) - 2x^2 + 2y^2 &= 0 \\ (x^2+y^2)^2 &= 2(x^2-y^2) \end{aligned}$$



Let $y = 0$. Then $x^4 = 2x^2 \Rightarrow x = 0$ or $x^2 = 2$.

Thus, $(0, 0)$, $\left(\sqrt{2}, 0\right)$ and $\left(-\sqrt{2}, 0\right)$ are on the curve.

CHAPTER 1
Limits and Their Properties

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CHAPTER 1
Limits and Their Properties

Section 1.1 A Preview of Calculus

Solutions to Odd-Numbered Exercises

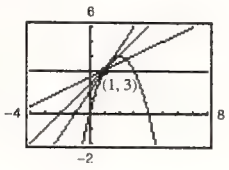
1. Precalculus: (20 ft/sec)(15 seconds) = 300 feet

3. Calculus required: slope of tangent line at $x = 2$ is rate of change, and equals about 0.16.

5. Precalculus: Area $= \frac{1}{2}bh = \frac{1}{2}(5)(3) = \frac{15}{2}$ sq. units

7. Precalculus: Volume = (2)(4)(3) = 24 cubic units

9. (a)



(b) The graphs of y_2 are approximations to the tangent line to y_1 at $x = 1$.

(c) The slope is approximately 2. For a better approximation make the list numbers smaller:
{0.2, 0.1, 0.01, 0.001}

11. (a) $D_1 = \sqrt{(5-1)^2 + (1-5)^2} = \sqrt{16+16} \approx 5.66$

(b) $D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{5}{3}\right)^2} + \sqrt{1 + \left(\frac{5}{3} - \frac{5}{4}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - 1\right)^2}$
$\approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11$

(c) Increase the number of line segments.

Section 1.2 Finding Limits Graphically and Numerically

1.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.3448	0.3344	0.3334	0.3332	0.3322	0.3226

$\lim_{x \to 2} \dfrac{x-2}{x^2 - x - 2} \approx 0.3333$ (Actual limit is $\frac{1}{3}$.)

3.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$	0.2911	0.2889	0.2887	0.2887	0.2884	0.2863

$\lim_{x \to 0} \dfrac{\sqrt{x+3} - \sqrt{3}}{x} \approx 0.2887$ (Actual limit is $1/(2\sqrt{3})$.)

5.

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$	−0.0641	−0.0627	−0.0625	−0.0625	−0.0623	−0.0610

$$\lim_{x\to 3}\frac{[1/(x+1)]-(1/4)}{x-3}\approx -0.0625 \quad \left(\text{Actual limit is } -\tfrac{1}{16}.\right)$$

7.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$	0.9983	0.99998	1.0000	1.0000	0.99998	0.9983

$$\lim_{x\to 0}\frac{\sin x}{x}\approx 1.0000$$ (Actual limit is 1.) (Make sure you use radian mode.)

9. $\lim_{x\to 3}(4-x)=1$

11. $\lim_{x\to 2} f(x)=\lim_{x\to 2}(4-x)=2$

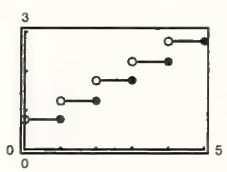
13. $\lim_{x\to 5}\dfrac{|x-5|}{x-5}$ does not exist. For values of x to the left of 5, $|x-5|/(x-5)$ equals -1, whereas for values of x to the right of 5, $|x-5|/(x-5)$ equals 1.

15. $\lim_{x\to \pi/2}\tan x$ does not exist since the function increases and decreases without bound as x approaches $\pi/2$.

17. $\lim_{x\to 0}\cos(1/x)$ does not exist since the function oscillates between -1 and 1 as x approaches 0.

19. $C(t)=0.75-0.50[\![-(t-1)]\!]$

(a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	1.75	2.25	2.25	2.25	2.25	2.25	2.25

$\lim_{t\to 3.5} C(t)=2.25$

(c)

t	2	2.5	2.9	3	3.1	3.5	4
C	1.25	1.75	1.75	1.75	2.25	2.25	2.25

$\lim_{t\to 3} C(t)$ does not exist. The values of C jump from 1.75 to 2.25 at $t=3$.

21. You need to find δ such that $0<|x-1|<\delta$ implies $|f(x)-1|=\left|\dfrac{1}{x}-1\right|<0.1$. That is,

$$-0.1<\frac{1}{x}-1<0.1$$

$$1-0.1<\frac{1}{x}<1+0.1$$

$$\frac{9}{10}<\frac{1}{x}<\frac{11}{10}$$

$$\frac{10}{9}>x>\frac{10}{11}$$

$$\frac{10}{9}-1>x-1>\frac{10}{11}-1$$

$$\frac{1}{9}>x-1>-\frac{1}{11}.$$

So take $\delta=\dfrac{1}{11}$. Then $0<|x-1|<\delta$ implies

$$-\frac{1}{11}<x-1<\frac{1}{11}$$

$$-\frac{1}{11}<x-1<\frac{1}{9}.$$

Using the first series of equivalent inequalities, you obtain

$$|f(x)-1|=\left|\frac{1}{x}-1\right|<\epsilon<0.1.$$

23. $\lim_{x\to 2}(3x+2) = 8 = L$

$|(3x+2)-8| < 0.01$

$|3x-6| < 0.01$

$3|x-2| < 0.01$

$0 < |x-2| < \dfrac{0.01}{3} \approx 0.0033 = \delta$

Hence, if $0 < |x-2| < \delta = \dfrac{0.01}{3}$, you have

$3|x-2| < 0.01$

$|3x-6| < 0.01$

$|(3x+2)-8| < 0.01$

$|f(x)-L| < 0.01$

25. $\lim_{x\to 2}(x^2-3) = 1 = L$

$|(x^2-3)-1| < 0.01$

$|x^2-4| < 0.01$

$|(x+2)(x-2)| < 0.01$

$|x+2|\,|x-2| < 0.01$

$|x-2| < \dfrac{0.01}{|x+2|}$

If we assume $1 < x < 3$, then $\delta = 0.01/5 = 0.002$.

Hence, if $0 < |x-2| < \delta = 0.002$, you have

$|x-2| < 0.002 = \dfrac{1}{5}(0.01) < \dfrac{1}{|x+2|}(0.01)$

$|x+2||x-2| < 0.01$

$|x^2-4| < 0.01$

$|(x^2-3)-1| < 0.01$

$|f(x)-L| < 0.01$

27. $\lim_{x\to 2}(x+3) = 5$

Given $\epsilon > 0$:

$|(x+3)-5| < \epsilon$

$|x-2| < \epsilon = \delta$

Hence, let $\delta = \epsilon$.

Hence, if $0 < |x-2| < \delta = \epsilon$, you have

$|x-2| < \epsilon$

$|(x+3)-5| < \epsilon$

$|f(x)-L| < \epsilon$

29. $\lim_{x\to -4}\left(\frac{1}{2}x-1\right) = \frac{1}{2}(-4)-1 = -3$

Given $\epsilon > 0$:

$\left|\left(\frac{1}{2}x-1\right)-(-3)\right| < \epsilon$

$\left|\frac{1}{2}x+2\right| < \epsilon$

$\frac{1}{2}|x-(-4)| < \epsilon$

$|x-(-4)| < 2\epsilon$

Hence, let $\delta = 2\epsilon$.

Hence, if $0 < |x-(-4)| < \delta = 2\epsilon$, you have

$|x-(-4)| < 2\epsilon$

$\left|\frac{1}{2}x+2\right| < \epsilon$

$\left|\left(\frac{1}{2}x-1\right)+3\right| < \epsilon$

$|f(x)-L| < \epsilon$

31. $\lim_{x\to 6} 3 = 3$

Given $\epsilon > 0$:

$|3-3| < \epsilon$

$0 < \epsilon$

Hence, any $\delta > 0$ will work.

Hence, for any $\delta > 0$, you have

$|3-3| < \epsilon$

$|f(x)-L| < \epsilon$

33. $\lim_{x\to 0}\sqrt[3]{x} = 0$

Given $\epsilon > 0$: $\left|\sqrt[3]{x}-0\right| < \epsilon$

$\left|\sqrt[3]{x}\right| < \epsilon$

$|x| < \epsilon^3 = \delta$

Hence, let $\delta = \epsilon^3$.

Hence for $0 < |x-0| < \delta = \epsilon^3$, you have

$|x| < \epsilon^3$

$|\sqrt[3]{x}| < \epsilon$

$|\sqrt[3]{x}-0| < \epsilon$

$|f(x)-L| < \epsilon$

35. $\lim_{x\to -2} |x-2| = |(-2)-2| = 4$

Given $\epsilon > 0$:

$$||x-2|-4| < \epsilon$$
$$|-(x-2)-4| < \epsilon \quad (x-2<0)$$
$$|-x-2| = |x+2| = |x-(-2)| < \epsilon$$

Hence, $\delta = \epsilon$.

Hence for $0 < |x-(-2)| < \delta = \epsilon$, you have

$$|x+2| < \epsilon$$
$$|-(x+2)| < \epsilon$$
$$|-(x-2)-4| < \epsilon$$
$$||x-2|-4| < \epsilon \quad \text{(because } x-20\text{)}$$
$$|f(x)-L| < \epsilon$$

37. $\lim_{x\to 1} (x^2+1) = 2$

Given $\epsilon > 0$:

$$|(x^2+1)-2| < \epsilon$$
$$|x^2-1| < \epsilon$$
$$|(x+1)(x-1)| < \epsilon$$
$$|x-1| < \frac{\epsilon}{|x+1|}$$

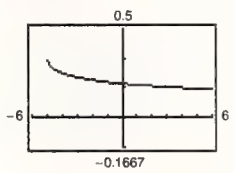
If we assume $0 < x < 2$, then $\delta = \epsilon/3$.

Hence for $0 < |x-1| < \delta = \frac{\epsilon}{3}$, you have

$$|x-1| < \frac{1}{3}\epsilon < \frac{1}{|x+1|}\epsilon$$
$$|x^2-1| < \epsilon$$
$$|(x^2+1)-2| < \epsilon$$
$$|f(x)-2| < \epsilon$$

39. $f(x) = \dfrac{\sqrt{x+5}-3}{x-4}$

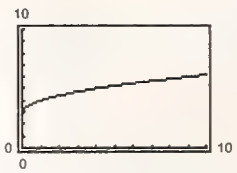
$\lim_{x\to 4} f(x) = \frac{1}{6}$



The domain is $[-5, 4) \cup (4, \infty)$.
The graphing utility does not show the hole at $\left(4, \frac{1}{6}\right)$.

41. $f(x) = \dfrac{x-9}{\sqrt{x}-3}$

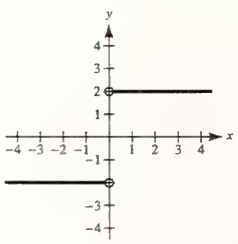
$\lim_{x\to 9} f(x) = 6$



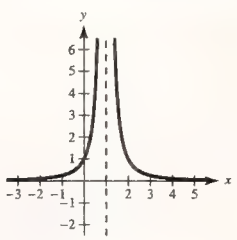
The domain is all $x \geq 0$ except $x = 9$. The graphing utility does not show the hole at $(9, 6)$.

43. $\lim_{x\to 8} f(x) = 25$ means that the values of f approach 25 as x gets closer and closer to 8.

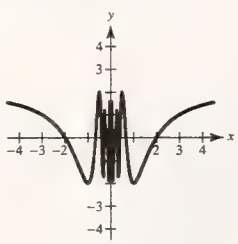
45. (i) The values of f approach different numbers as x approaches c from different sides of c:



(ii) The values of f increase without bound as x approaches c:

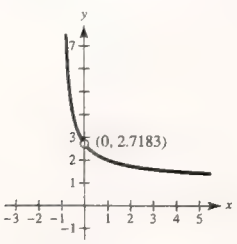


(iii) The values of f oscillate between two fixed numbers as x approaches c:



47. $f(x) = (1+x)^{1/x}$

$\lim_{x\to 0} (1+x)^{1/x} = e \approx 2.71828$



x	$f(x)$	x	$f(x)$
-0.1	2.867972	0.1	2.593742
-0.01	2.731999	0.01	2.704814
-0.001	2.719642	0.001	2.716942
-0.0001	2.718418	0.0001	2.718146
-0.00001	2.718295	0.00001	2.718268
-0.000001	2.718283	0.000001	2.718280

49. False; $f(x) = (\sin x)/x$ is undefined when $x = 0$.
From Exercise 7, we have

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

51. False; let

$$f(x) = \begin{cases} x^2 - 4x, & x \neq 4 \\ 10, & x = 4 \end{cases}.$$

$f(4) = 10$

$\lim_{x\to 4} f(x) = \lim_{x\to 4} (x^2 - 4x) = 0 \neq 10$

53. Answers will vary.

55. If $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} f(x) = L_2$, then for every $\epsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \epsilon$ and $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \epsilon$. Let δ equal the smaller of δ_1 and δ_2. Then for $|x - c| < \delta$, we have

$$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \epsilon + \epsilon.$$

Therefore, $|L_1 - L_2| < 2\epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

57. $\lim_{x\to c} [f(x) - L] = 0$ means that for every $\epsilon > 0$ there exists $\delta > 0$ such that if

$$0 < |x - c| < \delta,$$

then

$$|(f(x) - L) - 0| < \epsilon.$$

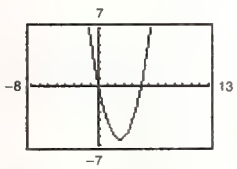
This means the same as $|f(x) - L| < \epsilon$ when

$$0 < |x - c| < \delta.$$

Thus, $\lim_{x\to c} f(x) = L$.

Section 1.3 Evaluating Limits Analytically

1.

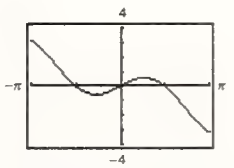


(a) $\lim_{x\to 5} h(x) = 0$

(b) $\lim_{x\to -1} h(x) = 6$

$h(x) = x^2 - 5x$

3.



(a) $\lim_{x\to 0} f(x) = 0$

(b) $\lim_{x\to \pi/3} f(x) \approx 0.524 \left(= \frac{\pi}{6}\right)$

$f(x) = x \cos x$

5. $\lim_{x\to 2} x^4 = 2^4 = 16$

7. $\lim_{x\to 0} (2x - 1) = 2(0) - 1 = -1$

9. $\lim_{x\to -3} (x^2 + 3x) = (-3)^2 + 3(-3) = 9 - 9 = 0$

11. $\lim_{x\to -3} (2x^2 + 4x + 1) = 2(-3)^2 + 4(-3) + 1 = 18 - 12 + 1 = 7$

13. $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$

15. $\lim_{x\to 1} \frac{x - 3}{x^2 + 4} = \frac{1 - 3}{1^2 + 4} = \frac{-2}{5} = -\frac{2}{5}$

17. $\lim_{x\to 7} \frac{5x}{\sqrt{x + 2}} = \frac{5(7)}{\sqrt{7 + 2}} = \frac{35}{\sqrt{9}} = \frac{35}{3}$

19. $\lim_{x\to 3} \sqrt{x + 1} = \sqrt{3 + 1} = 2$

21. $\lim_{x\to -4}(x+3)^2 = (-4+3)^2 = 1$

23. (a) $\lim_{x\to 1} f(x) = 5 - 1 = 4$

(b) $\lim_{x\to 4} g(x) = 4^3 = 64$

(c) $\lim_{x\to 1} g(f(x)) = g(f(1)) = g(4) = 64$

25. (a) $\lim_{x\to 1} f(x) = 4 - 1 = 3$

(b) $\lim_{x\to 3} g(x) = \sqrt{3+1} = 2$

(c) $\lim_{x\to 1} g(f(x)) = g(3) = 2$

27. $\lim_{x\to \pi/2} \sin x = \sin\frac{\pi}{2} = 1$

29. $\lim_{x\to 2} \cos\frac{\pi x}{3} = \cos\frac{\pi 2}{3} = -\frac{1}{2}$

31. $\lim_{x\to 0} \sec 2x = \sec 0 = 1$

33. $\lim_{x\to 5\pi/6} \sin x = \sin\frac{5\pi}{6} = \frac{1}{2}$

35. $\lim_{x\to 3} \tan\left(\frac{\pi x}{4}\right) = \tan\frac{3\pi}{4} = -1$

37. (a) $\lim_{x\to c}[5g(x)] = 5\lim_{x\to c} g(x) = 5(3) = 15$

(b) $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = 2 + 3 = 5$

(c) $\lim_{x\to c}[f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = (2)(3) = 6$

(d) $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{2}{3}$

39. (a) $\lim_{x\to c}[f(x)]^3 = \left[\lim_{x\to c} f(x)\right]^3 = (4)^3 = 64$

(b) $\lim_{x\to c}\sqrt{f(x)} = \sqrt{\lim_{x\to c} f(x)} = \sqrt{4} = 2$

(c) $\lim_{x\to c}[3f(x)] = 3\lim_{x\to c} f(x) = 3(4) = 12$

(d) $\lim_{x\to c}[f(x)]^{3/2} = \left[\lim_{x\to c} f(x)\right]^{3/2} = (4)^{3/2} = 8$

41. $f(x) = -2x + 1$ and $g(x) = \dfrac{-2x^2 + x}{x}$ agree except at $x = 0$.

(a) $\lim_{x\to 0} g(x) = \lim_{x\to 0} f(x) = 1$

(b) $\lim_{x\to -1} g(x) = \lim_{x\to -1} f(x) = 3$

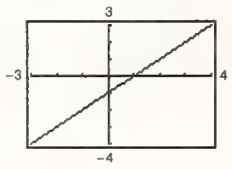
43. $f(x) = x(x+1)$ and $g(x) = \dfrac{x^3 - x}{x - 1}$ agree except at $x = 1$.

(a) $\lim_{x\to 1} g(x) = \lim_{x\to 1} f(x) = 2$

(b) $\lim_{x\to -1} g(x) = \lim_{x\to -1} f(x) = 0$

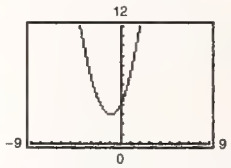
45. $f(x) = \dfrac{x^2 - 1}{x + 1}$ and $g(x) = x - 1$ agree except at $x = -1$.

$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = -2$



47. $f(x) = \dfrac{x^3 - 8}{x - 2}$ and $g(x) = x^2 + 2x + 4$ agree except at $x = 2$.

$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x) = 12$



49. $\lim_{x\to 5}\dfrac{x-5}{x^2-25} = \lim_{x\to 5}\dfrac{x-5}{(x+5)(x-5)}$

$= \lim_{x\to 5}\dfrac{1}{x+5} = \dfrac{1}{10}$

51. $\lim_{x\to -3}\dfrac{x^2+x-6}{x^2-9} = \lim_{x\to -3}\dfrac{(x+3)(x-2)}{(x+3)(x-3)}$

$= \lim_{x\to -3}\dfrac{x-2}{x-3} = \dfrac{-5}{-6} = \dfrac{5}{6}$

53. $\lim_{x\to 0}\dfrac{\sqrt{x+5}-\sqrt{5}}{x}=\lim_{x\to 0}\dfrac{\sqrt{x+5}-\sqrt{5}}{x}\cdot\dfrac{\sqrt{x+5}+\sqrt{5}}{\sqrt{x+5}+\sqrt{5}}$

$$=\lim_{x\to 0}\frac{(x+5)-5}{x\left(\sqrt{x+5}+\sqrt{5}\right)}=\lim_{x\to 0}\frac{1}{\sqrt{x+5}+\sqrt{5}}=\frac{1}{2\sqrt{5}}=\frac{\sqrt{5}}{10}$$

55. $\lim_{x\to 4}\dfrac{\sqrt{x+5}-3}{x-4}=\lim_{x\to 4}\dfrac{\sqrt{x+5}-3}{x-4}\cdot\dfrac{\sqrt{x+5}+3}{\sqrt{x+5}+3}$

$$=\lim_{x\to 4}\frac{(x+5)-9}{(x-4)\left(\sqrt{x+5}+3\right)}=\lim_{x\to 4}\frac{1}{\sqrt{x+5}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{6}$$

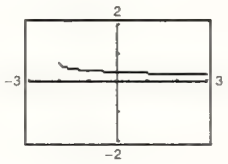
57. $\lim_{x\to 0}\dfrac{\dfrac{1}{2+x}-\dfrac{1}{2}}{x}=\lim_{x\to 0}\dfrac{\dfrac{2-(2+x)}{2(2+x)}}{x}=\lim_{x\to 0}\dfrac{-1}{2(2+x)}=-\dfrac{1}{4}$

59. $\lim_{\Delta x\to 0}\dfrac{2(x+\Delta x)-2x}{\Delta x}=\lim_{\Delta x\to 0}\dfrac{2x+2\Delta x-2x}{\Delta x}=\lim_{\Delta x\to 0}2=2$

61. $\lim_{\Delta x\to 0}\dfrac{(x+\Delta x)^2-2(x+\Delta x)+1-(x^2-2x+1)}{\Delta x}=\lim_{\Delta x\to 0}\dfrac{x^2+2x\Delta x+(\Delta x)^2-2x-2\Delta x+1-x^2+2x-1}{\Delta x}$

$$=\lim_{\Delta x\to 0}(2x+\Delta x-2)=2x-2$$

63. $\lim_{x\to 0}\dfrac{\sqrt{x+2}-\sqrt{2}}{x}\approx 0.354$

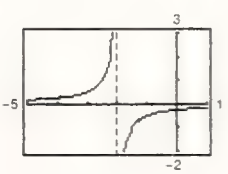


x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.345	?	0.354	0.353	0.349

Analytically, $\lim_{x\to 0}\dfrac{\sqrt{x+2}-\sqrt{2}}{x}=\lim_{x\to 0}\dfrac{\sqrt{x+2}-\sqrt{2}}{x}\cdot\dfrac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}$

$$=\lim_{x\to 0}\frac{x+2-2}{x\left(\sqrt{x+2}+\sqrt{2}\right)}=\lim_{x\to 0}\frac{1}{\sqrt{x+2}+\sqrt{2}}=\frac{1}{2\sqrt{2}}=\frac{\sqrt{2}}{4}\approx 0.354$$

65. $\lim_{x\to 0}\dfrac{\dfrac{1}{2+x}-\dfrac{1}{2}}{x}=-\dfrac{1}{4}$



x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.263	-0.251	-0.250	?	-0.250	-0.249	-0.238

Analytically, $\lim_{x\to 0}\dfrac{\dfrac{1}{2+x}-\dfrac{1}{2}}{x}=\lim_{x\to 0}\dfrac{2-(2+x)}{2(2+x)}\cdot\dfrac{1}{x}=\lim_{x\to 0}\dfrac{-x}{2(2+x)}\cdot\dfrac{1}{x}=\lim_{x\to 0}\dfrac{-1}{2(2+x)}=-\dfrac{1}{4}$.

67. $\lim_{x\to 0}\frac{\sin x}{5x} = \lim_{x\to 0}\left[\left(\frac{\sin x}{x}\right)\left(\frac{1}{5}\right)\right] = (1)\left(\frac{1}{5}\right) = \frac{1}{5}$

69. $\lim_{x\to 0}\frac{\sin x(1-\cos x)}{2x^2} = \lim_{x\to 0}\left[\frac{1}{2}\cdot\frac{\sin x}{x}\cdot\frac{1-\cos x}{x}\right]$

$= \frac{1}{2}(1)(0) = 0$

71. $\lim_{x\to 0}\frac{\sin^2 x}{x} = \lim_{x\to 0}\left[\frac{\sin x}{x}\sin x\right] = (1)\sin 0 = 0$

73. $\lim_{h\to 0}\frac{(1-\cos h)^2}{h} = \lim_{h\to 0}\left[\frac{1-\cos h}{h}(1-\cos h)\right]$

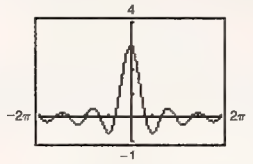
$= (0)(0) = 0$

75. $\lim_{x\to \pi/2}\frac{\cos x}{\cot x} = \lim_{x\to \pi/2}\sin x = 1$

77. $\lim_{t\to 0}\frac{\sin 3t}{2t} = \lim_{t\to 0}\left(\frac{\sin 3t}{3t}\right)\left(\frac{3}{2}\right) = (1)\left(\frac{3}{2}\right) = \frac{3}{2}$

79. $f(t) = \dfrac{\sin 3t}{t}$

t	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(t)$	2.96	2.9996	3	?	3	2.9996	2.96

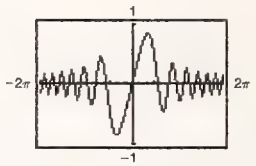


The limit appear to equal 3.

Analytically, $\lim_{t\to 0}\frac{\sin 3t}{t} = \lim_{t\to 0} 3\left(\frac{\sin 3t}{3t}\right) = 3(1) = 3$.

81. $f(x) = \dfrac{\sin x^2}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.099998	-0.01	-0.001	?	0.001	0.01	0.099998



Analytically, $\lim_{x\to 0}\frac{\sin x^2}{x} = \lim_{x\to 0} x\left(\frac{\sin x^2}{x^2}\right) = 0(1) = 0$.

83. $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{2(x+h)+3-(2x+3)}{h} = \lim_{h\to 0}\frac{2x+2h+3-2x-3}{h} = \lim_{h\to 0}\frac{2h}{h} = 2$

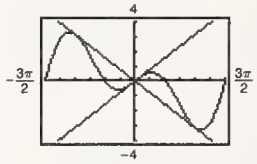
85. $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{4}{x+h}-\frac{4}{x}}{h} = \lim_{h\to 0}\frac{4x-4(x+h)}{(x+h)xh} = \lim_{h\to 0}\frac{-4}{(x+h)x} = \frac{-4}{x^2}$

87. $\lim_{x\to 0}(4-x^2) \le \lim_{x\to 0} f(x) \le \lim_{x\to 0}(4+x^2)$

$4 \le \lim_{x\to 0} f(x) \le 4$

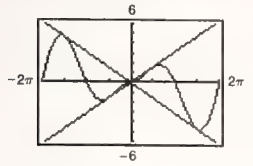
Therefore, $\lim_{x\to 0} f(x) = 4$.

89. $f(x) = x\cos x$



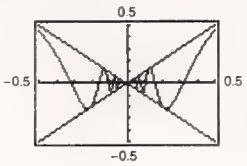
$\lim_{x\to 0}(x\cos x) = 0$

91. $f(x) = |x| \sin x$



$\lim_{x \to 0} |x| \sin x = 0$

93. $f(x) = x \sin \frac{1}{x}$



$\lim_{x \to 0} \left(x \sin \frac{1}{x} \right) = 0$

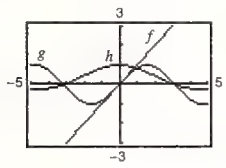
95. We say that two functions f and g agree at all but one point (on an open interval) if $f(x) = g(x)$ for all x in the interval except for $x = c$, where c is in the interval.

97. An indeterminant form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as $0/0$. That is,

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

for which $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$

99. $f(x) = x$, $g(x) = \sin x$, $h(x) = \dfrac{\sin x}{x}$



When you are "close to" 0 the magnitude of f is approximately equal to the magnitude of g. Thus, $|g|/|f| \approx 1$ when x is "close to" 0.

101. $s(t) = -16t^2 + 1000$

$$\lim_{t \to 5} \frac{s(5) - s(t)}{5 - t} = \lim_{t \to 5} \frac{600 - (-16t^2 + 1000)}{5 - t} = \lim_{t \to 5} \frac{16(t + 5)(t - 5)}{-(t - 5)} = \lim_{t \to 5} -16(t + 5) = -160 \text{ ft/sec.}$$

Speed = 160 ft/sec

103. $s(t) = -4.9t^2 + 150$

$$\lim_{t \to 3} \frac{s(3) - s(t)}{3 - t} = \lim_{t \to 3} \frac{-4.9(3^2) + 150 - (-4.9t^2 + 150)}{3 - t} = \lim_{t \to 3} \frac{-4.9(9 - t^2)}{3 - t}$$

$$= \lim_{x \to 3} \frac{-4.9(3 - t)(3 + t)}{3 - t} = \lim_{x \to 3} -4.9(3 + t) = -29.4 \text{ m/sec}$$

105. Let $f(x) = 1/x$ and $g(x) = -1/x$. $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ do not exist.

$$\lim_{x \to 0} [f(x) + g(x)] = \lim_{x \to 0} \left[\frac{1}{x} + \left(-\frac{1}{x} \right) \right] = \lim_{x \to 0} [0] = 0$$

107. Given $f(x) = b$, show that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - b| < \epsilon$ whenever $|x - c| < \delta$. Since $|f(x) - b| = |b - b| = 0 < \epsilon$ for any $\epsilon > 0$, then any value of $\delta > 0$ will work.

109. If $b = 0$, then the property is true because both sides are equal to 0. If $b \neq 0$, let $\epsilon > 0$ be given. Since $\lim_{x \to c} f(x) = L$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon/|b|$ whenever $0 < |x - c| < \delta$. Hence, wherever $0 < |x - c| < \delta$, we have

$$|b||f(x) - L| < \epsilon \quad \text{or} \quad |bf(x) - bL| < \epsilon$$

which implies that $\lim_{x \to c} [bf(x)] = bL$.

111. $-M|f(x)| \le f(x)g(x) \le M|f(x)|$

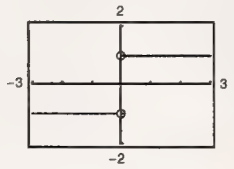
$$\lim_{x\to c}(-M|f(x)|) \le \lim_{x\to c} f(x)g(x) \le \lim_{x\to c}(M|f(x)|)$$

$$-M(0) \le \lim_{x\to c} f(x)g(x) \le M(0)$$

$$0 \le \lim_{x\to c} f(x)g(x) \le 0$$

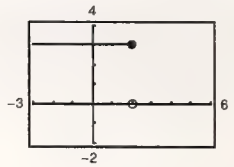
Therefore, $\lim_{x\to c} f(x)g(x) = 0$.

113. False. As x approaches 0 from the left, $\dfrac{|x|}{x} = -1$.



115. True.

117. False. The limit does not exist.



119. Let

$$f(x) = \begin{cases} 4, & \text{if } x \ge 0 \\ -4, & \text{if } x < 0 \end{cases}$$

$$\lim_{x\to 0}|f(x)| = \lim_{x\to 0} 4 = 4.$$

$\lim_{x\to 0} f(x)$ does not exist since for $x < 0, f(x) = -4$ and for $x \ge 0, f(x) = 4$.

121. $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$$

$\lim_{x\to 0} f(x)$ does not exist.

No matter how "close to" 0 x is, there are still an infinite number of rational and irrational numbers so that $\lim_{x\to 0} f(x)$ does not exist.

$\lim_{x\to 0} g(x) = 0$.

When x is "close to" 0, both parts of the function are "close to" 0.

123. (a) $\lim_{x\to 0}\dfrac{1-\cos x}{x^2} = \lim_{x\to 0}\dfrac{1-\cos x}{x^2}\cdot\dfrac{1+\cos x}{1+\cos x}$

$$= \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}$$

$$= \lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\frac{1}{1+\cos x}$$

$$= (1)\left(\frac{1}{2}\right) = \frac{1}{2}$$

(b) Thus, $\dfrac{1-\cos x}{x^2} \approx \dfrac{1}{2} \Rightarrow 1 - \cos x \approx \dfrac{1}{2}x^2$

$$\Rightarrow \cos x \approx 1 - \frac{1}{2}x^2 \text{ for } x \approx 0.$$

(c) $\cos(0.1) \approx 1 - \dfrac{1}{2}(0.1)^2 = 0.995$

(d) $\cos(0.1) \approx 0.9950$, which agrees with part (c).

Section 1.4 Continuity and One-Sided Limits

1. (a) $\lim_{x\to 3^+} f(x) = 1$

(b) $\lim_{x\to 3^-} f(x) = 1$

(c) $\lim_{x\to 3} f(x) = 1$

The function is continuous at $x = 3$.

3. (a) $\lim_{x\to 3^+} f(x) = 0$

(b) $\lim_{x\to 3^-} f(x) = 0$

(c) $\lim_{x\to 3} f(x) = 0$

The function is NOT continuous at $x = 3$.

5. (a) $\lim_{x\to 4^+} f(x) = 2$

(b) $\lim_{x\to 4^-} f(x) = -2$

(c) $\lim_{x\to 4} f(x)$ does not exist

The function is NOT continuous at $x = 4$.

7. $\lim_{x\to 5^+} \dfrac{x-5}{x^2-25} = \lim_{x\to 5^+} \dfrac{1}{x+5} = \dfrac{1}{10}$

9. $\lim_{x\to -3^-} \dfrac{x}{\sqrt{x^2-9}}$ does not exist because $\dfrac{x}{\sqrt{x^2-9}}$ grows without bound as $x \to -3^-$.

11. $\lim_{x\to 0^-} \dfrac{|x|}{x} = \lim_{x\to 0^-} \dfrac{-x}{x} = -1.$

13.
$$\lim_{\Delta x\to 0^-} \frac{\dfrac{1}{x+\Delta x} - \dfrac{1}{x}}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{x-(x+\Delta x)}{x(x+\Delta x)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{-\Delta x}{x(x+\Delta x)} \cdot \frac{1}{\Delta x}$$
$$= \lim_{\Delta x\to 0^-} \frac{-1}{x(x+\Delta x)}$$
$$= \frac{-1}{x(x+0)} = -\frac{1}{x^2}$$

15. $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \dfrac{x+2}{2} = \dfrac{5}{2}$

17. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x+1) = 2$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^3+1) = 2$

$\lim_{x\to 1} f(x) = 2$

19. $\lim_{x\to \pi} \cot x$ does not exist since

$\lim_{x\to \pi^+} \cot x$ and $\lim_{x\to \pi^-} \cot x$ do not exist.

21. $\lim_{x\to 4^-} (3[\![x]\!] - 5) = 3(3) - 5 = 4$

$([\![x]\!] = 3$ for $3 < x < 4)$

23. $\lim_{x\to 3} (2 - [\![-x]\!])$ does not exist because

$\lim_{x\to 3^-} (2 - [\![-x]\!]) = 2 - (-3) = 5$

and

$\lim_{x\to 3^+} (2 - [\![-x]\!]) = 2 - (-4) = 6.$

25. $f(x) = \dfrac{1}{x^2-4}$

has discontinuities at $x = -2$ and $x = 2$ since $f(-2)$ and $f(2)$ are not defined.

27. $f(x) = \dfrac{[\![x]\!]}{2} + x$

has discontinuities at each integer k since $\lim_{x\to k} f(x) \neq \lim_{x\to k^-} f(x)$.

29. $g(x) = \sqrt{25-x^2}$ is continuous on $[-5, 5]$.

31. $\lim_{x\to 0^-} f(x) = 3 = \lim_{x\to 0^+} f(x)$.

f is continuous on $[-1, 4]$.

33. $f(x) = x^2 - 2x + 1$ is continuous for all real x.

35. $f(x) = 3x - \cos x$ is continuous for all real x.

37. $f(x) = \dfrac{x}{x^2 - x}$ is not continuous at $x = 0, 1$. Since

$$\frac{x}{x^2 - x} = \frac{1}{x - 1} \text{ for } x \neq 0, x = 0 \text{ is a removable discontinuity, whereas } x = 1 \text{ is a nonremovable discontinuity.}$$

39. $f(x) = \dfrac{x}{x^2 + 1}$ is continuous for all real x.

41. $f(x) = \dfrac{x + 2}{(x + 2)(x - 5)}$

has a nonremovable discontinuity at $x = 5$ since $\lim_{x \to 5} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ since

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{1}{x - 5} = -\frac{1}{7}.$$

43. $f(x) = \dfrac{|x + 2|}{x + 2}$ has a nonremovable discontinuity at $x = -2$ since $\lim_{x \to -2} f(x)$ does not exist.

45. $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1$
2. $\left.\begin{array}{l} \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1 \\ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 = 1 \end{array}\right\} \lim_{x \to 1} f(x) = 1$
3. $f(1) = \lim_{x \to 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x.

47. $f(x) = \begin{cases} \frac{x}{2} + 1, & x \leq 2 \\ 3 - x, & x > 2 \end{cases}$ has a **possible** discontinuity at $x = 2$.

1. $f(2) = \dfrac{2}{2} + 1 = 2$
2. $\left.\begin{array}{l} \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \left(\frac{x}{2} + 1\right) = 2 \\ \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3 - x) = 1 \end{array}\right\} \lim_{x \to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

49. $f(x) = \begin{cases} \tan \frac{\pi x}{4}, & |x| < 1 \\ x, & |x| \geq 1 \end{cases} = \begin{cases} \tan \frac{\pi x}{4}, & -1 < x < 1 \\ x, & x \leq -1 \text{ or } x \geq 1 \end{cases}$ has **possible** discontinuities at $x = -1, x = 1$.

1. $f(-1) = -1 \qquad f(1) = 1$
2. $\lim_{x \to -1} f(x) = -1 \qquad \lim_{x \to 1} f(x) = 1$
3. $f(-1) = \lim_{x \to -1} f(x) \qquad f(1) = \lim_{x \to 1} f(x)$

f is continuous at $x = \pm 1$, therefore, f is continuous for all real x.

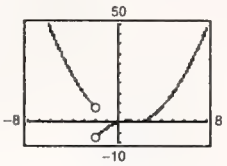
51. $f(x) = \csc 2x$ has nonremovable discontinuities at integer multiples of $\pi/2$.

53. $f(x) = [\![x - 1]\!]$ has nonremovable discontinuities at each integer k.

55. $\lim_{x\to 0^+} f(x) = 0$

$\lim_{x\to 0} f(x) = 0$

f is not continuous at $x = -2$.



57. $f(2) = 8$

Find a so that $\lim_{x\to 2^+} ax^2 = 8 \Rightarrow a = \dfrac{8}{2^2} = 2$.

59. Find a and b such that $\lim_{x\to -1^+} (ax + b) = -a + b = 2$ and $\lim_{x\to 3^-} (ax + b) = 3a + b = -2$.

$$\begin{aligned} a - b &= -2 \\ (+)\ 3a + b &= -2 \\ \hline 4a &= -4 \\ a &= -1 \\ b &= 2 + (-1) = 1 \end{aligned}$$

$$f(x) = \begin{cases} 2, & x \le -1 \\ -x + 1, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$$

61. $f(g(x)) = (x - 1)^2$

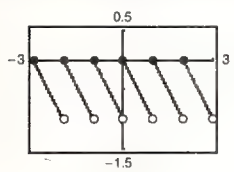
Continuous for all real x.

63. $f(g(x)) = \dfrac{1}{(x^2 + 5) - 6} = \dfrac{1}{x^2 - 1}$

Nonremovable discontinuities at $x = \pm 1$

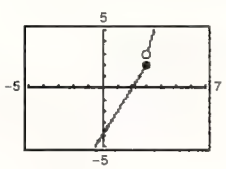
65. $y = [\![x]\!] - x$

Nonremovable discontinuity at each integer



67. $f(x) = \begin{cases} 2x - 4, & x \le 3 \\ x^2 - 2x, & x > 3 \end{cases}$

Nonremovable discontinuity at $x = 3$



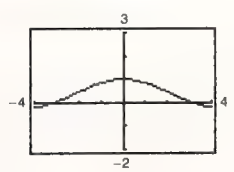
69. $f(x) = \dfrac{x}{x^2 + 1}$

Continuous on $(-\infty, \infty)$

71. $f(x) = \sec \dfrac{\pi x}{4}$

Continuous on:
$\ldots, (-6, -2), (-2, 2), (2, 6), (6, 10), \ldots$

73. $f(x) = \dfrac{\sin x}{x}$



The graph **appears** to be continuous on the interval $[-4, 4]$. Since $f(0)$ is not defined, we know that f has a discontinuity at $x = 0$. This discontinuity is removable so it does not show up on the graph.

75. $f(x) = \frac{1}{16}x^4 - x^3 + 3$ is continuous on $[1, 2]$.

$f(1) = \frac{33}{16}$ and $f(2) = -4$. By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 1 and 2.

77. $f(x) = x^2 - 2 - \cos x$ is continuous on $[0, \pi]$.

$f(0) = -3$ and $f(\pi) = \pi^2 - 1 > 0$. By the Intermediate Value Theorem, $f(c) = 0$ for the least one value of c between 0 and π.

79. $f(x) = x^3 + x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 1$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1. Using a graphing utility, we find that $x \approx 0.6823$.

81. $g(t) = 2\cos t - 3t$

g is continuous on $[0, 1]$.

$g(0) = 2 > 0$ and $g(1) \approx -1.9 < 0$.

By the Intermediate Value Theorem, $g(t) = 0$ for at least one value c between 0 and 1. Using a graphing utility, we find that $t \approx 0.5636$.

83. $f(x) = x^2 + x - 1$

f is continuous on $[0, 5]$.

$f(0) = -1$ and $f(5) = 29$

$-1 < 11 < 29$

The Intermediate Value Theorem applies.

$$x^2 + x - 1 = 11$$
$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
$$x = -4 \text{ or } x = 3$$
$$c = 3 \ (x = -4 \text{ is not in the interval.})$$

Thus, $f(3) = 11$.

85. $f(x) = x^3 - x^2 + x - 2$

f is continuous on $[0, 3]$.

$f(0) = -2$ and $f(3) = 19$

$-2 < 4 < 19$

The Intermediate Value Theorem applies.

$$x^3 - x^2 + x - 2 = 4$$
$$x^3 - x^2 + x - 6 = 0$$
$$(x - 2)(x^2 + x + 3) = 0$$
$$x = 2$$

($x^2 + x + 3$ has no real solution.)

$$c = 2$$

Thus, $f(2) = 4$.

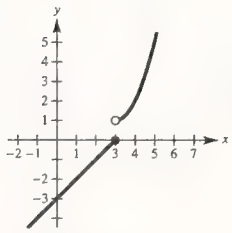
87. (a) The limit does not exist at $x = c$.

(b) The function is not defined at $x = c$.

(c) The limit exists at $x = c$, but it is not equal to the value of the function at $x = c$.

(d) The limit does not exist at $x = c$.

89.



The function is not continuous at $x = 3$ because $\lim_{x \to 3^+} f(x) = 1 \neq 0 = \lim_{x \to 3^-} f(x)$.

91. The functions agree for integer values of x:

$$\left.\begin{aligned} g(x) &= 3 - [\![-x]\!] = 3 - (-x) = 3 + x \\ f(x) &= 3 + [\![x]\!] = 3 + x \end{aligned}\right\} \text{ for } x \text{ an integer}$$

However, for non-integer values of x, the functions differ by 1.

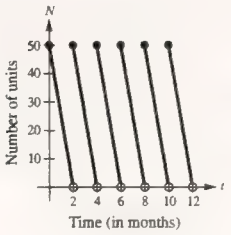
$f(x) = 3 + [\![x]\!] = g(x) - 1 = 2 - [\![-x]\!]$.

For example, $f\left(\frac{1}{2}\right) = 3 + 0 = 3$, $g\left(\frac{1}{2}\right) = 3 - (-1) = 4$.

93. $N(t) = 25\left(2\left[\left[\frac{t+2}{2}\right]\right] - t\right)$

t	0	1	1.8	2	3	3.8
$N(t)$	50	25	5	50	25	5

Discontinuous at every positive even integer. The company replenishes its inventory every two months.



95. Let $V = \frac{4}{3}\pi r^3$ be the volume of a sphere of radius r.

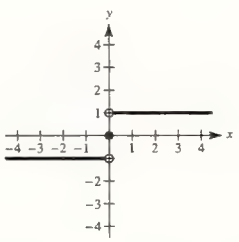
$V(1) = \frac{4}{3}\pi \approx 4.19$

$V(5) = \frac{4}{3}\pi(5^3) \approx 523.6$

Since $4.19 < 275 < 523.6$, the Intermediate Value Theorem implies that there is at least one value r between 1 and 5 such that $V(r) = 275$. (In fact, $r \approx 4.0341$.)

97. Let c be any real number. Then $\lim_{x \to c} f(x)$ does not exist since there are both rational and irrational numbers arbitrarily close to c. Therefore, f is not continuous at c.

99. $\text{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$



(a) $\lim_{x \to 0^-} \text{sgn}(x) = -1$

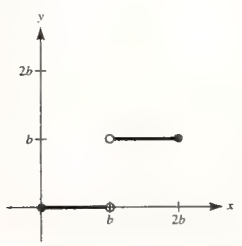
(b) $\lim_{x \to 0^+} \text{sgn}(x) = 1$

(c) $\lim_{x \to 0} \text{sgn}(x)$ does not exist.

101. True; if $f(x) = g(x)$, $x \neq c$, then $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$ and at least one of these limits (if they exist) does not equal the corresponding function at $x = c$.

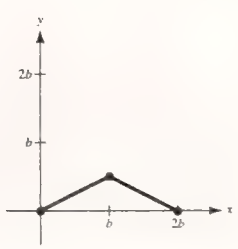
103. False; $f(1)$ is not defined and $\lim_{x \to 1} f(x)$ does not exist.

105. (a) $f(x) = \begin{cases} 0 & 0 \leq x < b \\ b & b < x \leq 2b \end{cases}$



NOT continuous at $x = b$.

(b) $g(x) = \begin{cases} \frac{x}{2} & 0 \leq x \leq b \\ b - \frac{x}{2} & b < x \leq 2b \end{cases}$



Continuous on $[0, 2b]$.

107. $f(x) = \dfrac{\sqrt{x + c^2} - c}{x}$, $c > 0$

Domain: $x + c^2 \geq 0 \Rightarrow x \geq -c^2$ and $x \neq 0$, $[-c^2, 0) \cup (0, \infty)$

$$\lim_{x\to 0} \frac{\sqrt{x + c^2} - c}{x} = \lim_{x\to 0} \frac{\sqrt{x + c^2} - c}{x} \cdot \frac{\sqrt{x + c^2} + c}{\sqrt{x + c^2} + c}$$

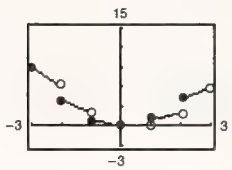
$$= \lim_{x\to 0} \frac{(x + c^2) - c^2}{x\left[\sqrt{x + c^2} + c\right]} = \lim_{x\to 0} \frac{1}{\sqrt{x + c^2} + c} = \frac{1}{2c}$$

Define $f(0) = 1/(2c)$ to make f continuous at $x = 0$.

109. $h(x) = x[\![x]\!]$

h has nonremovable discontinuities at

$x = \pm 1, \pm 2, \pm 3, \ldots$.



Section 1.5 Infinite Limits

1. $\lim_{x\to -2^+} 2\left|\dfrac{x}{x^2 - 4}\right| = \infty$

$\lim_{x\to -2^-} 2\left|\dfrac{x}{x^2 - 4}\right| = \infty$

3. $\lim_{x\to -2^+} \tan\dfrac{\pi x}{4} = -\infty$

$\lim_{x\to -2^-} \tan\dfrac{\pi x}{4} = \infty$

5. $f(x) = \dfrac{1}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	0.308	1.639	16.64	166.6	-166.7	-16.69	-1.695	-0.364

$\lim_{x\to -3^-} f(x) = \infty$

$\lim_{x\to -3^+} f(x) = -\infty$

7. $f(x) = \dfrac{x^2}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	3.769	15.75	150.8	1501	-1499	-149.3	-14.25	-2.273

$\lim_{x\to -3^-} f(x) = \infty$

$\lim_{x\to -3^+} f(x) = -\infty$

9. $\lim_{x\to 0^+} \frac{1}{x^2} = \infty = \lim_{x\to 0^-} \frac{1}{x^2}$

Therefore, $x = 0$ is a vertical asymptote.

11. $\lim_{x\to 2^+} \frac{x^2 - 2}{(x-2)(x+1)} = \infty$

$\lim_{x\to 2^-} \frac{x^2 - 2}{(x-2)(x+1)} = -\infty$

Therefore, $x = 2$ is a vertical asymptote.

$\lim_{x\to -1^+} \frac{x^2 - 2}{(x-2)(x+1)} = \infty$

$\lim_{x\to -1^-} \frac{x^2 - 2}{(x-2)(x+1)} = -\infty$

Therefore, $x = -1$ is a vertical asymptote.

13. $\lim_{x\to -2^-} \frac{x^2}{x^2-4} = \infty$ and $\lim_{x\to -2^+} \frac{x^2}{x^2-4} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\lim_{x\to 2^-} \frac{x^2}{x^2-4} = -\infty$ and $\lim_{x\to 2^+} \frac{x^2}{x^2-4} = \infty$

Therefore, $x = 2$ is a vertical asymptote.

15. No vertical asymptote since the denominator is never zero.

17. $f(x) = \tan 2x = \frac{\sin 2x}{\cos 2x}$ has vertical asymptotes at

$$x = \frac{(2n+1)\pi}{4} = \frac{\pi}{4} + \frac{n\pi}{2},\ n \text{ any integer.}$$

19. $\lim_{t\to 0^+} \left(1 - \frac{4}{t^2}\right) = -\infty = \lim_{t\to 0^-} \left(1 - \frac{4}{t^2}\right)$

Therefore, $t = 0$ is a vertical asymptote.

21. $\lim_{x\to -2^+} \frac{x}{(x+2)(x-1)} = \infty$

$\lim_{x\to -2^-} \frac{x}{(x+2)(x-1)} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\lim_{x\to 1^+} \frac{x}{(x+2)(x-1)} = \infty$

$\lim_{x\to 1^-} \frac{x}{(x+2)(x-1)} = -\infty$

Therefore, $x = 1$ is a vertical asymptote.

23. $f(x) = \frac{x^3+1}{x+1} = \frac{(x+1)(x^2-x+1)}{x+1}$

has no vertical asymptote since

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} (x^2 - x + 1) = 3$$

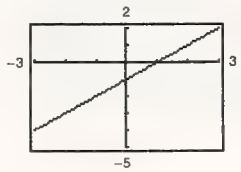
25. $f(x) = \frac{(x-5)(x+3)}{(x-5)(x^2+1)} = \frac{x+3}{x^2+1}, x \neq 5$

No vertical asymptotes. The graph has a hole at $x = 5$.

27. $s(t) = \frac{t}{\sin t}$ has vertical asymptotes at $t = n\pi$, n a nonzero integer. There is no vertical asymptote at $t = 0$ since

$$\lim_{t\to 0} \frac{t}{\sin t} = 1.$$

29. $\lim_{x\to -1} \dfrac{x^2-1}{x+1} = \lim_{x\to -1} (x-1) = -2$

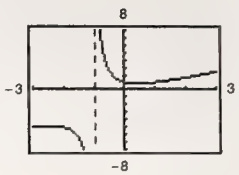


Removable discontinuity at $x = -1$

31. $\lim_{x\to -1^+} \dfrac{x^2+1}{x+1} = \infty$

$\lim_{x\to -1^-} \dfrac{x^2+1}{x+1} = -\infty$

Vertical asymptote at $x = -1$



33. $\lim_{x\to 2^+} \dfrac{x-3}{x-2} = -\infty$

35. $\lim_{x\to 3^+} \dfrac{x^2}{(x-3)(x+3)} = \infty$

37. $\lim_{x\to -3} \dfrac{x^2+2x-3}{x^2+x-6} = \lim_{x\to -3^-} \dfrac{x-1}{x-2} = \dfrac{4}{5}$

39. $\lim_{x\to 1} \dfrac{x^2-x}{(x^2+1)(x-1)} = \lim_{x\to 1} \dfrac{x}{x^2+1} = \dfrac{1}{2}$

41. $\lim_{x\to 0^-} \left(1 + \dfrac{1}{x}\right) = -\infty$

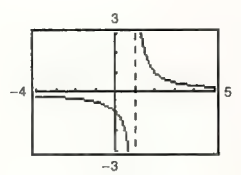
43. $\lim_{x\to 0^+} \dfrac{2}{\sin x} = \infty$

45. $\lim_{x\to \pi} \dfrac{\sqrt{x}}{\csc x} = \lim_{x\to \pi} \left(\sqrt{x} \sin x\right) = 0$

47. $\lim_{x\to (1/2)^-} x \sec(\pi x) = \infty$ and $\lim_{x\to (1/2)^+} x \sec(\pi x) = -\infty$.
Therefore, $\lim_{x\to (1/2)} x \sec(\pi x)$ does not exist.

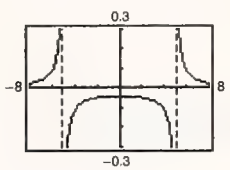
49. $f(x) = \dfrac{x^2+x+1}{x^3-1}$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \dfrac{1}{x-1} = \infty$



51. $f(x) = \dfrac{1}{x^2-25}$

$\lim_{x\to 5^-} f(x) = -\infty$



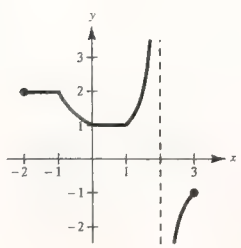
53. A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an infinite limit. ∞ is not a number. Rather, the symbol

$$\lim_{x\to c} f(x) = \infty$$

says how the limit fails to exist.

55. One answer is $f(x) = \dfrac{x-3}{(x-6)(x+2)} = \dfrac{x-3}{x^2-4x-12}$.

57.



59. $S = \dfrac{k}{1-r}$, $0 < |r| < 1$. Assume $k \neq 0$.

$\lim_{r\to 1^-} S = \lim_{r\to 1^-} \dfrac{k}{1-r} = \infty$ (or $-\infty$ if $k < 0$)

61. $C = \dfrac{528x}{100 - x}, \ 0 \le x < 100$

(a) $C(25) = \$176$ million

(b) $C(50) = \$528$ million

(c) $C(75) = \$1584$ million

(d) $\lim\limits_{x \to 100} \dfrac{528}{100 - x} = \infty$ Thus, it is not possible.

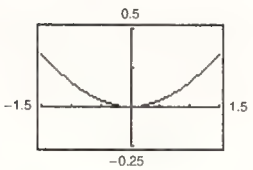
63. (a) $r = \dfrac{2(7)}{\sqrt{625 - 49}} = \dfrac{7}{12}$ ft/sec

(b) $r = \dfrac{2(15)}{\sqrt{625 - 225}} = \dfrac{3}{2}$ ft/sec

(c) $\lim\limits_{x \to 25} \dfrac{2x}{\sqrt{625 - x^2}} = \infty$

65. (a)

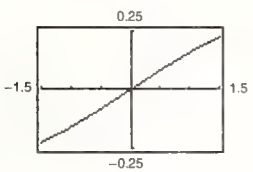
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0411	0.0067	0.0017	≈ 0	≈ 0	≈ 0



$\lim\limits_{x \to 0^+} \dfrac{x - \sin x}{x} = 0$

(b)

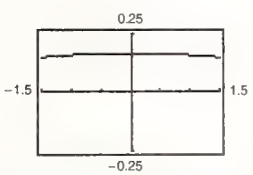
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0823	0.0333	0.0167	0.0017	≈ 0	≈ 0



$\lim\limits_{x \to 0^+} \dfrac{x - \sin x}{x^2} = 0$

(c)

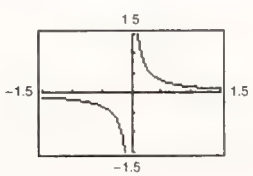
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.1646	0.1663	0.1666	0.1667	0.1667	0.1667



$\lim\limits_{x \to 0^+} \dfrac{x - \sin x}{x^3} = 0.1167 \ (1/6)$

(d)

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.3292	0.8317	1.6658	16.67	166.7	1667.0



$\lim\limits_{x \to 0^+} \dfrac{x - \sin x}{x^4} = \infty$

For $n \ge 3$, $\lim\limits_{x \to 0^+} \dfrac{x - \sin x}{x^n} = \infty$.

67. (a) Because the circumference of the motor is half that of the saw arbor, the saw makes $1700/2 = 850$ revolutions per minute.

(b) The direction of rotation is reversed.

(c) $2(20 \cot \phi) + 2(10 \cot \phi)$: straight sections. The angle subtended in each circle is

$$2\pi - \left(2\left(\frac{\pi}{2} - \phi\right)\right) = \pi + 2\phi.$$

Thus, the length of the belt around the pulleys is

$$20(\pi + 2\phi) + 10(\pi + 2\phi) = 30(\pi + 2\phi).$$

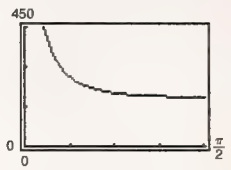
Total length $= 60 \cot \phi + 30(\pi + 2\phi)$

Domain: $\left(0, \frac{\pi}{2}\right)$

(d)

ϕ	0.3	0.6	0.9	1.2	1.5
L	306.2	217.9	195.9	189.6	188.5

(e)



(f) $\lim_{\phi \to (\pi/2)^-} L = 60\pi \approx 188.5$

(All the belts are around pulleys.)

(g) $\lim_{\phi \to 0^+} L = \infty$

69. False; for instance, let

$$f(x) = \frac{x^2 - 1}{x - 1} \text{ or}$$

$$g(x) = \frac{x}{x^2 + 1}.$$

71. False; let

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 3, & x = 0. \end{cases}$$

The graph of f has a vertical asymptote at $x = 0$, but $f(0) = 3$.

73. Given $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = L$:

(2) Product:

If $L > 0$, then for $\epsilon = L/2 > 0$ there exists $\delta_1 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_1$. Thus, $L/2 < g(x) < 3L/2$. Since $\lim_{x \to c} f(x) = \infty$ then for $M > 0$, there exists $\delta_2 > 0$ such that $f(x) > M(2/L)$ whenever $|x - c| < \delta_2$. Let δ be the smaller of δ_1 and δ_2. Then for $0 < |x - c| < \delta$, we have $f(x)g(x) > M(2/L)(L/2) = M$. Therefore $\lim_{x \to c} f(x)g(x) = \infty$. The proof is similar for $L < 0$.

(3) Quotient: Let $\epsilon > 0$ be given.

There exists $\delta_1 > 0$ such that $f(x) > 3L/2\epsilon$ whenever $0 < |x - c| < \delta_1$ and there exists $\delta_2 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_2$. This inequality gives us $L/2 < g(x) < 3L/2$. Let δ be the smaller of δ_1 and δ_2. Then for $0 < |x - c| < \delta$, we have

$$\left|\frac{g(x)}{f(x)}\right| < \frac{3L/2}{3L/2\epsilon} = \epsilon.$$

Therefore, $\lim_{x \to c} \frac{g(x)}{f(x)} = 0$.

75. Given $\lim_{x \to c} \frac{1}{f(x)} = 0$.

Suppose $\lim_{x \to c} f(x)$ exists and equals L. Then,

$$\lim_{x \to c} \frac{1}{f(x)} = \frac{\lim_{x \to c} 1}{\lim_{x \to c} f(x)} = \frac{1}{L} = 0.$$

This is not possible. Thus, $\lim_{x \to c} f(x)$ does not exist.

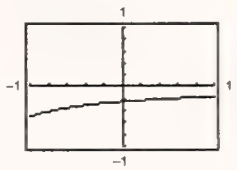
Review Exercises for Chapter 1

1. Calculus required. Using a graphing utility, you can estimate the length to be 8.3.
Or, the length is slightly longer than the distance between the two points, 8.25.

3.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	-0.26	-0.25	-0.250	-0.2499	-0.249	-0.24

$\lim_{x\to 0} f(x) \approx -0.25$



5. $h(x) = \dfrac{x^2 - 2x}{x}$

(a) $\lim_{x\to 0} h(x) = -2$

(b) $\lim_{x\to -1} h(x) = -3$

7. $\lim_{x\to 1} (3 - x) = 3 - 1 = 2$

Let $\epsilon > 0$ be given. Choose $\delta = \epsilon$. Then for $0 < |x - 1| < \delta = \epsilon$, you have

$$|x - 1| < \epsilon$$
$$|1 - x| < \epsilon$$
$$|(3 - x) - 2| < \epsilon$$
$$|f(x) - L| < \epsilon$$

9. $\lim_{x\to 2} (x^2 - 3) = 1$

Let $\epsilon > 0$ be given. We need $|x^2 - 3 - 1| < \epsilon \Rightarrow |x^2 - 4| = |(x - 2)(x + 2)| < \epsilon \Rightarrow |x - 2| < \dfrac{1}{|x + 2|}\epsilon$.

Assuming, $1 < x < 3$, you can choose $\delta = \epsilon/5$. Hence, for $0 < |x - 2| < \delta = \epsilon/5$ you have

$$|x - 2| < \frac{\epsilon}{5} < \frac{1}{|x + 2|}\epsilon$$
$$|x - 2||x + 2| < \epsilon$$
$$|x^2 - 4| < \epsilon$$
$$|(x^2 - 3) - 1| < \epsilon$$
$$|f(x) - L| < \epsilon$$

11. $\lim_{t\to 4} \sqrt{t + 2} = \sqrt{4 + 2} = \sqrt{6} \approx 2.45$

13. $\lim_{t\to -2} \dfrac{t + 2}{t^2 - 4} = \lim_{t\to -2} \dfrac{1}{t - 2} = -\dfrac{1}{4}$

15. $\lim_{x\to 4} \dfrac{\sqrt{x} - 2}{x - 4} = \lim_{x\to 4} \dfrac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)}$

$= \lim_{x\to 4} \dfrac{1}{\sqrt{x} + 2} = \dfrac{1}{\sqrt{4} + 2} = \dfrac{1}{4}$

17. $\lim_{x\to 0} \dfrac{[1/(x + 1)] - 1}{x} = \lim_{x\to 0} \dfrac{1 - (x + 1)}{x(x + 1)}$

$= \lim_{x\to 0} \dfrac{-1}{x + 1} = -1$

19. $\lim_{x\to -5} \dfrac{x^3 + 125}{x + 5} = \lim_{x\to -5} \dfrac{(x + 5)(x^2 - 5x + 25)}{x + 5}$

$= \lim_{x\to -5} (x^2 - 5x + 25)$

$= 75$

21. $\lim_{x\to 0} \dfrac{1 - \cos x}{\sin x} = \lim_{x\to 0} \left(\dfrac{x}{\sin x}\right)\left(\dfrac{1 - \cos x}{x}\right) = (1)(0) = 0$

23. $\lim_{\Delta x\to 0}\dfrac{\sin[(\pi/6)+\Delta x]-(1/2)}{\Delta x}=\lim_{\Delta x\to 0}\dfrac{\sin(\pi/6)\cos\Delta x+\cos(\pi/6)\sin\Delta x-(1/2)}{\Delta x}$

$$=\lim_{\Delta x\to 0}\frac{1}{2}\cdot\frac{(\cos\Delta x-1)}{\Delta x}+\lim_{\Delta x\to 0}\frac{\sqrt{3}}{2}\cdot\frac{\sin\Delta x}{\Delta x}$$

$$=0+\frac{\sqrt{3}}{2}(1)=\frac{\sqrt{3}}{2}$$

25. $\lim_{x\to c}[f(x)\cdot g(x)]=\left(-\frac{3}{4}\right)\left(\frac{2}{3}\right)=-\frac{1}{2}$

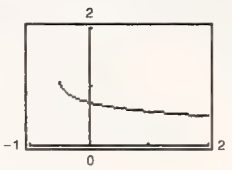
27. $f(x)=\dfrac{\sqrt{2x+1}-\sqrt{3}}{x-1}$

(a)

x	1.1	1.01	1.001	1.0001
$f(x)$	0.5680	0.5764	0.5773	0.5773

$\lim_{x\to 1^+}\dfrac{\sqrt{2x+1}-\sqrt{3}}{x-1}\approx 0.577$ (Actual limit is $\sqrt{3}/3$.)

(b)



(c) $\lim_{x\to 1^+}\dfrac{\sqrt{2x+1}-\sqrt{3}}{x-1}=\lim_{x\to 1^+}\dfrac{\sqrt{2x+1}-\sqrt{3}}{x-1}\cdot\dfrac{\sqrt{2x+1}+\sqrt{3}}{\sqrt{2x+1}+\sqrt{3}}$

$$=\lim_{x\to 1^+}\frac{(2x+1)-3}{(x-1)\left(\sqrt{2x+1}+\sqrt{3}\right)}$$

$$=\lim_{x\to 1^+}\frac{2}{\sqrt{2x+1}+\sqrt{3}}$$

$$=\frac{2}{2\sqrt{3}}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$$

29. $\lim_{t\to a}\dfrac{s(a)-s(t)}{a-t}=\lim_{t\to 4}\dfrac{(-4.9(4)^2+200)-(-4.9t^2+200)}{4-t}$

$$=\lim_{t\to 4}\frac{4.9(t-4)(t+4)}{4-t}$$

$$=\lim_{t\to 4}-4.9(t+4)=-39.2\text{ m/sec}$$

31. $\lim_{x\to 3^-}\dfrac{|x-3|}{x-3}=\lim_{x\to 3^-}\dfrac{-(x-3)}{x-3}=-1$

33. $\lim_{x\to 2}f(x)=0$

35. $\lim_{t\to 1}h(t)$ does not exist because $\lim_{t\to 1^-}h(t)=1+1=2$ and $\lim_{t\to 1^+}h(t)=\frac{1}{2}(1+1)=1$.

37. $f(x)=[\![x+3]\!]$

$\lim_{x\to k^+}[\![x+3]\!]=k+3$ where k is an integer.

$\lim_{x\to k^-}[\![x+3]\!]=k+2$ where k is an integer.

Nonremovable discontinuity at each integer k

Continuous on $(k, k+1)$ for all integers k

39. $f(x)=\dfrac{3x^2-x-2}{x-1}=\dfrac{(3x+2)(x-1)}{x-1}$

$\lim_{x\to 1}f(x)=\lim_{x\to 1}(3x+2)=5$

Removable discontinuity at $x=1$

Continuous on $(-\infty, 1)\cup(1,\infty)$

41. $f(x)=\dfrac{1}{(x-2)^2}$

$\lim_{x\to 2}\dfrac{1}{(x-2)^2}=\infty$

Nonremovable discontinuity at $x=2$

Continuous on $(-\infty, 2)\cup(2,\infty)$

43. $f(x)=\dfrac{3}{x+1}$

$\lim_{x\to 1^-}f(x)=-\infty$

$\lim_{x\to 1^+}f(x)=\infty$

Nonremovable discontinuity at $x=-1$

Continuous on $(-\infty, -1)\cup(-1,\infty)$

45. $f(x) = \csc\dfrac{\pi x}{2}$

Nonremovable discontinuities at each even integer. Continuous on

$(2k, 2k + 2)$

for all integers k.

47. $f(2) = 5$

Find c so that $\lim_{x\to 2^+}(cx + 6) = 5$.

$$c(2) + 6 = 5$$
$$2c = -1$$
$$c = -\frac{1}{2}$$

49. f is continuous on $[1, 2]$. $f(1) = -1 < 0$ and $f(2) = 13 > 0$. Therefore by the Intermediate Value Theorem, there is at least one value c in $(1, 2)$ such that $2c^3 - 3 = 0$.

51. $f(x) = \dfrac{x^2 - 4}{|x - 2|} = (x + 2)\left[\dfrac{x - 2}{|x - 2|}\right]$

(a) $\lim_{x\to 2^-} f(x) = -4$

(b) $\lim_{x\to 2^+} f(x) = 4$

(c) $\lim_{x\to 2} f(x)$ does not exist.

53. $g(x) = 1 + \dfrac{2}{x}$

Vertical asymptote at $x = 0$

55. $f(x) = \dfrac{8}{(x - 10)^2}$

Vertical asymptote at $x = 10$

57. $\lim_{x\to -2^-} \dfrac{2x^2 + x + 1}{x + 2} = -\infty$

59. $\lim_{x\to -1^+} \dfrac{x + 1}{x^3 + 1} = \lim_{x\to -1^+} \dfrac{1}{x^2 - x + 1} = \dfrac{1}{3}$

61. $\lim_{x\to 1^-} \dfrac{x^2 + 2x + 1}{x - 1} = -\infty$

63. $\lim_{x\to 0^+} \left(x - \dfrac{1}{x^3}\right) = -\infty$

65. $\lim_{x\to 0^+} \dfrac{\sin 4x}{5x} = \lim_{x\to 0^+} \left[\dfrac{4}{5}\left(\dfrac{\sin 4x}{4x}\right)\right] = \dfrac{4}{5}$

67. $\lim_{x\to 0^+} \dfrac{\csc 2x}{x} = \lim_{x\to 0^+} \dfrac{1}{x \sin 2x} = \infty$

69. $C = \dfrac{80{,}000p}{100 - p}$, $0 \le 0 < 100$

(a) $C(15) \approx \$14{,}117.65$ (b) $C(50) = \$80.000$

(c) $C(90) = \$720{,}000$ (d) $\lim_{p\to 100^-} \dfrac{80{,}000p}{100 - p} = \infty$

Problem Solving for Chapter 1

1. (a) Perimeter $\triangle PAO = \sqrt{x^2 + (y - 1)^2} + \sqrt{x^2 + y^2} + 1$

$$= \sqrt{x^2 + (x^2 - 1)^2} + \sqrt{x^2 + x^4} + 1$$

Perimeter $\triangle PBO = \sqrt{(x - 1)^2 + y^2} + \sqrt{x^2 + y^2} + 1$

$$= \sqrt{(x - 1)^2 + x^4} + \sqrt{x^2 + x^4} + 1$$

(b) $r(x) = \dfrac{\sqrt{x^2 + (x^2 - 1)^2} + \sqrt{x^2 + x^4} + 1}{\sqrt{(x - 1)^2 + x^4} + \sqrt{x^2 + x^4} + 1}$

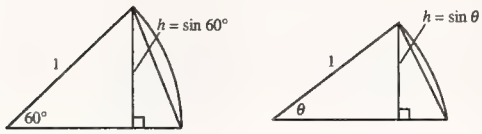
x	4	2	1	0.1	0.01
Perimeter $\triangle PAO$	33.02	9.08	3.41	2.10	2.01
Perimeter $\triangle PBO$	33.77	9.60	3.41	2.00	2.00
$r(x)$	0.98	0.95	1	1.05	1.005

(c) $\lim_{x\to 0^+} r(x) = \dfrac{1 + 0 + 1}{1 + 0 + 1} = \dfrac{2}{2} = 1$

3. (a) There are 6 triangles, each with a central angle of $60° = \pi/3$. Hence,

$$\text{Area hexagon} = 6\left[\frac{1}{2}bh\right] = 6\left[\frac{1}{2}(1)\sin\frac{\pi}{3}\right]$$

$$= \frac{3\sqrt{3}}{2} \approx 2.598.$$



Error: $\pi - \frac{3\sqrt{3}}{2} \approx 0.5435$.

(b) There are n triangles, each with central angle of $\theta = 2\pi/n$. Hence,

$$An = n\left[\frac{1}{2}bh\right] = n\left[\frac{1}{2}(1)\sin\frac{2\pi}{n}\right] = \frac{n\sin(2\pi/n)}{2}.$$

(c)

n	6	12	24	48	96
An	2.598	3	3.106	3.133	3.139

(d) As n gets larger and larger, $2\pi/n$ approaches 0.

Letting $x = 2\pi/n$,

$$An = \frac{\sin(2\pi/n)}{2/n} = \frac{\sin(2\pi/n)}{(2\pi/n)}\pi = \frac{\sin x}{x}\pi$$

which approaches $(1)\pi = \pi$.

5. (a) Slope $= -\frac{12}{5}$

(b) Slope of tangent line is $\frac{5}{12}$.

$$y + 12 = \frac{5}{12}(x - 5)$$

$$y = \frac{5}{12}x - \frac{169}{12} \quad \text{Tangent line}$$

(c) $Q = (x, y) = \left(x, \sqrt{169 - x^2}\right)$

$$m_x = \frac{-\sqrt{169 - x^2} + 12}{x - 5}$$

(d) $$\lim_{x\to 5} m_x = \lim_{x\to 5} \frac{12 - \sqrt{169 - x^2}}{x - 5} \cdot \frac{12 + \sqrt{169 - x^2}}{12 + \sqrt{169 - x^2}}$$

$$= \lim_{x\to 5} \frac{144 - (169 - x^2)}{(x - 5)\left(12 + \sqrt{169 - x^2}\right)}$$

$$= \lim_{x\to 5} \frac{x^2 - 25}{(x - 5)\left(12 + \sqrt{169 - x^2}\right)}$$

$$= \lim_{x\to 5} \frac{(x + 5)}{12 + \sqrt{169 - x^2}}$$

$$= \frac{10}{12 + 12} = \frac{5}{12}$$

This is the same slope as part (b).

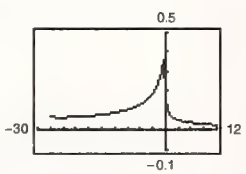
7. (a) $3 + x^{1/3} \geq 0$

$$x^{1/3} \geq -3$$

$$x \geq -27$$

Domain: $x \geq -27, x \neq 1$

(b)



(c) $$\lim_{x\to -27^+} f(x) = \frac{\sqrt{3 + (-27)^{1/3}} - 2}{-27 - 1}$$

$$= \frac{-2}{-28} = \frac{1}{14} \approx 0.0714$$

(d) $$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{\sqrt{3 + x^{1/3}} - 2}{x - 1} \cdot \frac{\sqrt{3 + x^{1/3}} + 2}{\sqrt{3 + x^{1/3}} + 2}$$

$$= \lim_{x\to 1} \frac{3 + x^{1/3} - 4}{(x - 1)\left(\sqrt{3 + x^{1/3}} + 2\right)}$$

$$= \lim_{x\to 1} \frac{x^{1/3} - 1}{(x^{1/3} - 1)(x^{2/3} + x^{1/3} + 1)\left(\sqrt{3 + x^{1/3}} + 2\right)}$$

$$= \lim_{x\to 1} \frac{1}{(x^{2/3} + x^{1/3} + 1)\left(\sqrt{3 + x^{1/3}} + 2\right)}$$

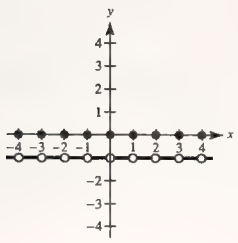
$$= \frac{1}{(1 + 1 + 1)(2 + 2)} = \frac{1}{12}$$

9. (a) $\lim_{x\to 2} f(x) = 3$: g_1, g_4

(b) f continuous at 2: g_1

(c) $\lim_{x\to 2^-} f(x) = 3$: g_1, g_3, g_4

11.



(a) $f(1) = [\![1]\!] + [\![-1]\!] = 1 + (-1) = 0$

$f(0) = 0$

$f(\frac{1}{2}) = 0 + (-1) = -1$

$f(-2.7) = -3 + 2 = -1$

(b) $\lim_{x \to 1^-} f(x) = -1$

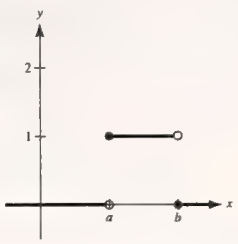
$\lim_{x \to 1^+} f(x) = -1$

$\lim_{x \to 1/2} f(x) = -1$

(c) f is continuous for all real numbers except

$x = 0, \pm 1, \pm 2, \pm 3, \ldots$

13. (a)



(b) (i) $\lim_{x \to a^+} P_{a,b}(x) = 1$

(ii) $\lim_{x \to a^-} P_{a,b}(x) = 0$

(iii) $\lim_{x \to b^+} P_{a,b}(x) = 0$

(iv) $\lim_{x \to b^-} P_{a,b}(x) = 1$

(c) $P_{a,b}$ is continuous for all positive real numbers except $x = a, b$.

(d) The area under the graph of u, and above the x-axis, is 1.

CHAPTER 2
Differentiation

Section 2.1 The Derivative and the Tangent Line Problem . . . **53**

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CHAPTER 2
Differentiation

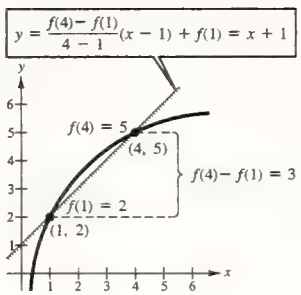
Section 2.1 The Derivative and the Tangent Line Problem

Solutions to Odd-Numbered Exercises

1. (a) $m = 0$

(b) $m = -3$

3. (a), (b)



(c) $y = \dfrac{f(4) - f(1)}{4 - 1}(x - 1) + f(1)$

$= \dfrac{3}{3}(x - 1) + 2$

$= 1(x - 1) + 2$

$= x + 1$

5. $f(x) = 3 - 2x$ is a line. Slope $= -2$

7. Slope at $(1, -3) = \lim_{\Delta x \to 0} \dfrac{g(1 + \Delta x) - g(1)}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{(1 + \Delta x)^2 - 4 - (-3)}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{1 + 2(\Delta x) + (\Delta x)^2 - 1}{\Delta x}$

$= \lim_{\Delta x \to 0} [2 + 2(\Delta x)] = 2$

9. Slope at $(0, 0) = \lim_{\Delta t \to 0} \dfrac{f(0 + \Delta t) - f(0)}{\Delta t}$

$= \lim_{\Delta t \to 0} \dfrac{3(\Delta t) - (\Delta t)^2 - 0}{\Delta t}$

$= \lim_{\Delta t \to 0} (3 - \Delta t) = 3$

11. $f(x) = 3$

$f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{3 - 3}{\Delta x}$

$= \lim_{\Delta x \to 0} 0 = 0$

13. $f(x) = -5x$

$f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{-5(x + \Delta x) - (-5x)}{\Delta x}$

$= \lim_{\Delta x \to 0} -5 = -5$

15. $h(s) = 3 + \dfrac{2}{3}s$

$h'(s) = \lim_{\Delta s \to 0} \dfrac{h(s + \Delta s) - h(s)}{\Delta s}$

$= \lim_{\Delta s \to 0} \dfrac{3 + \frac{2}{3}(s + \Delta s) - \left(3 + \frac{2}{3}s\right)}{\Delta s}$

$= \lim_{\Delta s \to 0} \dfrac{\frac{2}{3}\Delta s}{\Delta s} = \dfrac{2}{3}$

17. $f(x) = 2x^2 + x - 1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[2(x + \Delta x)^2 + (x + \Delta x) - 1] - [2x^2 + x - 1]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(2x^2 + 4x\Delta x + 2(\Delta x)^2 + x + \Delta x - 1) - (2x^2 + x - 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{4x\Delta x + 2(\Delta x)^2 + \Delta x}{\Delta x} = \lim_{\Delta x \to 0} (4x + 2\Delta x + 1) = 4x + 1$$

19. $f(x) = x^3 - 12x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^3 - 12(x + \Delta x)] - [x^3 - 12x]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12x - 12\Delta x - x^3 + 12x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2 - 12) = 3x^2 - 12$$

21. $f(x) = \dfrac{1}{x - 1}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{1}{x + \Delta x - 1} - \dfrac{1}{x - 1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x - 1) - (x + \Delta x - 1)}{\Delta x(x + \Delta x - 1)(x - 1)}$$

$$= \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x(x + \Delta x - 1)(x - 1)}$$

$$= \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x - 1)(x - 1)}$$

$$= -\frac{1}{(x - 1)^2}$$

23. $f(x) = \sqrt{x + 1}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x + 1} - \sqrt{x + 1}}{\Delta x} \cdot \left(\frac{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}}{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}} \right)$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x + 1) - (x + 1)}{\Delta x\left[\sqrt{x + \Delta x + 1} + \sqrt{x + 1}\right]}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}}$$

$$= \frac{1}{\sqrt{x + 1} + \sqrt{x + 1}} = \frac{1}{2\sqrt{x + 1}}$$

25. (a) $f(x) = x^2 + 1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 + 1] - [x^2 + 1]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

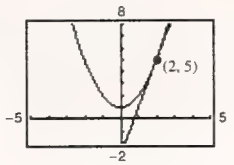
At $(2, 5)$, the slope of the tangent line is $m = 2(2) = 4$. The equation of the tangent line is

$$y - 5 = 4(x - 2)$$

$$y - 5 = 4x - 8$$

$$y = 4x - 3.$$

(b)



27. (a) $f(x) = x^3$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$

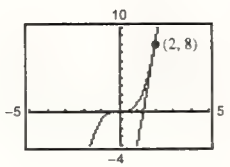
$$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2$$

At $(2, 8)$, the slope of the tangent is $m = 3(2)^2 = 12$. The equation of the tangent line is

$$y - 8 = 12(x - 2)$$

$$y = 12x - 16.$$

(b)



29. (a) $f(x) = \sqrt{x}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

At $(1, 1)$, the slope of the tangent line is

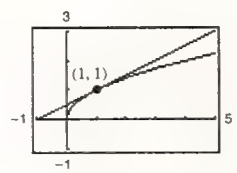
$$m = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

The equation of the tangent line is

$$y - 1 = \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2}x + \frac{1}{2}.$$

(b)



31. (a) $f(x) = x + \dfrac{4}{x}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) + \dfrac{4}{x + \Delta x} - \left(x + \dfrac{4}{x}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x(x + \Delta x)(x + \Delta x) + 4x - x^2(x + \Delta x) - 4(x + \Delta x)}{x(\Delta x)(x + \Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 2x^2(\Delta x) + x(\Delta x)^2 - x^3 - x^2(\Delta x) - 4(\Delta x)}{x(\Delta x)(x + \Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{x^2(\Delta x) + x(\Delta x)^2 - 4(\Delta x)}{x(\Delta x)(x + \Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + x(\Delta x) - 4}{x(x + \Delta x)}$$

$$= \frac{x^2 - 4}{x^2} = 1 - \frac{4}{x^2}$$

At $(4, 5)$, the slope of the tangent line is

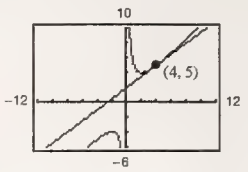
$$m = 1 - \frac{4}{16} = \frac{3}{4}.$$

The equation of the tangent line is

$$y - 5 = \frac{3}{4}(x - 4)$$

$$y = \frac{3}{4}x + 2.$$

(b)



33. From Exercise 27 we know that $f'(x) = 3x^2$. Since the slope of the given line is 3, we have

$$3x^2 = 3$$

$$x = \pm 1.$$

Therefore, at the points $(1, 1)$ and $(-1, -1)$ the tangent lines are parallel to $3x - y + 1 = 0$. These lines have equations

$$y - 1 = 3(x - 1) \quad \text{and} \quad y + 1 = 3(x + 1)$$

$$y = 3x - 2 \qquad\qquad y = 3x + 2.$$

35. Using the limit definition of derivative,

$$f'(x) = \frac{-1}{2x\sqrt{x}}.$$

Since the slope of the given line is $-\frac{1}{2}$, we have

$$-\frac{1}{2x\sqrt{x}} = -\frac{1}{2}$$

$$x = 1.$$

Therefore, at the point $(1, 1)$ the tangent line is parallel to $x + 2y - 6 = 0$. The equation of this line is

$$y - 1 = -\frac{1}{2}(x - 1)$$

$$y - 1 = -\frac{1}{2}x + \frac{1}{2}$$

$$y = -\frac{1}{2}x + \frac{3}{2}.$$

37. $g(5) = 2$ because the tangent line passes through $(5, 2)$.

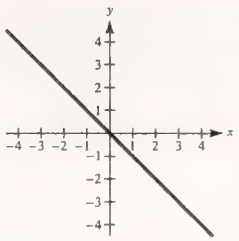
$$g'(5) = \frac{2 - 0}{5 - 9} = \frac{2}{-4} = -\frac{1}{2}$$

39. $f(x) = x \Rightarrow f'(x) = 1$ matches (b)

41. $f(x) = \sqrt{x} \Rightarrow f'(x)$ matches (a)

(decreasing slope as $x \to \infty$)

43.



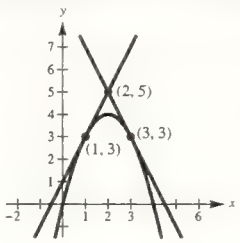
Answers will vary.

Sample answer: $y = -x$

45. (a) If $f'(c) = 3$ and f is odd, then $f'(-c) = f'(c) = 3$

(b) If $f'(c) = 3$ and f is even, then $f'(-c) = -f(c) = -3$

47. Let (x_0, y_0) be a point of tangency on the graph of f. By the limit definition for the derivative, $f'(x) = 4 - 2x$. The slope of the line through $(2, 5)$ and (x_0, y_0) equals the derivative of f at x_0:



$$\frac{5 - y_0}{2 - x_0} = 4 - 2x_0$$

$$5 - y_0 = (2 - x_0)(4 - 2x_0)$$

$$5 - (4x_0 - x_0^2) = 8 - 8x_0 + 2x_0^2$$

$$0 = x_0^2 - 4x_0 + 3$$

$$0 = (x_0 - 1)(x_0 - 3) \Rightarrow x_0 = 1, 3$$

Therefore, the points of tangency are $(1, 3)$ and $(3, 3)$, and the corresponding slopes are 2 and -2. The equations of the tangent lines are

$$y - 5 = 2(x - 2) \qquad y - 5 = -2(x - 2)$$

$$y = 2x + 1 \qquad y = -2x + 9$$

49. (a) $g'(0) = -3$

(b) $g'(3) = 0$

(c) Because $g'(1) = -\frac{8}{3}$, g is decreasing (falling) at $x = 1$.

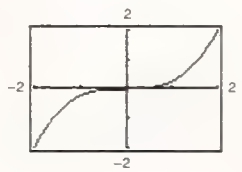
(d) Because $g'(-4) = \frac{7}{3}$, g is increasing (rising) at $x = -4$.

(e) Because $g'(4)$ and $g'(6)$ are both positive, $g(6)$ is greater than $g(4)$, and $g(6) - g(4) > 0$.

(f) No, it is not possible. All you can say is that g is decreasing (falling) at $x = 2$.

51. $f(x) = \frac{1}{4}x^3$

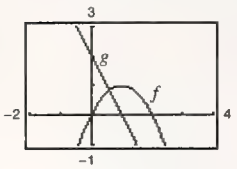
By the limit definition of the derivative we have $f'(x) = \frac{3}{4}x^2$.



x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	-2	$-\frac{27}{32}$	$-\frac{1}{4}$	$-\frac{1}{32}$	0	$\frac{1}{32}$	$\frac{1}{4}$	$\frac{27}{32}$	2
$f'(x)$	3	$\frac{27}{16}$	$\frac{3}{4}$	$\frac{3}{16}$	0	$\frac{3}{16}$	$\frac{3}{4}$	$\frac{27}{16}$	3

53. $g(x) = \dfrac{f(x + 0.01) - f(x)}{0.01}$

$= (2(x + 0.01) - (x + 0.01)^2 - 2x + x^2) + 100$

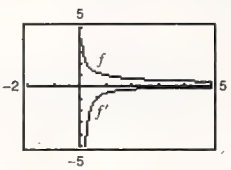


The graph of $g(x)$ is approximately the graph of $f'(x)$.

55. $f(2) = 2(4 - 2) = 4$, $f(2.1) = 2.1(4 - 2.1) = 3.99$

$f'(2) \approx \dfrac{3.99 - 4}{2.1 - 2} = -0.1$ [Exact: $f'(2) = 0$]

57. $f(x) = \dfrac{1}{\sqrt{x}}$ and $f'(x) = \dfrac{-1}{2x^{3/2}}$.



As $x \to \infty$, f is nearly horizontal and thus $f' \approx 0$.

59. $f(x) = 4 - (x - 3)^2$

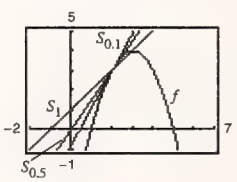
$S_{\Delta x}(x) = \dfrac{f(2 + \Delta x) - f(2)}{\Delta x}(x - 2) + f(2)$

$= \dfrac{4 - (2 + \Delta x - 3)^2 - 3}{\Delta x}(x - 2) + 3 = \dfrac{1 - (\Delta x - 1)^2}{\Delta x}(x - 2) + 3 = (-\Delta x + 2)(x - 2) + 3$

(a) $\Delta x = 1$: $S_{\Delta x} = (x - 2) + 3 = x + 1$

$\Delta x = 0.5$: $S_{\Delta x} = \left(\dfrac{3}{2}\right)(x - 2) + 3 = \dfrac{3}{2}x$

$\Delta x = 0.1$: $S_{\Delta x} = \left(\dfrac{19}{10}\right)(x - 2) + 3 = \dfrac{19}{10}x - \dfrac{4}{5}$



(b) As $\Delta x \to 0$, the line approaches the tangent line to f at (2, 3).

61. $f(x) = x^2 - 1, c = 2$

$f'(2) = \lim\limits_{x \to 2} \dfrac{f(x) - f(2)}{x - 2} = \lim\limits_{x \to 2} \dfrac{(x^2 - 1) - 3}{x - 2} = \lim\limits_{x \to 2} \dfrac{(x - 2)(x + 2)}{x - 2} = \lim\limits_{x \to 2} (x + 2) = 4$

63. $f(x) = x^3 + 2x^2 + 1, c = -2$

$f'(-2) = \lim\limits_{x \to -2} \dfrac{f(x) - f(-2)}{x + 2} = \lim\limits_{x \to -2} \dfrac{x^2(x + 2)}{x + 2} = \lim\limits_{x \to -2} \dfrac{(x^3 + 2x^2 + 1) - 1}{x + 2} = \lim\limits_{x \to -2} x^2 = 4$

65. $g(x) = \sqrt{|x|}, c = 0$

$g'(0) = \lim\limits_{x \to 0} \dfrac{g(x) - g(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{\sqrt{|x|}}{x}$. Does not exist.

As $x \to 0^-$, $\dfrac{\sqrt{|x|}}{x} = \dfrac{-1}{\sqrt{x}} \to -\infty$

As $x \to 0^+$, $\dfrac{\sqrt{|x|}}{x} = \dfrac{1}{\sqrt{x}} \to \infty$

67. $f(x) = (x - 6)^{2/3}, c = 6$

$f'(6) = \lim\limits_{x \to 6} \dfrac{f(x) - f(6)}{x - 6}$

$= \lim\limits_{x \to 6} \dfrac{(x - 6)^{2/3} - 0}{x - 6}$

$= \lim\limits_{x \to 6} \dfrac{1}{(x - 6)^{1/3}}$

Does not exist.

69. $h(x) = (x + 5), c = -5$

$$h'(-5) = \lim_{x \to -5} \frac{h(x) - h(-5)}{x - (-5)}$$

$$= \lim_{x \to -5} \frac{|x + 5| - 0}{x + 5}$$

$$= \lim_{x \to -5} \frac{|x + 5|}{x + 5}$$

Does not exist.

71. $f(x)$ is differentiable everywhere except at $x = -3$. (Sharp turn in the graph.)

73. $f(x)$ is differentiable everywhere except at $x = -1$. (Discontinuity)

75. $f(x)$ is differentiable everywhere except at $x = 3$. (Sharp turn in the graph)

77. $f(x)$ is differentiable on the interval $(1, \infty)$. (At $x = 1$ the tangent line is vertical)

79. $f(x)$ is differentiable everywhere except at $x = 0$. (Discontinuity)

81. $f(x) = |x - 1|$

The derivative from the left is

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{|x - 1| - 0}{x - 1} = -1.$$

The derivative from the right is

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{|x - 1| - 0}{x - 1} = 1.$$

The one-sided limits are not equal. Therefore, f is not differentiable at $x = 1$.

83. $f(x) = \begin{cases} (x - 1)^3, & x \leq 1 \\ (x - 1)^2, & x > 1 \end{cases}$

The derivative from the left is

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{(x - 1)^3 - 0}{x - 1}$$

$$= \lim_{x \to 1^-} (x - 1)^2 = 0.$$

The derivative from the right is

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{(x - 1)^2 - 0}{x - 1}$$

$$= \lim_{x \to 1^+} (x - 1) = 0.$$

These one-sided limits are equal. Therefore, f is differentiable at $x = 1$. $(f'(1) = 0)$

85. Note that f is continuous at $x = 2$. $f(x) = \begin{cases} x^2 + 1, & x \leq 2 \\ 4x - 3, & x > 2 \end{cases}$

The derivative from the left is $\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{(x^2 + 1) - 5}{x - 2} = \lim_{x \to 2} (x + 2) = 4.$

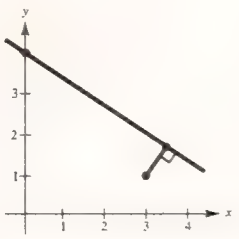
The derivative from the right is $\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{(4x - 3) - 5}{x - 2} = \lim_{x \to 2^+} 4 = 4.$

The one-sided limits are equal. Therefore, f is differentiable at $x = 2$. $(f'(2) = 4)$

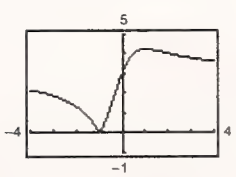
87. (a) The distance from (3, 1) to the line $mx - y + 4 = 0$ is

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

$$= \frac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}} = \frac{|3m + 3|}{\sqrt{m^2 + 1}}$$



(b)



The function d is not differentiable at $m = -1$. This corresponds to the line $y = -x + 4$, which passes through the point (3, 1).

89. False. The slope is $\lim_{\Delta x \to 0} \dfrac{f(2 + \Delta x) - f(2)}{\Delta x}$.

91. False. If the derivative from the left of a point does not equal the derivative from the right of a point, then the derivative does not exist at that point. For example, if $f(x) = |x|$, then the derivative from the left at $x = 0$ is -1 and the derivative from the right at $x = 0$ is 1. At $x = 0$, the derivative does not exist.

93. $f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Using the Squeeze Theorem, we have $-|x| \leq x \sin(1/x) \leq |x|$, $x \neq 0$. Thus, $\lim_{x \to 0} x \sin(1/x) = 0 = f(0)$ and f is continuous at $x = 0$. Using the alternative form of the derivative we have

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} \left(\sin \frac{1}{x} \right).$$

Since this limit does not exist (it oscillates between -1 and 1), the function is not differentiable at $x = 0$.

$$g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Using the Squeeze Theorem again we have $-x^2 \leq x^2 \sin(1/x) \leq x^2$, $x \neq 0$. Thus, $\lim_{x \to 0} x^2 \sin(1/x) = 0 = f(0)$ and f is continuous at $x = 0$. Using the alternative form of the derivative again we have

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} x \sin \frac{1}{x} = 0.$$

Therefore, g is differentiable at $x = 0$, $g'(0) = 0$.

Section 2.2 Basic Differentiation Rules and Rates of Change

1. (a) $y = x^{1/2}$, $y' = \frac{1}{2}x^{-1/2}$, $y'(1) = \frac{1}{2}$

(b) $y = x^{3/2}$, $y' = \frac{3}{2}x^{1/2}$, $y'(1) = \frac{3}{2}$

(c) $y = x^2$, $y' = 2x$, $y'(1) = 2$

(d) $y = x^3$, $y' = 3x^2$, $y'(1) = 3$

3. $y = 8$, $y' = 0$

5. $y = x^6$, $y' = 6x^5$

7. $y = \dfrac{1}{x^7} = x^{-7}$, $y' = -7x^{-8} = \dfrac{-7}{x^8}$

9. $y = \sqrt[5]{x} = x^{1/5}$, $y' = \dfrac{1}{5}x^{-4/5} = \dfrac{1}{5x^{4/5}}$

11. $f(x) = x + 1$, $f'(x) = 1$

13. $f(t) = -2t^2 + 3t - 6$, $f'(t) = -4t + 3$

15. $g(x) = x^2 + 4x^3$, $g'(x) = 2x + 12x^2$

17. $s(t) = t^3 - 2t + 4$, $s'(t) = 3t^2 - 2$

19. $y = \dfrac{\pi}{2} \sin\theta - \cos\theta$, $y' = \dfrac{\pi}{2} \cos\theta + \sin\theta$

21. $y = x^2 - \dfrac{1}{2} \cos x$, $y' = 2x + \dfrac{1}{2} \sin x$

23. $y = \dfrac{1}{x} - 3 \sin x$, $y' = -\dfrac{1}{x^2} - 3 \cos x$

	Function	Rewrite	Derivative	Simplify
25.	$y = \dfrac{5}{2x^2}$	$y = \dfrac{5}{2}x^{-2}$	$y' = -5x^{-3}$	$y' = \dfrac{-5}{x^3}$
27.	$y = \dfrac{3}{(2x)^3}$	$y = \dfrac{3}{8}x^{-3}$	$y' = \dfrac{-9}{8}x^{-4}$	$y' = \dfrac{-9}{8x^4}$
29.	$y = \dfrac{\sqrt{x}}{x}$	$y = x^{-1/2}$	$y' = -\dfrac{1}{2}x^{-3/2}$	$y' = -\dfrac{1}{2x^{3/2}}$

31. $f(x) = \dfrac{3}{x^2} = 3x^{-2}, (1, 3)$

$f'(x) = -6x^{-3} = \dfrac{-6}{x^3}$

$f'(1) = -6$

33. $f(x) = -\dfrac{1}{2} + \dfrac{7}{5}x^3, \left(0, -\dfrac{1}{2}\right)$

$f'(x) = \dfrac{21}{5}x^2$

$f'(0) = 0$

35. $y = (2x + 1)^2, (0, 1)$

$= 4x^2 + 4x + 1$

$y' = 8x + 4$

$y'(0) = 4$

37. $f(\theta) = 4 \sin \theta - \theta, (0, 0)$

$f'(\theta) = 4 \cos \theta - 1$

$f'(0) = 4(1) - 1 = 3$

39. $f(x) = x^2 + 5 - 3x^{-2}$

$f'(x) = 2x + 6x^{-3} = 2x + \dfrac{6}{x^3}$

41. $g(t) = t^2 - \dfrac{4}{t^3} = t^2 - 4t^{-3}$

$g'(t) = 2t + 12t^{-4} = 2t + \dfrac{12}{t^4}$

43. $f(x) = \dfrac{x^3 - 3x^2 + 4}{x^2} = x - 3 + 4x^{-2}$

$f'(x) = 1 - \dfrac{8}{x^3} = \dfrac{x^3 - 8}{x^3}$

45. $y = x(x^2 + 1) = x^3 + x$

$y' = 3x^2 + 1$

47. $f(x) = \sqrt{x} - 6\sqrt[3]{x} = x^{1/2} - 6x^{1/3}$

$f'(x) = \dfrac{1}{2}x^{-1/2} + 2x^{-2/3} = \dfrac{1}{2\sqrt{x}} + \dfrac{2}{x^{2/3}}$

49. $h(s) = s^{4/5} - s^{2/3}$

$h'(s) = \dfrac{4}{5}s^{-4/5} - \dfrac{2}{3}s^{-1/3} = \dfrac{4}{5s^{1/5}} - \dfrac{2}{3s^{1/3}}$

51. $f(x) = 6\sqrt{x} + 5 \cos x = 6x^{1/2} + 5 \cos x$

$f'(x) = 3x^{-1/2} - 5 \sin x = \dfrac{3}{\sqrt{x}} - 5 \sin x$

53. (a) $y = x^4 - 3x^2 + 2$

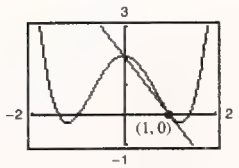
$y' = 4x^3 - 6x$

At $(1, 0)$: $y' = 4(1)^3 - 6(1) = -2$.

Tangent line: $y - 0 = -2(x - 1)$

$2x + y - 2 = 0$

(b)



55. (a) $f(x) = \dfrac{2}{\sqrt[4]{x^3}} = 2x^{-3/4}$

$f'(x) = \dfrac{-3}{2}x^{-7/4} = \dfrac{-3}{2x^{7/4}}$

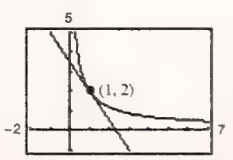
At $(1, 2)$, $f'(1) = \dfrac{-3}{2}$

Tangent line: $y - 2 = -\dfrac{3}{2}(x - 1)$

$y = -\dfrac{3}{2}x + \dfrac{7}{2}$

$3x + 2y - 7 = 0$

(b)



57. $y = x^4 - 8x^2 + 2$

$y' = 4x^3 - 16x$

$= 4x(x^2 - 4)$

$= 4x(x - 2)(x + 2)$

$y' = 0 \Rightarrow x = 0, \pm 2$

Horizontal tangents: $(0, 2), (2, -14), (-2, -14)$

59. $y = \dfrac{1}{x^2} = x^{-2}$

$y' = -2x^{-3} = \dfrac{-2}{x^3}$ cannot equal zero.

Therefore, there are no horizontal tangents.

61. $y = x + \sin x, 0 \le x < 2\pi$

$y' = 1 + \cos x = 0$

$\cos x = -1 \Rightarrow x = \pi$

At $x = \pi, y = \pi$.

Horizontal tangent: (π, π)

63. $x^2 - kx = 4x - 9$ Equate functions

$2x - k = 4$ Equate derivatives

Hence, $k = 2x - 4$ and

$x^2 - (2x - 4)x = 4x - 9 \Rightarrow -x^2 = -9 \Rightarrow x = \pm 3$.

For $x = 3, k = 2$ and for $x = -3, k = -10$.

65. $\dfrac{k}{x} = -\dfrac{3}{4}x + 3$ Equate functions

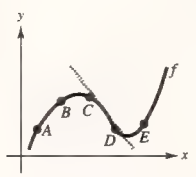
$-\dfrac{k}{x^2} = -\dfrac{3}{4}$ Equate derivatives

Hence, $k = \dfrac{3}{4}x^2$ and $\dfrac{\frac{3}{4}x^2}{x} = \dfrac{-3}{4}x + 3 \Rightarrow \dfrac{3}{4}x = -\dfrac{3}{4}x + 3 \Rightarrow \dfrac{3}{2}x = 3 \Rightarrow x = 2 \Rightarrow k = 3$.

67. (a) The slope appears to be steepest between A and B.

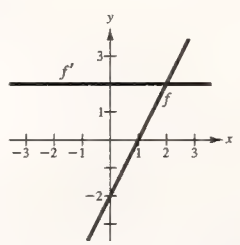
(b) The average rate of change between A and B is **greater** than the instantaneous rate of change at B.

(c)



69. $g(x) = f(x) + 6 \Rightarrow g'(x) = f'(x)$

71.



If f is linear then its derivative is a constant function.

$f(x) = ax + b$

$f'(x) = a$

73. Let (x_1, y_1) and (x_2, y_2) be the points of tangency on $y = x^2$ and $y = -x^2 + 6x - 5$, respectively. The derivatives of these functions are

$$y' = 2x \Rightarrow m = 2x_1 \quad \text{and} \quad y' = -2x + 6 \Rightarrow m = -2x_2 + 6.$$

$$m = 2x_1 = -2x_2 + 6$$

$$x_1 = -x_2 + 3$$

Since $y_1 = x_1^2$ and $y_2 = -x_2^2 + 6x_2 - 5$,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-x_2^2 + 6x_2 - 5) - (-x_1^2)}{x_2 - x_1} = -2x_2 + 6.$$

$$\frac{(-x_2^2 + 6x_2 - 5) - (-x_2 + 3)^2}{x_2 - (-x_2 + 3)} = -2x_2 + 6$$

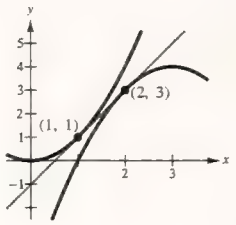
$$(-x_2^2 + 6x_2 - 5) - (x_2^2 - 6x_2 + 9) = (-2x_2 + 6)(2x_2 - 3)$$

$$-2x_2^2 + 12x_2 - 14 = -4x_2^2 + 18x_2 - 18$$

$$2x_2^2 - 6x_2 + 4 = 0$$

$$2(x_2 - 2)(x_2 - 1) = 0$$

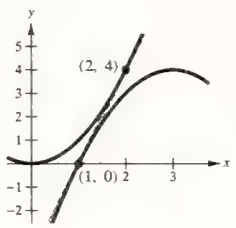
$$x_2 = 1 \text{ or } 2$$



$x_2 = 1 \Rightarrow y_2 = 0, x_1 = 2$ and $y_1 = 4$

Thus, the tangent line through (1, 0) and (2, 4) is

$$y - 0 = \left(\frac{4 - 0}{2 - 1}\right)(x - 1) \Rightarrow y = 4x - 4.$$



$x_2 = 2 \Rightarrow y_2 = 3, x_1 = 1$ and $y_1 = 1$

Thus, the tangent line through (2, 3) and (1, 1) is

$$y - 1 = \left(\frac{3 - 1}{2 - 1}\right)(x - 1) \Rightarrow y = 2x - 1.$$

75. $f(x) = \sqrt{x}$, $(-4, 0)$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{1}{2\sqrt{x}} = \frac{0 - y}{-4 - x}$$

$$4 + x = 2\sqrt{x}y$$

$$4 + x = 2\sqrt{x}\sqrt{x}$$

$$4 + x = 2x$$

$$x = 4, y = 2$$

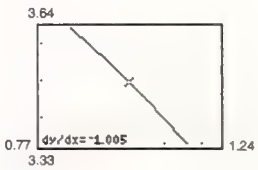
The point (4, 2) is on the graph of f.

Tangent line: $y - 2 = \dfrac{0 - 2}{-4 - 4}(x - 4)$

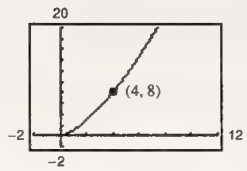
$$4y - 8 = x - 4$$

$$0 = x - 4y + 4$$

77. $f'(1) = -1$



79. (a) One possible secant is between (3.9, 7.7019) and (4, 8):



$$y - 8 = \frac{8 - 7.7019}{4 - 3.9}(x - 4)$$

$$y - 8 = 2.981(x - 4)$$

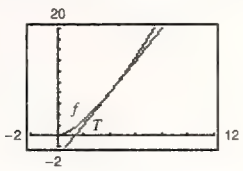
$$y = S(x) = 2.981x - 3.924$$

(b) $f'(x) = \frac{3}{2}x^{1/2} \Rightarrow f'(4) = \frac{3}{2}(2) = 3$

$T(x) = 3(x - 4) + 8 = 3x - 4$

$S(x)$ is an approximation of the tangent line $T(x)$.

(c) As you move further away from (4, 8), the accuracy of the approximation T gets worse.



(d)

Δx	−3	−2	−1	−0.5	−0.1	0	0.1	0.5	1	2	3
$f(4 + \Delta x)$	1	2.828	5.196	6.548	7.702	8	8.302	9.546	11.180	14.697	18.520
$T(4 + \Delta x)$	−1	2	5	6.5	7.7	8	8.3	9.5	11	14	17

81. False. Let $f(x) = x^2$ and $g(x) = x^2 + 4$. Then $f'(x) = g'(x) = 2x$, but $f(x) \neq g(x)$.

83. False. If $y = \pi^2$, then $dy/dx = 0$. (π^2 is a constant.)

85. True. If $g(x) = 3f(x)$, then $g'(x) = 3f'(x)$.

87. $f(t) = 2t + 7, [1, 2]$

$f'(t) = 2$

Instantaneous rate of change is the constant 2.
Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{[2(2) + 7] - [2(1) + 7]}{1} = 2$$

(These are the same because f is a line of slope 2.)

89. $f(x) = -\frac{1}{x}, [1, 2]$

$$f'(x) = \frac{1}{x^2}$$

Instantaneous rate of change:

$$(1, -1) \Rightarrow f'(1) = 1$$

$$\left(2, -\frac{1}{2}\right) \Rightarrow f'(2) = \frac{1}{4}$$

Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{(-1/2) - (-1)}{2 - 1} = \frac{1}{2}$$

91. (a) $s(t) = -16t^2 + 1362$

$v(t) = -32t$

(b) $\dfrac{s(2) - s(1)}{2 - 1} = 1298 - 1346 = -48$ ft/sec

(c) $v(t) = s'(t) = -32t$

When $t = 1$: $v(1) = -32$ ft/sec.

When $t = 2$: $v(2) = -64$ ft/sec.

(d) $-16t^2 + 1362 = 0$

$$t^2 = \frac{1362}{16} \Rightarrow t = \frac{\sqrt{1362}}{4} \approx 9.226 \text{ sec}$$

(e) $v\left(\dfrac{\sqrt{1362}}{4}\right) = -32\left(\dfrac{\sqrt{1362}}{4}\right)$

$= -8\sqrt{1362} \approx -295.242$ ft/sec

93. $s(t) = -4.9t^2 + v_0t + s_0$

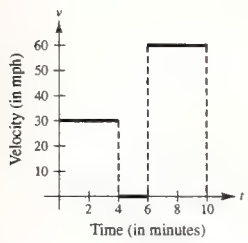
$= -4.9t^2 + 120t$

$v(t) = -9.8t + 120$

$v(5) = -9.8(5) + 120 = 71$ m/sec

$v(10) = -9.8(10) + 120 = 22$ m/sec

95.



(The velocity has been converted to miles per hour)

97. $v = 40$ mph $= \frac{2}{3}$ mi/min

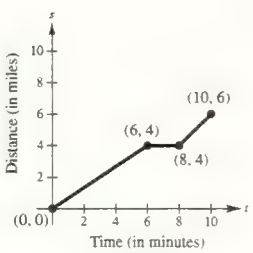
$(\frac{2}{3}$ mi/min$)(6$ min$) = 4$ mi

$v = 0$ mph $= 0$ mi/min

$(0$ mi/min$)(2$ min$) = 0$ mi

$v = 60$ mph $= 1$mi/min

$(1$ mi/min$)(2$ min$) = 2$ mi



99. (a) Using a graphing utility, you obtain

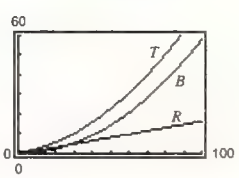
$R = 0.167v - 0.02.$

(b) Using a graphing utility, you obtain

$B = 0.00586v^2 - 0.0239v + 0.46.$

(c) $T = R + B = 0.00586v^2 + 0.1431v + 0.44$

(d)



(e) $\dfrac{dT}{dv} = 0.01172v + 0.1431$

For $v = 40$, $T'(40) \approx 0.612$.

For $v = 80$, $T'(80) \approx 1.081$.

For $v = 100$, $T'(100) \approx 1.315$.

(f) For increasing speeds, the total stopping distance increases.

101. $A = s^2, \dfrac{dA}{ds} = 2s$

When $s = 4$ m,

$\dfrac{dA}{ds} = 8$ square meters per meter change in s.

103. $$C = \frac{1{,}008{,}000}{Q} + 6.3Q$$

$$\frac{dC}{dQ} = -\frac{1{,}008{,}000}{Q^2} + 6.3$$

$C(351) - C(350) \approx 5083.095 - 5085 \approx -\$1.91$

When $Q = 350$, $\dfrac{dC}{dQ} \approx -\$1.93$.

105. (a) $f'(1.47)$ is the rate of change of the amount of gasoline sold when the price is \$1.47 per gallon.

(b) $f'(1.47)$ is usually negative. As prices go up, sales go down.

107. $y = ax^2 + bx + c$

Since the parabola passes through $(0, 1)$ and $(1, 0)$, we have

$(0, 1)$: $1 = a(0)^2 + b(0) + c \implies c = 1$

$(1, 0)$: $0 = a(1)^2 + b(1) + 1 \implies b = -a - 1$.

Thus, $y = ax^2 + (-a - 1)x + 1$. From the tangent line $y = x - 1$, we know that the derivative is 1 at the point $(1, 0)$.

$y' = 2ax + (-a - 1)$

$1 = 2a(1) + (-a - 1)$

$1 = a - 1$

$a = 2$

$b = -a - 1 = -3$

Therefore, $y = 2x^2 - 3x + 1$.

109. $y = x^3 - 9x$

$y' = 3x^2 - 9$

Tangent lines through $(1, -9)$:

$$y + 9 = (3x^2 - 9)(x - 1)$$

$$(x^3 - 9x) + 9 = 3x^3 - 3x^2 - 9x + 9$$

$$0 = 2x^3 - 3x^2 = x^2(2x - 3)$$

$$x = 0 \text{ or } x = \tfrac{3}{2}$$

The points of tangency are $(0, 0)$ and $\left(\frac{3}{2}, -\frac{81}{8}\right)$. At $(0, 0)$ the slope is $y'(0) = -9$. At $\left(\frac{3}{2}, -\frac{81}{8}\right)$ the slope is $y'\left(\frac{3}{2}\right) = -\frac{9}{4}$.

Tangent lines:

$y - 0 = -9(x - 0)$ and $y + \frac{81}{8} = -\frac{9}{4}\left(x - \frac{3}{2}\right)$

$y = -9x$ $y = -\frac{9}{4}x - \frac{27}{4}$

$9x + y = 0$ $9x + 4y + 27 = 0$

111. $f(x) = \begin{cases} ax^3, & x \le 2 \\ x^2 + b, & x > 2 \end{cases}$

f must be continuous at $x = 2$ to be differentiable at $x = 2$.

$$\left.\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} ax^3 = 8a \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} (x^2 + b) = 4 + b \end{aligned}\right\} \quad \begin{aligned} 8a &= 4 + b \\ 8a - 4 &= b \end{aligned}$$

$$f'(x) = \begin{cases} 3ax^2, & x < 2 \\ 2x, & x > 2 \end{cases}$$

For f to be differentiable at $x = 2$, the left derivative must equal the right derivative.

$3a(2)^2 = 2(2)$

$12a = 4$

$a = \frac{1}{3}$

$b = 8a - 4 = -\frac{4}{3}$

113. Let $f(x) = \cos x$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos x(\cos \Delta x - 1)}{\Delta x} - \lim_{\Delta x \to 0} \sin x \left(\frac{\sin \Delta x}{\Delta x}\right)$$

$$= 0 - \sin x(1) = -\sin x$$

Section 2.3 The Product and Quotient Rules and Higher-Order Derivatives

1. $g(x) = (x^2 + 1)(x^2 - 2x)$

$$g'(x) = (x^2 + 1)(2x - 2) + (x^2 - 2x)(2x)$$

$$= 2x^3 - 2x^2 + 2x - 2 + 2x^3 - 4x^2$$

$$= 4x^3 - 6x^2 + 2x - 2$$

3. $h(t) = \sqrt[3]{t}(t^2 + 4) = t^{1/3}(t^2 + 4)$

$$h'(t) = t^{1/3}(2t) + (t^2 + 4)\frac{1}{3}t^{-2/3}$$

$$= 2t^{4/3} + \frac{t^2 + 4}{3t^{2/3}}$$

$$= \frac{7t^2 + 4}{3t^{2/3}}$$

5. $f(x) = x^3 \cos x$

$$f'(x) = x^3(-\sin x) + \cos x(3x^2)$$

$$= 3x^2 \cos x - x^3 \sin x$$

7. $f(x) = \dfrac{x}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

9. $h(x) = \dfrac{\sqrt[3]{x}}{x^3 + 1} = \dfrac{x^{1/3}}{x^3 + 1}$

$$h'(x) = \frac{(x^3 + 1)\frac{1}{3}x^{-2/3} - x^{1/3}(3x^2)}{(x^3 + 1)^2}$$

$$= \frac{(x^3 + 1) - x(9x^2)}{3x^{2/3}(x^3 + 1)^2}$$

$$= \frac{1 - 8x^3}{3x^{2/3}(x^3 + 1)^2}$$

11. $g(x) = \dfrac{\sin x}{x^2}$

$$g'(x) = \frac{x^2(\cos x) - \sin x(2x)}{(x^2)^2} = \frac{x \cos x - 2 \sin x}{x^3}$$

13. $f(x) = (x^3 - 3x)(2x^2 + 3x + 5)$

$$f'(x) = (x^3 - 3x)(4x + 3) + (2x^2 + 3x + 5)(3x^2 - 3)$$

$$= 10x^4 + 12x^3 - 3x^2 - 18x - 15$$

$$f'(0) = -15$$

15. $f(x) = \dfrac{x^2 - 4}{x - 3}$

$$f'(x) = \frac{(x - 3)(2x) - (x^2 - 4)(1)}{(x - 3)^2} = \frac{2x^2 - 6x - x^2 + 4}{(x - 3)^2}$$

$$= \frac{x^2 - 6x + 4}{(x - 3)^2}$$

$$f'(1) = \frac{1 - 6 + 4}{(1 - 3)^2} = -\frac{1}{4}$$

17. $f(x) = x \cos x$

$$f'(x) = (x)(-\sin x) + (\cos x)(1) = \cos x - x \sin x$$

$$f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\pi}{4}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{8}(4 - \pi)$$

Function	Rewrite	Derivative	Simplify
19. $y = \dfrac{x^2 + 2x}{3}$	$y = \dfrac{1}{3}x^2 + \dfrac{2}{3}x$	$y' = \dfrac{2}{3}x + \dfrac{2}{3}$	$y' = \dfrac{2x + 2}{3}$
21. $y = \dfrac{7}{3x^3}$	$y = \dfrac{7}{3}x^{-3}$	$y' = -7x^{-4}$	$y' = -\dfrac{7}{x^4}$
23. $y = \dfrac{4x^{3/2}}{x}$	$y = 4\sqrt{x}, x > 0$	$y' = 2x^{-1/2}$	$y' = \dfrac{2}{\sqrt{x}}$

25. $f(x) = \dfrac{3 - 2x - x^2}{x^2 - 1}$

$$f'(x) = \frac{(x^2 - 1)(-2 - 2x) - (3 - 2x - x^2)(2x)}{(x^2 - 1)^2}$$

$$= \frac{2x^2 - 4x + 2}{(x^2 - 1)^2} = \frac{2(x - 1)^2}{(x^2 - 1)^2}$$

$$= \frac{2}{(x + 1)^2}, x \neq 1$$

27. $f(x) = x\left(1 - \dfrac{4}{x + 3}\right) = x - \dfrac{4x}{x + 3}$

$$f'(x) = 1 - \frac{(x + 3)4 - 4x(1)}{(x + 3)^2} = \frac{(x^2 + 6x + 9) - 12}{(x + 3)^2}$$

$$= \frac{x^2 + 6x - 3}{(x + 3)^2}$$

29. $f(x) = \dfrac{2x + 5}{\sqrt{x}} = 2x^{1/2} + 5x^{-1/2}$

$$f'(x) = x^{-1/2} - \frac{5}{2}x^{-3/2} = x^{-3/2}\left[x - \frac{5}{2}\right]$$

$$= \frac{2x - 5}{2x\sqrt{x}} = \frac{2x - 5}{2x^{3/2}}$$

31. $h(s) = (s^3 - 2)^2 = s^6 - 4s^3 + 4$

$$h'(s) = 6s^5 - 12s^2 = 6s^2(s^3 - 2)$$

33. $f(x) = \dfrac{2 - \dfrac{1}{x}}{x - 3} = \dfrac{2x - 1}{x(x - 3)} = \dfrac{2x - 1}{x^2 - 3x}$

$$f'(x) = \frac{(x^2 - 3x)2 - (2x - 1)(2x - 3)}{(x^2 - 3x)^2} = \frac{2x^2 - 6x - 4x^2 + 8x - 3}{(x^2 - 3x)^2}$$

$$= \frac{-2x^2 + 2x - 3}{(x^2 - 3x)^2} = -\frac{2x^2 - 2x + 3}{x^2(x - 3)^2}$$

35. $f(x) = (3x^3 + 4x)(x - 5)(x + 1)$

$$f'(x) = (9x^2 + 4)(x - 5)(x + 1) + (3x^3 + 4x)(1)(x + 1) + (3x^3 + 4x)(x - 5)(1)$$

$$= (9x^2 + 4)(x^2 - 4x - 5) + 3x^4 + 3x^3 + 4x^2 + 4x + 3x^4 - 15x^3 + 4x^2 - 20x$$

$$= 9x^4 - 36x^3 - 41x^2 - 16x - 20 + 6x^4 - 12x^3 + 8x^2 - 16x$$

$$= 15x^4 - 48x^3 - 33x^2 - 32x - 20$$

37. $f(x) = \dfrac{x^2 + c^2}{x^2 - c^2}$

$$f'(x) = \frac{(x^2 - c^2)(2x) - (x^2 + c^2)(2x)}{(x^2 - c^2)^2}$$

$$= \frac{-4xc^2}{(x^2 - c^2)^2}$$

39. $f(x) = t^2 \sin t$

$$f'(t) = t^2 \cos t + 2t \sin t$$

$$= t(t \cos t + 2 \sin t)$$

41. $f(t) = \dfrac{\cos t}{t}$

$$f'(t) = \frac{-t\sin t - \cos t}{t^2} = -\frac{t\sin t + \cos t}{t^2}$$

43. $f(x) = -x + \tan x$

$$f'(x) = -1 + \sec^2 x = \tan^2 x$$

45. $g(t) = \sqrt[4]{t} + 8\sec t = t^{1/4} + 8\sec t$

$$g'(t) = \frac{1}{4}t^{-3/4} + 8\sec t\tan t = \frac{1}{4t^{3/4}} + 8\sec t\tan t$$

47. $y = \dfrac{3(1-\sin x)}{2\cos x} = \dfrac{3}{2}(\sec x - \tan x)$

$$y' = \frac{3}{2}(\sec x\tan x - \sec^2 x) = \frac{3}{2}\sec x(\tan x - \sec x)$$

$$= \frac{3}{2}(\sec x\tan x - \tan^2 x - 1)$$

49. $y = -\csc x - \sin x$

$$y' = \csc x\cot x - \cos x$$

$$= \frac{\cos x}{\sin^2 x} - \cos x$$

$$= \cos x(\csc^2 x - 1)$$

$$= \cos x\cot^2 x$$

51. $f(x) = x^2\tan x$

$$f'(x) = x^2\sec^2 x + 2x\tan x$$

$$= x(x\sec^2 x + 2\tan x)$$

53. $y = 2x\sin x + x^2\cos x$

$$y' = 2x\cos x + 2\sin x + x^2(-\sin x) + 2x\cos x$$

$$= 4x\cos x + 2\sin x - x^2\sin x$$

55. $g(x) = \left(\dfrac{x+1}{x+2}\right)(2x-5)$

$$g'(x) = \frac{2x^2 + 8x - 1}{(x+2)^2}$$ (form of answer may vary)

57. $g(\theta) = \dfrac{\theta}{1-\sin\theta}$

$$g'(\theta) = \frac{1 - \sin\theta + \theta\cos\theta}{(\sin\theta - 1)^2}$$ (form of answer may vary)

59. $y = \dfrac{1+\csc x}{1-\csc x}$

$$y' = \frac{(1-\csc x)(-\csc x\cot x) - (1+\csc x)(\csc x\cot x)}{(1-\csc x)^2} = \frac{-2\csc x\cot x}{(1-\csc x)^2}$$

$$y'\left(\frac{\pi}{6}\right) = \frac{-2(2)(\sqrt{3})}{(1-2)^2} = -4\sqrt{3}$$

61. $h(t) = \dfrac{\sec t}{t}$

$$h'(t) = \frac{t(\sec t\tan t) - (\sec t)(1)}{t^2}$$

$$= \frac{\sec t(t\tan t - 1)}{t^2}$$

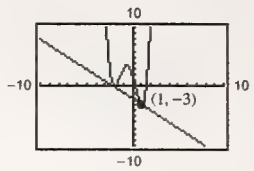
$$h'(\pi) = \frac{\sec\pi(\pi\tan\pi - 1)}{\pi^2} = \frac{1}{\pi^2}$$

63. (a) $f(x) = (x^3 - 3x + 1)(x + 2), \quad (1, -3)$

$$f'(x) = (x^3 - 3x + 1)(1) + (x + 2)(3x^2 - 3)$$
$$= 4x^3 + 6x^2 - 6x - 5$$
$$f'(1) = -1 = \text{slope at } (1, -3).$$

Tangent line: $y + 3 = -1(x - 1) \Rightarrow y = -x - 2$

(b)



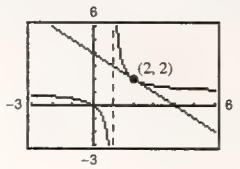
65. (a) $f(x) = \dfrac{x}{x - 1}, (2, 2)$

$$f'(x) = \frac{(x - 1)(1) - x(1)}{(x - 1)^2} = \frac{-1}{(x - 1)^2}$$

$$f'(2) = \frac{-1}{(2 - 1)^2} = -1 = \text{ slope at } (2, 2).$$

Tangent line: $y - 2 = -1(x - 2) \Rightarrow y = -x + 4$

(b)



67. (a) $f(x) = \tan x, \quad \left(\dfrac{\pi}{4}, 1\right)$

$$f'(x) = \sec^2 x$$

$$f'\left(\frac{\pi}{4}\right) = 2 = \text{slope at } \left(\frac{\pi}{4}, 1\right).$$

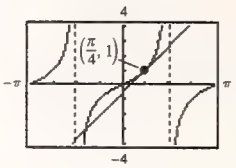
Tangent line:

$$y - 1 = 2\left(x - \frac{\pi}{4}\right)$$

$$y - 1 = 2x - \frac{\pi}{2}$$

$$4x - 2y - \pi + 2 = 0$$

(b)



69. $f(x) = \dfrac{x^2}{x - 1}$

$$f'(x) = \frac{(x - 1)(2x) - x^2(1)}{(x - 1)^2}$$

$$= \frac{x^2 - 2x}{(x - 1)^2} = \frac{x(x - 2)}{(x - 1)^2}$$

$f'(x) = 0$ when $x = 0$ or $x = 2$.

Horizontal tangents are at $(0, 0)$ and $(2, 4)$.

71. $f'(x) = \dfrac{(x + 2)3 - 3x(1)}{(x + 2)^2} = \dfrac{6}{(x + 2)^2}$

$$g'(x) = \frac{(x + 2)5 - (5x + 4)(1)}{(x + 2)^2} = \frac{6}{(x + 2)^2}$$

$$g(x) = \frac{5x + 4}{(x + 2)} = \frac{3x}{(x + 2)} + \frac{2x + 4}{(x + 2)} = f(x) + 2$$

f and g differ by a constant.

73. $f(x) = x^n \sin x$

$$f'(x) = x^n \cos x + nx^{n-1} \sin x$$
$$= x^{n-1}(x \cos x + n \sin x)$$

When $n = 1$: $f'(x) = x \cos x + \sin x$.

When $n = 2$: $f'(x) = x(x \cos x + 2 \sin x)$.

When $n = 3$: $f'(x) = x^2(x \cos x + 3 \sin x)$.

When $n = 4$: $f'(x) = x^3(x \cos x + 4 \sin x)$.

For general n, $f'(x) = x^{n-1}(x \cos x + n \sin x)$.

75. Area $= A(t) = (2t + 1)\sqrt{t} = 2t^{3/2} + t^{1/2}$

$$A'(t) = 2\left(\frac{3}{2}t^{1/2}\right) + \frac{1}{2}t^{-1/2}$$

$$= 3t^{1/2} + \frac{1}{2}t^{-1/2}$$

$$= \frac{6t + 1}{2\sqrt{t}} \text{ cm}^2/\text{sec}$$

77. $C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad 1 \le x$

$$\frac{dC}{dx} = 100\left(-\frac{400}{x^3} + \frac{30}{(x + 30)^2}\right)$$

(a) When $x = 10$: $\frac{dC}{dx} = -\$38.13$.

(b) When $x = 15$: $\frac{dC}{dx} = -\$10.37$.

(c) When $x = 20$: $\frac{dC}{dx} = -\$3.80$.

As the order size increases, the cost per item decreases.

79. $P(t) = 500\left[1 + \frac{4t}{50 + t^2}\right]$

$$P'(t) = 500\left[\frac{(50 + t^2)(4) - (4t)(2t)}{(50 + t^2)^2}\right]$$

$$= 500\left[\frac{200 - 4t^2}{(50 + t^2)^2}\right]$$

$$= 2000\left[\frac{50 - t^2}{(50 + t^2)^2}\right]$$

$P'(2) \approx 31.55$ bacteria per hour

81. (a) $\sec x = \frac{1}{\cos x}$

$$\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right] = \frac{(\cos x)(0) - (1)(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos x \cos x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

(b) $\csc x = \frac{1}{\sin x}$

$$\frac{d}{dx}[\csc x] = \frac{d}{dx}\left[\frac{1}{\sin x}\right] = \frac{(\sin x)(0) - (1)(\cos x)}{(\sin x)^2} = -\frac{\cos x}{\sin x \sin x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$

(c) $\cot x = \frac{\cos x}{\sin x}$

$$\frac{d}{dx}[\cot x] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{\sin x(-\sin x) - (\cos x)(\cos x)}{(\sin x)^2} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

83. $f(x) = 4x^{3/2}$

$f'(x) = 6x^{1/2}$

$f''(x) = 3x^{-1/2} = \frac{3}{\sqrt{x}}$

85. $f(x) = \frac{x}{x - 1}$

$$f'(x) = \frac{(x - 1)(1) - x(1)}{(x - 1)^2} = \frac{-1}{(x - 1)^2}$$

$$f''(x) = \frac{2}{(x - 1)^3}$$

87. $f(x) = 3 \sin x$

$f'(x) = 3 \cos x$

$f''(x) = -3 \sin x$

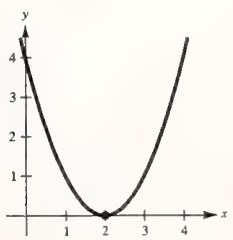
89. $f'(x) = x^2$

$f''(x) = 2x$

91. $f'''(x) = 2\sqrt{x}$

$f^{(4)}(x) = \frac{1}{2}(2)x^{-1/2} = \frac{1}{\sqrt{x}}$

93.



$f(2) = 0$

One such function is $f(x) = (x - 2)^2$.

95. $f(x) = 2g(x) + h(x)$

$f'(x) = 2g'(x) + h'(x)$

$f'(2) = 2g'(2) + h'(2)$

$= 2(-2) + 4$

$= 0$

97. $f(x) = \frac{g(x)}{h(x)}$

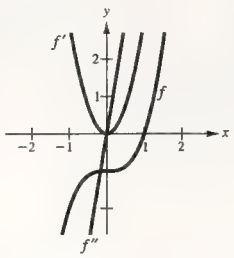
$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$

$$f'(2) = \frac{h(2)g'(2) - g(2)h'(2)}{[h(2)]^2}$$

$$= \frac{(-1)(-2) - (3)(4)}{(-1)^2}$$

$$= -10$$

99.



It appears that f is cubic; so f' would be quadratic and f'' would be linear.

101. $v(t) = 36 - t^2,\ 0 \le t \le 6$

$a(t) = -2t$

$v(3) = 27$ m/sec

$a(3) = -6$ m/sec

The speed of the object is decreasing.

103. $v(t) = \dfrac{100t}{2t + 15}$

$$a(t) = \frac{(2t + 15)(100) - (100t)(2)}{(2t + 15)^2} = \frac{1500}{(2t + 15)^2}$$

(a) $a(5) = \dfrac{1500}{[2(5) + 15]^2} = 2.4$ ft/sec^2

(b) $a(10) = \dfrac{1500}{[2(10) + 15]^2} \approx 1.2$ ft/sec^2

(c) $a(20) = \dfrac{1500}{[2(20) + 15]^2} \approx 0.5$ ft/sec^2

105. $f(x) = g(x)h(x)$

(a) $f'(x) = g(x)h'(x) + h(x)g'(x)$

$$f''(x) = g(x)h''(x) + g'(x)h'(x) + h(x)g''(x) + h'(x)g'(x)$$
$$= g(x)h''(x) + 2g'(x)h'(x) + h(x)g''(x)$$

$$f'''(x) = g(x)h'''(x) + g'(x)h''(x) + 2g'(x)h''(x) + 2g''(x)h'(x) + h(x)g'''(x) + h'(x)g''(x)$$
$$= g(x)h'''(x) + 3g'(x)h''(x) + 3g''(x)h'(x) + g'''(x)h(x)$$

$$f^{(4)}(x) = g(x)h^{(4)}(x) + g'(x)h'''(x) + 3g'(x)h'''(x) + 3g''(x)h''(x) + 3g''(x)h''(x) + 3g'''(x)h'(x)$$
$$+ g'''(x)h'(x) + g^{(4)}(x)h(x)$$
$$= g(x)h^{(4)}(x) + 4g'(x)h'''(x) + 6g''(x)h''(x) + 4g'''(x)h'(x) + g^{(4)}(x)h(x)$$

(b) $$f^{(n)}(x) = g(x)h^{(n)}(x) + \frac{n(n-1)(n-2)\cdots(2)(1)}{1[(n-1)(n-2)\cdots(2)(1)]}g'(x)h^{(n-1)}(x) + \frac{n(n-1)(n-2)\cdots(2)(1)}{(2)(1)[(n-2)(n-3)\cdots(2)(1)]}g''(x)h^{(n-2)}(x)$$
$$+ \frac{n(n-1)(n-2)\cdots(2)(1)}{(3)(2)(1)[(n-3)(n-4)\cdots(2)(1)]}g'''(x)h^{(n-3)}(x) + \cdots$$
$$+ \frac{n(n-1)(n-2)\cdots(2)(1)}{[(n-1)(n-2)\cdots(2)(1)](1)}g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)$$
$$= g(x)h^{(n)}(x) + \frac{n!}{1!(n-1)!}g'(x)h^{(n-1)}(x) + \frac{n!}{2!(n-2)!}g''(x)h^{(n-2)}(x) + \cdots$$
$$+ \frac{n!}{(n-1)!1!}g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)$$

Note: $n! = n(n-1)\ldots 3 \cdot 2 \cdot 1$ (read "n factorial.")

107. $f(x) = \cos x \qquad f\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

$f'(x) = -\sin x \qquad f'\left(\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

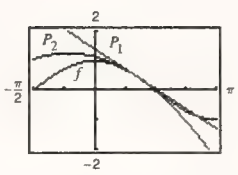
$f''(x) = -\cos x \qquad f''\left(\frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$

(a) $P_1(x) = f'(a)(x - a) + f(a) = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2}$

$P_2(x) = \frac{1}{2}f''(a)(x - a)^2 + f'(a)(x - a) + f(a)$

$= -\frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2}$

(b)



(c) P_2 is a better approximation.

(d) The accuracy worsens as you move farther away from $x = a = (\pi/3)$.

109. False. If $y = f(x)g(x)$, then

$$\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x).$$

111. True

$h'(c) = f(c)g'(c) + g(c)f'(c)$

$= f(c)(0) + g(c)(0)$

$= 0$

113. True

115. $f(x) = x|x| = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$

$f'(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ -2x, & \text{if } x < 0 \end{cases} = 2|x|$

$f''(x) = \begin{cases} 2, & \text{if } x > 0 \\ -2, & \text{if } x < 0 \end{cases}$

$f''(0)$ does not exist since the left and right derivatives are not equal.

Section 2.4 The Chain Rule

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
1. $y = (6x - 5)^4$	$u = 6x - 5$	$y = u^4$
3. $y = \sqrt{x^2 - 1}$	$u = x^2 - 1$	$y = \sqrt{u}$
5. $y = \csc^3 x$	$u = \csc x$	$y = u^3$

7. $y = (2x - 7)^3$

$y' = 3(2x - 7)^2(2) = 6(2x - 7)^2$

9. $g(x) = 3(4 - 9x)^4$

$g'(x) = 12(4 - 9x)^3(-9) = -108(4 - 9x)^3$

11. $f(x) = (9 - x^2)^{2/3}$

$f'(x) = \frac{2}{3}(9 - x^2)^{-1/3}(-2x) = -\frac{4x}{3(9 - x^2)^{1/3}}$

13. $f(t) = (1 - t)^{1/2}$

$f'(t) = \frac{1}{2}(1 - t)^{-1/2}(-1) = -\frac{1}{2\sqrt{1 - t}}$

15. $y = (9x^2 + 4)^{1/3}$

$$y' = \frac{1}{3}(9x^2 + 4)^{-2/3}(18x) = \frac{6x}{(9x^2 + 4)^{2/3}}$$

17. $y = 2(4 - x^2)^{1/4}$

$$y' = 2\left(\frac{1}{4}\right)(4 - x^2)^{-3/4}(-2x)$$

$$= \frac{-x}{\sqrt[4]{(4 - x^2)^3}}$$

19. $y = (x - 2)^{-1}$

$$y' = -1(2 - x)^{-2}(1) = \frac{-1}{(x - 2)^2}$$

21. $f(t) = (t - 3)^{-2}$

$$f'(t) = -2(t - 3)^{-3} = \frac{-2}{(t - 3)^3}$$

23. $y = (x + 2)^{-1/2}$

$$\frac{dy}{dx} = -\frac{1}{2}(x + 2)^{-3/2} = -\frac{1}{2(x + 2)^{3/2}}$$

25. $f(x) = x^2(x - 2)^4$

$$f'(x) = x^2[4(x - 2)^3(1)] + (x - 2)^4(2x)$$

$$= 2x(x - 2)^3[2x + (x - 2)]$$

$$= 2x(x - 2)^3(3x - 2)$$

27. $y = x\sqrt{1 - x^2} = x(1 - x^2)^{1/2}$

$$y' = x\left[\frac{1}{2}(1 - x^2)^{-1/2}(-2x)\right] + (1 - x^2)^{1/2}(1)$$

$$= -x^2(1 - x^2)^{-1/2} + (1 - x^2)^{1/2}$$

$$= (1 - x^2)^{-1/2}[-x^2 + (1 - x^2)]$$

$$= \frac{1 - 2x^2}{\sqrt{1 - x^2}}$$

29. $y = \dfrac{x}{\sqrt{x^2 + 1}} = x(x^2 + 1)^{-1/2}$

$$y' = x\left[-\frac{1}{2}(x^2 + 1)^{-3/2}(2x)\right] + (x^2 + 1)^{-1/2}(1)$$

$$= -x^2(x^2 + 1)^{-3/2} + (x^2 + 1)^{-1/2}$$

$$= (x^2 + 1)^{-3/2}[-x^2 + (x^2 + 1)]$$

$$= \frac{1}{(x^2 + 1)^{3/2}}$$

31. $g(x) = \left(\dfrac{x + 5}{x^2 + 2}\right)^2$

$$g'(x) = 2\left(\frac{x + 5}{x^2 + 2}\right)\left(\frac{(x^2 + 2) - (x + 5)(2x)}{(x^2 + 2)^2}\right)$$

$$= \frac{2(x + 5)(2 - 10x - x^2)}{(x^2 + 2)^3}$$

33. $f(v) = \left(\dfrac{1 - 2v}{1 + v}\right)^3$

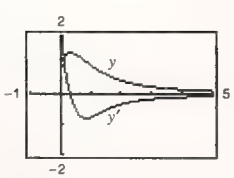
$$f'(v) = 3\left(\frac{1 - 2v}{1 + v}\right)^2\left(\frac{(1 + v)(-2) - (1 - 2v)}{(1 + v)^2}\right)$$

$$= \frac{-9(1 - 2v)^2}{(1 + v)^4}$$

35. $y = \dfrac{\sqrt{x} + 1}{x^2 + 1}$

$$y' = \frac{1 - 3x^2 - 4x^{3/2}}{2\sqrt{x}(x^2 + 1)^2}$$

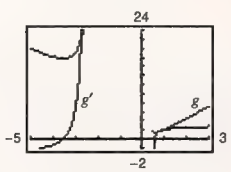
The zero of y' corresponds to the point on the graph of y where the tangent line is horizontal.



37. $g(t) = \dfrac{3t^2}{\sqrt{t^2 + 2t - 1}}$

$$g'(t) = \frac{3t(t^2 + 3t - 2)}{(t^2 + 2t - 1)^{3/2}}$$

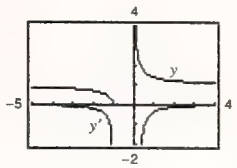
The zeros of g' correspond to the points on the graph of g where the tangent lines are horizontal.



39. $y = \sqrt{\dfrac{x+1}{x}}$

$y' = -\dfrac{\sqrt{(x+1)/x}}{2x(x+1)}$

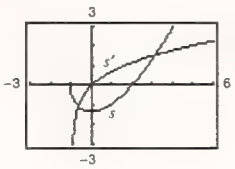
y' has no zeros.



41. $s(t) = \dfrac{-2(2-t)\sqrt{1+t}}{3}$

$s'(t) = \dfrac{t}{\sqrt{1+t}}$

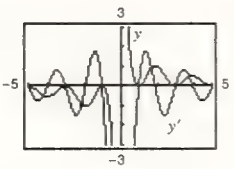
The zero of $s'(t)$ corresponds to the point on the graph of $s(t)$ where the tangent line is horizontal.



43. $y = \dfrac{\cos \pi x + 1}{x}$

$\dfrac{dy}{dx} = \dfrac{-\pi x \sin \pi x - \cos \pi x - 1}{x^2}$

$= -\dfrac{\pi x \sin \pi x + \cos \pi x + 1}{x^2}$



The zeros of y' correspond to the points on the graph of y where the tangent lines are horizontal.

45. (a) $y = \sin x$

$y' = \cos x$

$y'(0) = 1$

1 cycle in $[0, 2\pi]$

(b) $y = \sin 2x$

$y' = 2 \cos 2x$

$y'(0) = 2$

2 cycles in $[0, 2\pi]$

The slope of $\sin ax$ at the origin is a.

47. $y = \cos 3x$

$\dfrac{dy}{dx} = -3 \sin 3x$

49. $g(x) = 3 \tan 4x$

$g'(x) = 12 \sec^2 4x$

51. $y = \sin (\pi x)^2 = \sin (\pi^2 x^2)$

$y' = \cos (\pi x^2)[2\pi^2 x] = 2\pi^2 x \cos(\pi^2 x^2)$

53. $h(x) = \sin 2x \cos 2x$

$h'(x) = \sin 2x(-2 \sin 2x) + \cos 2x(2 \cos 2x)$

$= 2 \cos^2 2x - 2 \sin^2 2x$

$= 2 \cos 4x.$

Alternate solution: $h(x) = \dfrac{1}{2} \sin 4x$

$h'(x) = \dfrac{1}{2} \cos 4x(4) = 2 \cos 4x$

55. $f(x) = \dfrac{\cot x}{\sin x} = \dfrac{\cos x}{\sin^2 x}$

$f'(x) = \dfrac{\sin^2 x(-\sin x) - \cos x(2 \sin x \cos x)}{\sin^4 x}$

$= \dfrac{-\sin^2 x - 2 \cos^2 x}{\sin^3 x} = \dfrac{-1 - \cos^2 x}{\sin^3 x}$

57. $y = 4\sec^2 x$

$y' = 8\sec x \cdot \sec x \tan x = 8\sec^2 x \tan x$

59. $f(\theta) = \frac{1}{4}\sin^2 2\theta = \frac{1}{4}(\sin 2\theta)^2$

$f'(\theta) = 2\left(\frac{1}{4}\right)(\sin 2\theta)(\cos 2\theta)(2)$

$= \sin 2\theta \cos 2\theta = \frac{1}{2}\sin 4\theta$

61. $f(x) = 3\sec^2(\pi t - 1)$

$f'(t) = 6\sec(\pi t - 1)\sec(\pi t - 1)\tan(\pi t - 1)(\pi)$

$= 6\pi\sec^2(\pi t - 1)\tan(\pi t - 1) = \dfrac{6\pi\sin(\pi t - 1)}{\cos^3(\pi t - 1)}$

63. $y = \sqrt{x} + \frac{1}{4}\sin(2x)^2$

$= \sqrt{x} + \frac{1}{4}\sin(4x^2)$

$\dfrac{dy}{dx} = \frac{1}{2}x^{-1/2} + \frac{1}{4}\cos(4x^2)(8x)$

$= \dfrac{1}{2\sqrt{x}} + 2x\cos(2x)^2$

65. $y = \sin(\cos x)$

$\dfrac{dy}{dx} = \cos(\cos x) \cdot (-\sin x)$

$= -\sin x \cos(\cos x)$

67. $s(t) = (t^2 + 2t + 8)^{1/2}, \quad (2, 4)$

$s'(t) = \frac{1}{2}(t^2 + 2t + 8)^{-1/2}(2t + 2)$

$= \dfrac{t + 1}{\sqrt{t^2 + 2t + 8}}$

$s'(2) = \dfrac{3}{4}$

69. $f(x) = \dfrac{3}{x^3 - 4} = 3(x^3 - 4)^{-1}, \quad \left(-1, -\dfrac{3}{5}\right)$

$f'(x) = -3(x^3 - 4)^{-2}(3x^2) = -\dfrac{9x^2}{(x^3 - 4)^2}$

$f'(-1) = -\dfrac{9}{25}$

71. $f(t) = \dfrac{3t + 2}{t - 1}, \quad (0, -2)$

$f'(t) = \dfrac{(t - 1)(3) - (3t + 2)(1)}{(t - 1)^2} = \dfrac{-5}{(t - 1)^2}$

$f'(0) = -5$

73. $y = 37 - \sec^3(2x), \quad (0, 36)$

$y' = -3\sec^2(2x)[2\sec(2x)\tan(2x)]$

$= -6\sec^3(2x)\tan(2x)$

$y'(0) = 0$

75. (a) $f(x) = \sqrt{3x^2 - 2}, \quad (3, 5)$

$f'(x) = \frac{1}{2}(3x^2 - 2)^{-1/2}(6x)$

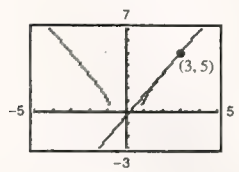
$= \dfrac{3x}{\sqrt{3x^2 - 2}}$

$f'(3) = \dfrac{9}{5}$

Tangent line:

$y - 5 = \dfrac{9}{5}(x - 3) \Rightarrow 9x - 5y - 2 = 0$

(b)



77. (a) $f(x) = \sin 2x, \quad (\pi, 0)$

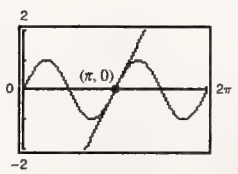
$f'(x) = 2\cos 2x$

$f'(\pi) = 2$

Tangent line:

$y = 2(x - \pi) \Rightarrow 2x - y - 2\pi = 0$

(b)



79. $f(x) = 2(x^2 - 1)^3$

$f'(x) = 6(x^2 - 1)^2(2x)$

$= 12x(x^4 - 2x^2 + 1)$

$= 12x^5 - 24x^3 + 12x$

$f''(x) = 60x^4 - 72x^2 + 12$

$= 12(5x^2 - 1)(x^2 - 1)$

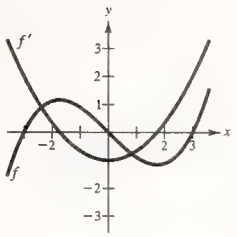
81. $f(x) = \sin x^2$

$f'(x) = 2x \cos x^2$

$f''(x) = 2x[2x(-\sin x^2)] + 2\cos x^2$

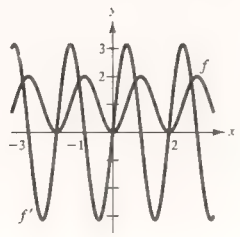
$= 2[\cos x^2 - 2x^2 \sin x^2]$

83.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

85.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

87. $g(x) = f(3x)$

$g'(x) = f'(3x)(3) \Rightarrow g'(x) = 3f'(3x)$

89. (a) $f(x) = g(x)h(x)$

$f'(x) = g(x)h'(x) + g'(x)h(x)$

$f'(5) = (-3)(-2) + (6)(3) = 24$

(b) $f(x) = g(h(x))$

$f'(x) = g'(h(x))h'(x)$

$f'(5) = g'(3)(-2) = -2g'(3)$

Need $g'(3)$ to find $f'(5)$.

(c) $f(x) = \dfrac{g(x)}{h(x)}$

$f'(x) = \dfrac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$

$f'(5) = \dfrac{(3)(6) - (-3)(-2)}{(3)^2} = \dfrac{12}{9} = \dfrac{4}{3}$

(d) $f(x) = [g(x)]^3$

$f'(x) = 3[g(x)]^2 g'(x)$

$f'(5) = 3(-3)^2(6) = 162$

91. (a) $f = 132{,}400(331 - v)^{-1}$

$f' = (-1)(132{,}400)(331 - v)^{-2}(-1)$

$= \dfrac{132{,}400}{(331 - v)^2}$

When $v = 30$, $f' \approx 1.461$.

(b) $f = 132{,}400(331 + v)^{-1}$

$f' = (-1)(132{,}400)(331 + v)^{-2}(1)$

$= \dfrac{-132.400}{(331 + v)^2}$

When $v = 30$, $f' \approx -1.016$.

93. $\theta = 0.2 \cos 8t$

The maximum angular displacement is $\theta = 0.2$ (since $-1 \leq \cos 8t \leq 1$).

$\dfrac{d\theta}{dt} = 0.2[-8 \sin 8t] = -1.6 \sin 8t$

When $t = 3$, $d\theta/dt = -1.6 \sin 24 \approx 1.4489$ radians per second.

95. $S = C(R^2 - r^2)$

$\dfrac{dS}{dt} = C\left(2R\dfrac{dR}{dt} - 2r\dfrac{dr}{dt}\right)$

Since r is constant, we have $dr/dt = 0$ and

$\dfrac{dS}{dt} = (1.76 \times 10^5)(2)(1.2 \times 10^{-2})(10^{-5})$

$= 4.224 \times 10^{-2} = 0.04224.$

97. (a) $x = -1.6372t^3 + 19.3120t^2 - 0.5082t - 0.6161$

(b) $C = 60x + 1350$

$= 60(-1.6372t^3 + 19.3120t^2 - 0.5082t - 0.6161) + 1350$

$\dfrac{dC}{dt} = 60(-4.9116t^2 + 38.624t - 0.5082)$

$= -294.696t^2 + 2317.44t - 30.492$

The function $\dfrac{dC}{dt}$ is quadratic, not linear. The cost function levels off at the end of the day, perhaps due to fatigue.

99. $f(x) = \sin \beta x$

(a) $f'(x) = \beta \cos \beta x$

$f''(x) = -\beta^2 \sin \beta x$

$f'''(x) = -\beta^3 \cos \beta x$

$f^{(4)} = \beta^4 \sin \beta x$

(b) $f''(x) + \beta^2 f(x) = -\beta^2 \sin \beta x + \beta^2(\sin \beta x) = 0$

(c) $f^{(2k)}(x) = (-1)^k \beta^{2k} \sin \beta x$

$f^{(2k-1)}(x) = (-1)^{k+1} \beta^{2k-1} \cos \beta x$

101. (a) $r'(x) = f'(g(x))g'(x)$

$r'(1) = f'(g(1))g'(1)$

Note that $g(1) = 4$ and $f'(4) = \dfrac{5-0}{6-2} = \dfrac{5}{4}$

Also, $g'(1) = 0$. Thus, $r'(1) = 0$

(b) $s'(x) = g'(f(x))f'(x)$

$s'(4) = g'(f(4))f'(4)$

Note that $f(4) = \dfrac{5}{2}$, $g'\left(\dfrac{5}{2}\right) = \dfrac{6-4}{6-2} = \dfrac{1}{2}$ and

$f'(4) = \dfrac{5}{4}$.

Thus, $s'(4) = \dfrac{1}{2}\left(\dfrac{5}{4}\right) = \dfrac{5}{8}$.

103. $g = \sqrt{x(x+n)}$

$= \sqrt{x^2 + nx}$

$\dfrac{dg}{dx} = \dfrac{1}{2}(x^2 + nx)^{-1/2}(2x + n)$

$= \dfrac{2x + n}{2\sqrt{x^2 + nx}}$

$= \dfrac{(2x + n)/2}{\sqrt{x(x+n)}}$

$= \dfrac{[x + (x + n)]/2}{\sqrt{x(x+n)}}$

$= \dfrac{a}{g}$

105. $g(x) = |2x - 3|$

$g'(x) = 2\left(\dfrac{2x-3}{|2x-3|}\right), \quad x \neq \dfrac{3}{2}$

107. $h(x) = |x| \cos x$

$h'(x) = -|x| \sin x + \dfrac{x}{|x|} \cos x, \quad x \neq 0$

109. (a) $f(x) = \tan\dfrac{\pi x}{4}$ $\qquad f(1) = 1$

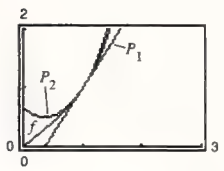
$f'(x) = \dfrac{\pi}{4}\sec^2\dfrac{\pi x}{4}$ $\qquad f'(1) = \dfrac{\pi}{4}(2) = \dfrac{\pi}{2}$

$f''(x) = \dfrac{\pi}{2}\sec^2\dfrac{\pi x}{4}\cdot\tan\dfrac{\pi x}{4}\left(\dfrac{\pi}{4}\right)$ $\qquad f''(1) = \dfrac{\pi}{8}(2)(1) = \dfrac{\pi}{4}$

$P_1(x) = f'(1)(x-1) + f(1) = \dfrac{\pi}{2}(x-1) + 1.$

$P_2(x) = \dfrac{1}{2}\left(\dfrac{\pi}{4}\right)(x-1)^2 + f'(1)(x-1) + f(1) = \dfrac{\pi}{8}(x-1)^2 + \dfrac{\pi}{2}(x-1) + 1$

(b)



(c) P_2 is a better approximation than P_1

(d) The accuracy worsens as you move away from $x = c = 1$.

111. False. If $y = (1-x)^{1/2}$, then $y' = \frac{1}{2}(1-x)^{-1/2}(-1)$.

113. True

Section 2.5 Implicit Differentiation

1. $x^2 + y^2 = 36$

$2x + 2yy' = 0$

$$y' = \frac{-x}{y}$$

3. $x^{1/2} + y^{1/2} = 9$

$\dfrac{1}{2}x^{-1/2} + \dfrac{1}{2}y^{-1/2}y' = 0$

$$y' = -\frac{x^{-1/2}}{y^{-1/2}} = -\sqrt{\frac{y}{x}}$$

5. $x^3 - xy + y^2 = 4$

$3x^2 - xy' - y + 2yy' = 0$

$(2y - x)y' = y - 3x^2$

$$y' = \frac{y - 3x^2}{2y - x}$$

7. $x^3y^3 - y - x = 0$

$3x^3y^2y' + 3x^2y^3 - y' - 1 = 0$

$(3x^3y^2 - 1)y' = 1 - 3x^2y^3$

$$y' = \frac{1 - 3x^2y^3}{3x^3y^2 - 1}$$

9. $x^3 - 3x^2 + 2xy^2 = 12$

$3x^2 - 3x^2y' - 6xy + 4xyy' + 2y^2 = 0$

$(4xy - 3x^2)y' = 6xy - 3x^2 - 2y^2$

$$y' = \frac{6xy - 3x^2 - 2y^2}{4xy - 3x^2}$$

11. $\sin x + 2\cos 2y = 1$

$\cos x - 4(\sin 2y)y' = 0$

$$y' = \frac{\cos x}{4\sin 2y}$$

13. $\sin x = x(1 + \tan y)$

$\cos x = x(\sec^2 y)y' + (1 + \tan y)(1)$

$y' = \dfrac{\cos x - \tan y - 1}{x \sec^2 y}$

15. $y = \sin(xy)$

$y' = [xy' + y]\cos(xy)$

$y' - x\cos(xy)y' = y\cos(xy)$

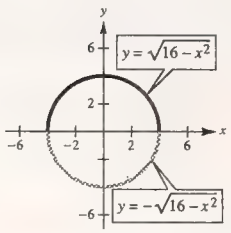
$y' = \dfrac{y\cos(xy)}{1 - x\cos(xy)}$

17. (a) $x^2 + y^2 = 16$

$y^2 = 16 - x^2$

$y = \pm\sqrt{16 - x^2}$

(b)



(c) Explicitly:

$\dfrac{dy}{dx} = \pm\dfrac{1}{2}(16 - x^2)^{-1/2}(-2x)$

$= \dfrac{\mp x}{\sqrt{16 - x^2}} = \dfrac{-x}{\pm\sqrt{16 - x^2}} = \dfrac{-x}{y}$

(d) Implicitly:

$2x + 2yy' = 0$

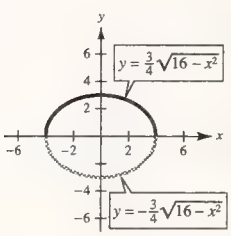
$y' = -\dfrac{x}{y}$

19. (a) $16y^2 = 144 - 9x^2$

$y^2 = \dfrac{1}{16}(144 - 9x^2) = \dfrac{9}{16}(16 - x^2)$

$y = \pm\dfrac{3}{4}\sqrt{16 - x^2}$

(b)



(c) Explicitly:

$\dfrac{dy}{dx} = \pm\dfrac{3}{8}(16 - x^2)^{-1/2}(-2x)$

$= \mp\dfrac{3x}{4\sqrt{16 - x^2}} = \dfrac{-3x}{4(4/3)y} = \dfrac{-9x}{16y}$

(d) Implicitly:

$18x + 32yy' = 0$

$y' = \dfrac{-9x}{16y}$

21. $xy = 4$

$xy' + y(1) = 0$

$xy' = -y$

$y' = \dfrac{-y}{x}$

At $(-4, -1)$: $y' = -\dfrac{1}{4}$

23. $y^2 = \dfrac{x^2 - 4}{x^2 + 4}$

$2yy' = \dfrac{(x^2 + 4)(2x) - (x^2 - 4)(2x)}{(x^2 + 4)^2}$

$2yy' = \dfrac{16x}{(x^2 + 4)^2}$

$y' = \dfrac{8x}{y(x^2 + 4)^2}$

At $(2, 0)$, y' is undefined.

25. $x^{2/3} + y^{2/3} = 5$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = \frac{-x^{-1/3}}{y^{-1/3}} = -\sqrt[3]{\frac{y}{x}}$$

At (8, 1): $y' = -\frac{1}{2}$.

27. $\tan(x + y) = x$

$$(1 + y')\sec^2(x + y) = 1$$

$$y' = \frac{1 - \sec^2(x + y)}{\sec^2(x + y)}$$

$$= \frac{-\tan^2(x + y)}{\tan^2(x + y) + 1} = -\sin^2(x + y)$$

$$= -\frac{x^2}{x^2 + 1}$$

At (0, 0): $y' = 0$.

29. $(x^2 + 4)y = 8$

$$(x^2 + 4)y' + y(2x) = 0$$

$$y' = \frac{-2xy}{x^2 + 4}$$

$$= \frac{-2x[8/(x^2 + 4)]}{x^2 + 4}$$

$$= \frac{-16x}{(x^2 + 4)^2}$$

At (2, 1): $y' = \frac{-32}{64} = -\frac{1}{2}$

$\left(\text{Or, you could just solve for } y\text{: } y = \frac{8}{x^2 + 4}\right)$

31. $(x^2 + y^2)^2 = 4x^2y$

$$2(x^2 + y^2)(2x + 2yy') = 4x^2y' + y(8x)$$

$$4x^3 + 4x^2yy' + 4xy^2 + 4y^3y' = 4x^2y' + 8xy$$

$$4x^2yy' + 4y^3y' - 4x^2y' = 8xy - 4x^3 - 4xy^2$$

$$4y'(x^2y + y^3 - x^2) = 4(2xy - x^3 - xy^2)$$

$$y' = \frac{2xy - x^3 - xy^2}{x^2y + y^3 - x^2}$$

At (1, 1): $y' = 0$.

33. $\tan y = x$

$$y'\sec^2 y = 1$$

$$y' = \frac{1}{\sec^2 y} = \cos^2 y, -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$y' = \frac{1}{1 + x^2}$$

35. $x^2 + y^2 = 36$

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y}$$

$$y'' = \frac{y(-1) + xy'}{y^2} = \frac{-y + x\left(-\frac{x}{y}\right)}{y^2} = \frac{-y^2 - x^2}{y^3} = \frac{-36}{y^3}$$

37. $x^2 - y^2 = 16$

$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}$$

$$x - yy' = 0$$

$$1 - yy'' - (y')^2 = 0$$

$$1 - yy'' - \left(\frac{x}{y}\right)^2 = 0$$

$$y^2 - y^3y'' = x^2$$

$$y'' = \frac{y^2 - x^2}{y^3} = \frac{-16}{y^3}$$

39. $y^2 = x^3$

$$2yy' = 3x^2$$

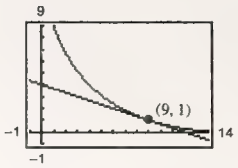
$$y' = \frac{3x^2}{2y} = \frac{3x^2}{2y} \cdot \frac{xy}{xy} = \frac{3y}{2x} \cdot \frac{x^3}{y^2} = \frac{3y}{2x}$$

$$y'' = \frac{2x(3y') - 3y(2)}{4x^2}$$

$$= \frac{2x[3 \cdot (3y/2x)] - 6y}{4x^2}$$

$$= \frac{3y}{4x^2} = \frac{3x}{4y}$$

41. $\sqrt{x} + \sqrt{y} = 4$



$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$$

$$y' = \frac{-\sqrt{y}}{\sqrt{x}}$$

At $(9, 1)$, $y' = -\frac{1}{3}$

Tangent line: $y - 1 = -\frac{1}{3}(x - 9)$

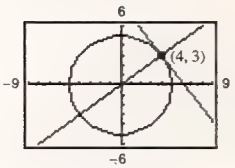
$$y = -\frac{1}{3}x + 4$$

$$x + 3y - 12 = 0$$

43. $x^2 + y^2 = 25$

$$y' = \frac{-x}{y}$$

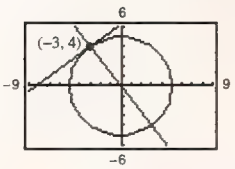
At $(4, 3)$:



Tangent line: $y - 3 = \frac{-4}{3}(x - 4) \Rightarrow 4x + 3y - 25 = 0$

Normal line: $y - 3 = \frac{3}{4}(x - 4) \Rightarrow 3x - 4y = 0.$

At $(-3, 4)$:



Tangent line: $y - 4 = \frac{3}{4}(x + 3) \Rightarrow 3x - 4y + 25 = 0$

Normal line: $y - 4 = \frac{-4}{3}(x + 3) \Rightarrow 4x + 3y = 0.$

45. $x^2 + y^2 = r^2$

$2x + 2yy' = 0$

$$y' = \frac{-x}{y} = \text{slope of tangent line}$$

$$\frac{y}{x} = \text{slope of normal line}$$

Let (x_0, y_0) be a point on the circle. If $x_0 = 0$, then the tangent line is horizontal, the normal line is vertical and, hence, passes through the origin. If $x_0 \neq 0$, then the equation of the normal line is

$$y - y_0 = \frac{y_0}{x_0}(x - x_0)$$

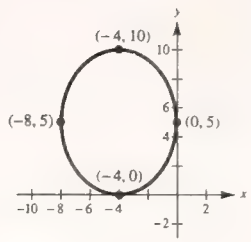
$$y = \frac{y_0}{x_0}x$$

which passes through the origin.

47. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

$$50x + 32yy' + 200 - 160y' = 0$$

$$y' = \frac{200 + 50x}{160 - 32y}$$



Horizontal tangents occur when $x = -4$:

$$25(16) + 16y^2 + 200(-4) - 160y + 400 = 0$$

$$y(y - 10) = 0 \Rightarrow y = 0, 10$$

Horizontal tangents: $(-4, 0), (-4, 10)$.

Vertical tangents occur when $y = 5$:

$$25x^2 + 400 + 200x - 800 + 400 = 0$$

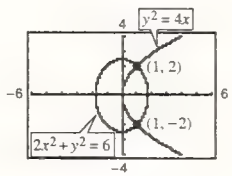
$$25x(x + 8) = 0 \Rightarrow x = 0, -8$$

Vertical tangents: $(0, 5), (-8, 5)$.

49. Find the points of intersection by letting $y^2 = 4x$ in the equation $2x^2 + y^2 = 6$.

$2x^2 + 4x = 6$ and $(x + 3)(x - 1) = 0$

The curves intersect at $(1, \pm 2)$.



Ellipse:	*Parabola:*
$4x + 2yy' = 0$	$2yy' = 4$
$y' = -\frac{2x}{y}$	$y' = \frac{2}{y}$

At $(1, 2)$, the slopes are:

$y' = -1$ $\qquad$ $y' = 1$.

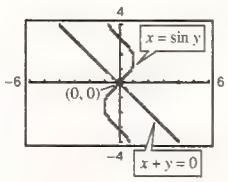
At $(1, -2)$, the slopes are:

$y' = 1$ $\qquad$ $y' = -1$.

Tangents are perpendicular.

51. $y = -x$ and $x = \sin y$

Point of intersection: $(0, 0)$



$y = -x$:	$x = \sin y$:
$y' = -1$	$1 = y' \cos y$
	$y' = \sec y$

At $(0, 0)$, the slopes are:

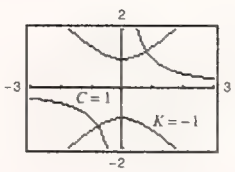
$y' = -1$ $\qquad$ $y' = 1$.

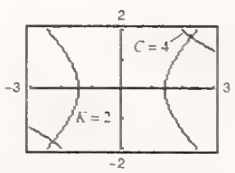
Tangents are perpendicular.

53. $xy = C$ $\qquad$ $x^2 - y^2 = K$

$xy' + y = 0$ $\qquad$ $2x - 2yy' = 0$

$$y' = -\frac{y}{x} \qquad y' = \frac{x}{y}$$





At any point of intersection (x, y) the product of the slopes is $(-y/x)(x/y) = -1$. The curves are orthogonal.

55. $2y^2 - 3x^4 = 0$

(a) $4yy' - 12x^3 = 0$

$4yy' = 12x^3$

$y' = \dfrac{12x^3}{4y} = \dfrac{3x^3}{y}$

(b) $4y\dfrac{dy}{dt} - 12x^3\dfrac{dx}{dt} = 0$

$y\dfrac{dy}{dt} = 3x^3\dfrac{dx}{dt}$

57. $\cos \pi y - 3 \sin \pi x = 1$

(a) $-\pi \sin(\pi y)y' - 3\pi \cos \pi x = 0$

$y' = \dfrac{-3 \cos \pi x}{\sin \pi y}$

(b) $-\pi \sin(\pi y)\dfrac{dy}{dt} - 3\pi \cos(\pi x)\dfrac{dx}{dt} = 0$

$-\sin(\pi y)\dfrac{dy}{dt} = 3 \cos(\pi x)\dfrac{dx}{dt}$

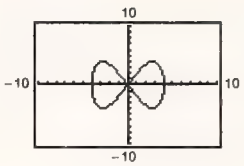
59. A function is in explicit form if y is written as a function of x: $y = f(x)$. For example, $y = x^3$. An implicit equation is not in the form $y = f(x)$. For example, $x^2 + y^2 = 5$.

61. (a) $x^4 = 4(4x^2 - y^2)$

$4y^2 = 16x^2 - x^4$

$y^2 = 4x^2 - \dfrac{1}{4}x^4$

$y = \pm\sqrt{4x^2 - \dfrac{1}{4}x^4}$



(b) $y = 3 \Rightarrow 9 = 4x^2 - \dfrac{1}{4}x^4$

$36 = 16x^2 - x^4$

$x^4 - 16x^2 + 36 = 0$

$x^2 = \dfrac{16 \pm \sqrt{256 - 144}}{2} = 8 \pm \sqrt{28}$

Note that $x^2 = 8 \pm \sqrt{28} = 8 \pm 2\sqrt{7} = (1 \pm \sqrt{7})^2$.

Hence, there are four values of x:

$-1-\sqrt{7}, 1-\sqrt{7}, -1+\sqrt{7}, 1 + \sqrt{7}$

To find the slope, $2yy' = 8x - x^3 \Rightarrow y' = \dfrac{x(8 - x^2)}{2(3)}$.

For $x = -1 - \sqrt{7}$, $y' = \frac{1}{3}(\sqrt{7} + 7)$, and the line is

$y_1 = \frac{1}{3}(\sqrt{7} + 7)(x + 1 + \sqrt{7}) + 3 = \frac{1}{3}[(\sqrt{7} + 7)x + 8\sqrt{7} + 23)]$.

For $x = 1 - \sqrt{7}$, $y' = \frac{1}{3}(\sqrt{7} - 7)$, and the line is

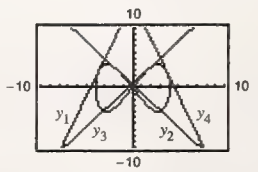
$y_2 = \frac{1}{3}(\sqrt{7} - 7)(x - 1 + \sqrt{7}) + 3 = \frac{1}{3}[(\sqrt{7} - 7)x + 23 - 8\sqrt{7}]$.

For $x = -1 + \sqrt{7}$, $y' = -\frac{1}{3}(\sqrt{7} - 7)$, and the line is

$y_3 = -\frac{1}{3}(\sqrt{7} - 7)(x + 1 - \sqrt{7}) + 3 = -\frac{1}{3}[(\sqrt{7} - 7)x - (23 - 8\sqrt{7})]$.

For $x = 1 + \sqrt{7}$, $y' = -\frac{1}{3}(\sqrt{7} + 7)$, and the line is

$y_4 = -\frac{1}{3}(\sqrt{7} + 7)(x - 1 - \sqrt{7}) + 3 = -\frac{1}{3}[(\sqrt{7} + 7)x - (8\sqrt{7} + 23)]$.



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61. —CONTINUED—

(c) Equating y_3 and y_4,

$$-\frac{1}{3}(\sqrt{7}-7)(x+1-\sqrt{7})+3=-\frac{1}{3}(\sqrt{7}+7)(x-1-\sqrt{7})+3$$

$$(\sqrt{7}-7)(x+1-\sqrt{7})=(\sqrt{7}+7)(x-1-\sqrt{7})$$

$$\sqrt{7}x+\sqrt{7}-7-7x-7+7\sqrt{7}=\sqrt{7}x-\sqrt{7}-7+7x-7-7\sqrt{7}$$

$$16\sqrt{7}=14x$$

$$x=\frac{8\sqrt{7}}{7}$$

If $x=\frac{8\sqrt{7}}{7}$, then $y=5$ and the lines intersect at $\left(\frac{8\sqrt{7}}{7},5\right)$.

63. Let $f(x)=x^n=x^{p/q}$, where p and q are nonzero integers and $q>0$. First consider the case where $p=1$. The derivative of $f(x)=x^{1/q}$ is given by

$$\frac{d}{dx}[x^{1/q}]=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}$$

where $t=x+\Delta x$. Observe that

$$\frac{f(t)-f(x)}{t-x}=\frac{t^{1/q}-x^{1/q}}{t-x}=\frac{t^{1/q}-x^{1/q}}{(t^{1/q})^q-(x^{1/q})^q}$$

$$=\frac{t^{1/q}-x^{1/q}}{(t^{1/q}-x^{1/q})(t^{1-(1/q)}+t^{1-(2/q)}x^{1/q}+\cdots+t^{1/q}x^{1-(2/q)}+x^{1-(1/q)})}$$

$$=\frac{1}{t^{1-(1/q)}+t^{1-(2/q)}x^{1/q}+\cdots+t^{1/q}x^{1-(2/q)}+x^{1-(1/q)}}.$$

As $t\to x$, the denominator approaches $qx^{1-(1/q)}$. That is,

$$\frac{d}{dx}[x^{1/q}]=\frac{1}{qx^{1-(1/q)}}=\frac{1}{q}x^{(1/q)-1}.$$

Now consider $f(x)=x^{p/q}=(x^p)^{1/q}$. From the Chain Rule,

$$f'(x)=\frac{1}{q}(x^p)^{(1/q)-1}\frac{d}{dx}[x^p]=\frac{1}{q}(x^p)^{(1/q)-1}px^{p-1}=\frac{p}{q}x^{[(p/q)-p]+(p-1)}=\frac{p}{q}x^{(p/q)-1}=nx^{n-1}\left(n=\frac{p}{q}\right).$$

Section 2.6 Related Rates

1. $y=\sqrt{x}$

$$\frac{dy}{dt}=\left(\frac{1}{2\sqrt{x}}\right)\frac{dx}{dt}$$

$$\frac{dx}{dt}=2\sqrt{x}\frac{dy}{dt}$$

(a) When $x=4$ and $dx/dt=3$,

$$\frac{dy}{dt}=\frac{1}{2\sqrt{4}}(3)=\frac{3}{4}.$$

(b) When $x=25$ and $dy/dt=2$,

$$\frac{dx}{dt}=2\sqrt{25}\,(2)=20.$$

3. $xy=4$

$$x\frac{dy}{dt}+y\frac{dx}{dt}=0$$

$$\frac{dy}{dt}=\left(-\frac{y}{x}\right)\frac{dx}{dt}$$

$$\frac{dx}{dt}=\left(-\frac{x}{y}\right)\frac{dy}{dt}$$

(a) When $x=8$, $y=1/2$, and $dx/dt=10$,

$$\frac{dy}{dt}=-\frac{1/2}{8}(10)=-\frac{5}{8}.$$

(b) When $x=1$, $y=4$, and $dy/dt=-6$,

$$\frac{dx}{dt}=-\frac{1}{4}(-6)=\frac{3}{2}.$$

5. $y = x^2 + 1$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = 2x\,\frac{dx}{dt}$$

(a) When $x = -1$,

$$\frac{dy}{dt} = 2(-1)(2) = -4 \text{ cm/sec.}$$

(b) When $x = 0$,

$$\frac{dy}{dt} = 2(0)(2) = 0 \text{ cm/sec.}$$

(c) When $x = 1$,

$$\frac{dy}{dt} = 2(1)(2) = 4 \text{ cm/sec.}$$

7. $y = \tan x$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = \sec^2 x\,\frac{dx}{dt}$$

(a) When $x = -\pi/3$,

$$\frac{dy}{dt} = (2)^2(2) = 8 \text{ cm/sec.}$$

(b) When $x = -\pi/4$,

$$\frac{dy}{dt} = \left(\sqrt{2}\right)^2(2) = 4 \text{ cm/sec.}$$

(c) When $x = 0$,

$$\frac{dy}{dt} = (1)^2(2) = 2 \text{ cm/sec.}$$

9. (a) $\dfrac{dx}{dt}$ negative $\Rightarrow \dfrac{dy}{dt}$ positive

(b) $\dfrac{dy}{dt}$ positive $\Rightarrow \dfrac{dx}{dt}$ negative

11. Yes, y changes at a constant rate: $\dfrac{dy}{dt} = a \cdot \dfrac{dx}{dt}$.

No, the rate $\dfrac{dy}{dt}$ is a multiple of $\dfrac{dx}{dt}$.

13. $D = \sqrt{x^2 + y^2} = \sqrt{x^2 + (x^2 + 1)^2} = \sqrt{x^4 + 3x^2 + 1}$

$$\frac{dx}{dt} = 2$$

$$\frac{dD}{dt} = \frac{1}{2}(x^4 + 3x^2 + 1)^{-1/2}(4x^3 + 6x)\frac{dx}{dt} = \frac{2x^3 + 3x}{\sqrt{x^4 + 3x^2 + 1}}\frac{dx}{dt} = \frac{4x^3 + 6x}{\sqrt{x^4 + 3x^2 + 1}}$$

15. $A = \pi r^2$

$$\frac{dr}{dt} = 3$$

$$\frac{dA}{dt} = 2\pi r\,\frac{dr}{dt}$$

(a) When $r = 6$,

$$\frac{dA}{dt} = 2\pi(6)(3) = 36\pi \text{ cm}^2/\text{min.}$$

(b) When $r = 24$,

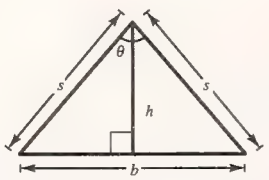
$$\frac{dA}{dt} = 2\pi(24)(3) = 144\pi \text{ cm}^2/\text{min.}$$

17. (a) $\sin\dfrac{\theta}{2} = \dfrac{(1/2)b}{s} \Rightarrow b = 2s\sin\dfrac{\theta}{2}$

$$\cos\frac{\theta}{2} = \frac{h}{s} \Rightarrow h = s\cos\frac{\theta}{2}$$

$$A = \frac{1}{2}bh = \frac{1}{2}\left(2s\sin\frac{\theta}{2}\right)\left(s\cos\frac{\theta}{2}\right)$$

$$= \frac{s^2}{2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = \frac{s^2}{2}\sin\theta$$



(b) $\dfrac{dA}{dt} = \dfrac{s^2}{2}\cos\theta\dfrac{d\theta}{dt}$ where $\dfrac{d\theta}{dt} = \dfrac{1}{2}$ rad/min.

When $\theta = \dfrac{\pi}{6}$, $\dfrac{dA}{dt} = \dfrac{s^2}{2}\left(\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{1}{2}\right) = \dfrac{\sqrt{3}s^2}{8}$

When $\theta = \dfrac{\pi}{3}$, $\dfrac{dA}{dt} = \dfrac{s^2}{2}\left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right) = \dfrac{s^2}{8}$

(c) If $d\theta/dt$ is constant, dA/dt is proportional to $\cos\theta$.

19. $V = \frac{4}{3}\pi r^3, \frac{dV}{dt} = 800$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\left(\frac{dV}{dt}\right) = \frac{1}{4\pi r^2}(800)$$

(a) When $r = 30$, $\frac{dr}{dt} = \frac{1}{4\pi(30)^2}(800) = \frac{2}{9\pi}$ cm/min.

(b) When $r = 60$, $\frac{dr}{dt} = \frac{1}{4\pi(60)^2}(800) = \frac{1}{18\pi}$ cm/min.

21. $s = 6x^2$

$$\frac{dx}{dt} = 3$$

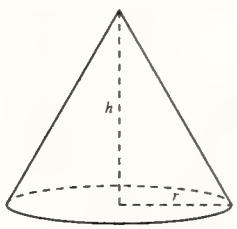
$$\frac{ds}{dt} = 12x \frac{dx}{dt}$$

(a) When $x = 1$,

$$\frac{ds}{dt} = 12(1)(3) = 36 \text{ cm}^2/\text{sec}.$$

(b) When $x = 10$,

$$\frac{ds}{dt} = 12(10)(3) = 360 \text{ cm}^2/\text{sec}.$$



23. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{9}{4}h^2\right)h$ [since $2r = 3h$]

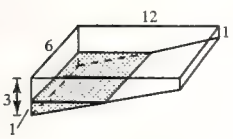
$$= \frac{3\pi}{4}h^3$$

$$\frac{dV}{dt} = 10$$

$$\frac{dV}{dt} = \frac{9\pi}{4}h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{4(dV/dt)}{9\pi h^2}$$

When $h = 15$, $\frac{dh}{dt} = \frac{4(10)}{9\pi(15)^2} = \frac{8}{405\pi}$ ft/min.

25.

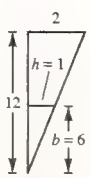


(a) Total volume of pool $= \frac{1}{2}(2)(12)(6) + (1)(6)(12) = 144 \text{ m}^3$

Volume of 1m. of water $= \frac{1}{2}(1)(6)(6) = 18 \text{ m}^3$

(see similar triangle diagram)

% pool filled $= \frac{18}{144}(100\%) = 12.5\%$



(b) Since for $0 \le h \le 2$, $b = 6h$, you have

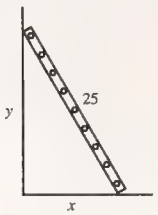
$$V = \frac{1}{2}bh(6) = 3bh = 3(6h)h = 18h^2$$

$$\frac{dV}{dt} = 36h\frac{dh}{dt} = \frac{1}{4} \Rightarrow \frac{dh}{dt} = \frac{1}{144h} = \frac{1}{144(1)} = \frac{1}{144} \text{ m/min}.$$

27. $x^2 + y^2 = 25^2$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-x}{y} \cdot \frac{dx}{dt} = \frac{-2x}{y} \text{ since } \frac{dx}{dt} = 2.$$



(a) When $x = 7, y = \sqrt{576} = 24, \frac{dy}{dt} = \frac{-2(7)}{24} = \frac{-7}{12}$ ft/sec.

When $x = 15, y = \sqrt{400} = 20, \frac{dy}{dt} = \frac{-2(15)}{20} = \frac{-3}{2}$ ft/sec.

When $x = 24, y = 7, \frac{dy}{dt} = \frac{-2(24)}{7} = \frac{-48}{7}$ ft/sec.

(b) $A = \frac{1}{2}xy$

$$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$$

From part (a) we have $x = 7, y = 24, \frac{dx}{dt} = 2$, and $\frac{dy}{dt} = -\frac{7}{12}$.

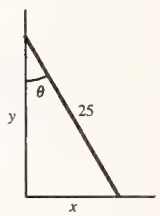
Thus, $\frac{dA}{dt} = \frac{1}{2}\left[7\left(-\frac{7}{12}\right) + 24(2)\right]$

$$= \frac{527}{24} \approx 21.96 \text{ ft}^2/\text{sec}.$$

(c) $\tan\theta = \frac{x}{y}$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{y} \cdot \frac{dx}{dt} - \frac{x}{y^2} \cdot \frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \cos^2\theta\left[\frac{1}{y} \cdot \frac{dx}{dt} - \frac{x}{y^2} \cdot \frac{dy}{dt}\right]$$

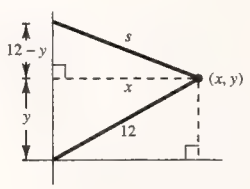


Using $x = 7, y = 24, \frac{dx}{dt} = 2, \frac{dy}{dt} = -\frac{7}{12}$ and $\cos\theta = \frac{24}{25}$, we have $\frac{d\theta}{dt} = \left(\frac{24}{25}\right)^2\left[\frac{1}{24}(2) - \frac{7}{(24)^2}\left(-\frac{7}{12}\right)\right] = \frac{1}{12}$ rad/sec.

29. When $y = 6, x = \sqrt{12^2 - 6^2} = 6\sqrt{3}$, and

$$s = \sqrt{x^2 + (12 - y)^2}$$

$$= \sqrt{108 + 36} = 12.$$



$$x^2 + (12 - y)^2 = s^2$$

$$2x\frac{dx}{dt} + 2(12 - y)(-1)\frac{dy}{dt} = 2s\frac{ds}{dt}$$

$$x\frac{dx}{dt} + (y - 12)\frac{dy}{dt} = s\frac{ds}{dt}$$

Also, $x^2 + y^2 = 12^2$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}.$$

Thus, $x\frac{dx}{dt} + (y - 12)\left(\frac{-x}{y}\frac{dx}{dt}\right) = s\frac{ds}{dt}$

$$\frac{dx}{dt}\left[x - x + \frac{12x}{y}\right] = s\frac{ds}{dt} \Rightarrow \frac{dx}{dt} = \frac{sy}{12x} \cdot \frac{ds}{dt} = \frac{(12)(6)}{(12)(6\sqrt{3})}(-0.2) = \frac{-1}{5\sqrt{3}} = \frac{-\sqrt{3}}{15} \text{ m/sec (horizontal)}$$

$$\frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt} = \frac{-6\sqrt{3}}{6} \cdot \frac{(-\sqrt{3})}{15} = \frac{1}{5} \text{ m/sec (vertical)}.$$

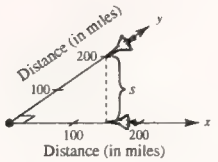
31. (a) $s^2 = x^2 + y^2$

$$\frac{dx}{dt} = -450$$

$$\frac{dy}{dt} = -600$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{x(dx/dt) + y(dy/dt)}{s}$$



When $x = 150$ and $y = 200$, $s = 250$ and

$$\frac{ds}{dt} = \frac{150(-450) + 200(-600)}{250} = -750 \text{ mph.}$$

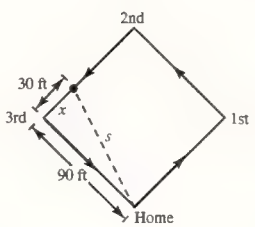
(b) $t = \dfrac{250}{750} = \dfrac{1}{3}\text{ hr} = 20 \text{ min}$

33. $s^2 = 90^2 + x^2$

$$x = 30$$

$$\frac{dx}{dt} = -28$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} \Rightarrow \frac{ds}{dt} = \frac{x}{s} \cdot \frac{dx}{dt}$$



When $x = 30$,

$$s = \sqrt{90^2 + 30^2} = 30\sqrt{10}$$

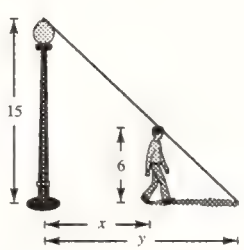
$$\frac{ds}{dt} = \frac{30}{30\sqrt{10}}(-28) = \frac{-28}{\sqrt{10}} \approx -8.85 \text{ ft/sec.}$$

35. (a) $\dfrac{15}{6} = \dfrac{y}{y - x} \Rightarrow 15y - 15x = 6y$

$$y = \frac{5}{3}x$$

$$\frac{dx}{dt} = 5$$

$$\frac{dy}{dt} = \frac{5}{3} \cdot \frac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3}\text{ ft/sec}$$



(b) $\dfrac{d(y - x)}{dt} = \dfrac{dy}{dt} - \dfrac{dx}{dt} = \dfrac{25}{3} - 5 = \dfrac{10}{3}\text{ ft/sec}$

37. $x(t) = \frac{1}{2}\sin\frac{\pi t}{6}$, $x^2 + y^2 = 1$

(a) Period: $\frac{2\pi}{\pi/6} = 12$ seconds

(b) When $x = \frac{1}{2}$, $y = \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$ m.

Lowest point: $\left(0, \frac{\sqrt{3}}{2}\right)$

(c) When $x = \frac{1}{4}$, $y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$ and $t = 1$

$$\frac{dx}{dt} = \frac{1}{2}\left(\frac{\pi}{6}\right)\cos\frac{\pi t}{6} = \frac{\pi}{12}\cos\frac{\pi t}{6}$$

$$x^2 + y^2 = 1$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}.$$

Thus,

$$\frac{dy}{dt} = -\frac{1/4}{\sqrt{15}/4} \cdot \frac{\pi}{12}\cos\left(\frac{\pi}{6}\right)$$

$$= \frac{-\pi}{\sqrt{15}}\left(\frac{1}{12}\right)\frac{\sqrt{3}}{2} = \frac{-\pi}{24}\frac{1}{\sqrt{5}} = \frac{-\sqrt{5}\pi}{120}.$$

$$\text{Speed} = \left|\frac{-\sqrt{5}\pi}{120}\right| = \frac{\sqrt{5}\pi}{120} \text{ m/sec}$$

39. Since the evaporation rate is proportional to the surface area, $dV/dt = k(4\pi r^2)$. However, since $V = (4/3)\pi r^3$, we have

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}.$$

Therefore,

$$k(4\pi r^2) = 4\pi r^2\frac{dr}{dt} \Rightarrow k = \frac{dr}{dt}.$$

41.

$$pV^{1.3} = k$$

$$1.3\,pV^{0.3}\frac{dV}{dt} + V^{1.3}\frac{dp}{dt} = 0$$

$$V^{0.3}\left(1.3p\frac{dV}{dt} + V\frac{dp}{dt}\right) = 0$$

$$1.3p\frac{dV}{dt} = -V\frac{dp}{dt}$$

43.

$$\tan\theta = \frac{y}{30}$$

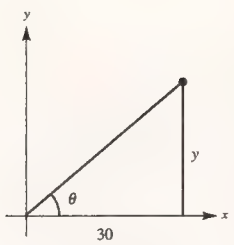
$$\frac{dy}{dt} = 3 \text{ m/sec.}$$

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{30}\frac{dy}{dt}$$

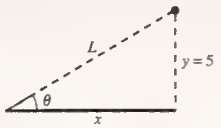
$$\frac{d\theta}{dt} = \frac{1}{30}\cos^2\theta \cdot \frac{dy}{dt}$$

When $y = 30$, $\theta = \pi/4$ and $\cos\theta = \sqrt{2}/2$. Thus,

$$\frac{d\theta}{dt} = \frac{1}{30}\left(\frac{1}{2}\right)(3) = \frac{1}{20} \text{ rad/sec.}$$



45. $\tan\theta = \dfrac{y}{x}, y = 5$



$$\frac{dx}{dt} = -600 \text{ mi/hr}$$

$$(\sec^2\theta)\frac{d\theta}{dt} = -\frac{5}{x^2}\cdot\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \cos^2\theta\left(-\frac{5}{x^2}\right)\frac{dx}{dt} = \frac{x^2}{L^2}\left(-\frac{5}{x^2}\right)\frac{dx}{dt}$$

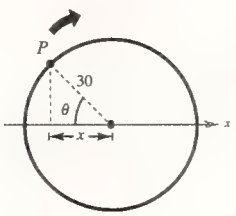
$$= \left(-\frac{5^2}{L^2}\right)\left(\frac{1}{5}\right)\frac{dx}{dt} = (-\sin^2\theta)\left(\frac{1}{5}\right)(-600) = 120\sin^2\theta$$

(a) When $\theta = 30°$, $\dfrac{d\theta}{dt} = \dfrac{120}{4} = 30 \text{ rad/hr} = \dfrac{1}{2}\text{ rad/min}.$

(b) When $\theta = 60°$, $\dfrac{d\theta}{dt} = 120\left(\dfrac{3}{4}\right) = 90 \text{ rad/hr} = \dfrac{3}{2}\text{ rad/min}.$

(c) When $\theta = 75°$, $\dfrac{d\theta}{dt} = 120\sin^2 75° \approx 111.96 \text{ rad/hr} \approx 1.87 \text{ rad/min}.$

47. $\dfrac{d\theta}{dt} = (10 \text{ rev/sec})(2\pi \text{ rad/rev}) = 20\pi \text{ rad/sec}$

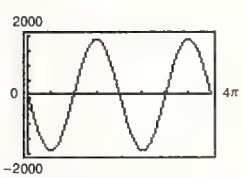


(a) $\cos\theta = \dfrac{x}{30}$

$$-\sin\theta\frac{d\theta}{dt} = \frac{1}{30}\frac{dx}{dt}$$

$$\frac{dx}{dt} = -30\sin\theta\frac{d\theta}{dt} = -30\sin\theta(20\pi) = -600\pi\sin\theta$$

(b)



(c) $|dx/dt| = |-600\pi\sin\theta|$ is greatest when $\sin\theta = 1 \implies \theta = (\pi/2) + n\pi$ (or $90° + n\cdot 180°$)

$|dx/dt|$ is least when $\theta = n\pi$ (or $n\cdot 180°$).

(d) For $\theta = 30°$, $\dfrac{dx}{dt} = -600\pi\sin(30°) = -600\pi\dfrac{1}{2} = -300\pi \text{ cm/sec}.$

For $\theta = 60°$, $\dfrac{dx}{dt} = -600\pi\sin(60°) = -600\pi\dfrac{\sqrt{3}}{2} = -300\sqrt{3}\,\pi \text{ cm/sec}$

49. $\tan\theta = \dfrac{x}{50} \Longrightarrow x = 50\tan\theta$

$$\frac{dx}{dt} = 50\sec^2\theta\,\frac{d\theta}{dt}$$

$$2 = 50\sec^2\theta\,\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{25}\cos^2\theta, -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$$

51. $x^2 + y^2 = 25$; acceleration of the top of the ladder $= \dfrac{d^2y}{dt^2}$

First derivative: $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

Second derivative: $x\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt}\cdot\dfrac{dx}{dt} + y\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt}\cdot\dfrac{dy}{dt} = 0$

$$\frac{d^2y}{dt^2} = \left(\frac{1}{y}\right)\left[-x\frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2\right]$$

When $x = 7$, $y = 24$, $\dfrac{dy}{dt} = -\dfrac{7}{12}$, and $\dfrac{dx}{dt} = 2$ (see Exercise 27). Since $\dfrac{dx}{dt}$ is constant, $\dfrac{d^2x}{dt^2} = 0$.

$$\frac{d^2y}{dt^2} = \frac{1}{24}\left[-7(0) - (2)^2 - \left(-\frac{7}{12}\right)^2\right] = \frac{1}{24}\left[-4 - \frac{49}{144}\right] = \frac{1}{24}\left[-\frac{625}{144}\right] \approx -0.1808 \text{ ft/sec}^2$$

53. (a) Using a graphing utility, you obtain $m(s) = -0.881s^2 + 29.10s - 206.2$

(b) $\dfrac{dm}{dt} = \dfrac{dm}{ds}\dfrac{ds}{dt} = (-1.762s + 29.10)\dfrac{ds}{dt}$

(c) If $t = s$ (1995), then $s = 15.5$ and $\dfrac{ds}{dt} = 1.2$.

Thus, $\dfrac{dm}{dt} = (-1.762(15.5) + 29.10)(1.2) \approx 2.15$ million.

Review Exercises for Chapter 2

1. $f(x) = x^2 - 2x + 3$

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{[(x+\Delta x)^2 - 2(x+\Delta x) + 3] - [x^2 - 2x + 3]}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{(x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) + 3) - (x^2 - 2x + 3)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{2x(\Delta x) + (\Delta x)^2 - 2(\Delta x)}{\Delta x} = \lim_{\Delta x\to 0}(2x + \Delta x - 2) = 2x - 2$$

3. $f(x) = \sqrt{x} + 1$

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{(\sqrt{x+\Delta x} + 1) - (\sqrt{x} + 1)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}\cdot\frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x\to 0}\frac{(x+\Delta x) - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})}$$

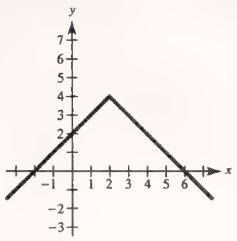
$$= \lim_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

5. f is differentiable for all $x \neq -1$.

7. $f(x) = 4 - |x - 2|$

(a) Continuous at $x = 2$.

(b) Not differentiable at $x = 2$ because of the sharp turn in the graph.



9. Using the limit definition, you obtain $g'(x) = \frac{4}{3}x - \frac{1}{6}$.

At $x = -1$, $g'(-1) = -\frac{4}{3} - \frac{1}{6} = \frac{-3}{2}$

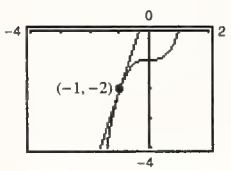
11. (a) Using the limit defintion, $f'(x) = 3x^2$.

At $x = -1$, $f'(-1) = 3$. The tangent line is

$y - (-2) = 3(x - (-1))$

$y = 3x + 1$

(b)



13. $g'(2) = \lim_{x \to 2} \frac{g(x) - g(2)}{x - 2}$

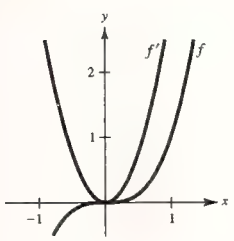
$= \lim_{x \to 2} \frac{x^2(x - 1) - 4}{x - 2}$

$= \lim_{x \to 2} \frac{x^3 - x^2 - 4}{x - 2}$

$= \lim_{x \to 2} \frac{(x - 2)(x^2 + x + 2)}{x - 2}$

$= \lim_{x \to 2} (x^2 + x + 2) = 8$

15.



17. $y = 25$

$y' = 0$

19. $f(x) = x^8$

$f'(x) = 8x^7$

21. $h(t) = 3t^4$

$h'(t) = 12t^3$

23. $f(x) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

25. $h(x) = 6\sqrt{x} + 3\sqrt[3]{x} = 6x^{1/2} + 3x^{1/3}$

$h'(x) = 3x^{-1/2} + x^{-2/3} = \frac{3}{\sqrt{x}} + \frac{1}{\sqrt[3]{x^2}}$

27. $g(t) = \frac{2}{3}t^{-2}$

$g'(x) = \frac{-4}{3}t^{-3} = \frac{-4}{3t^3}$

29. $f(\theta) = 2\theta - 3 \sin \theta$

$f'(\theta) = 2 - 3 \cos \theta$

31. $f(\theta) = 3 \cos \theta - \frac{\sin \theta}{4}$

$f'(\theta) = -3 \sin \theta - \frac{\cos \theta}{4}$

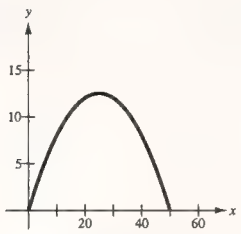
33. $F = 200\sqrt{T}$

$$F'(t) = \frac{100}{\sqrt{T}}$$

(a) When $T = 4$, $F'(4) = 50$ vibrations/sec/lb.

(b) When $T = 9$, $F'(9) = 33\frac{1}{3}$ vibrations/sec/lb.

37. (a)



Total horizontal distance: 50

(b) $0 = x - 0.02x^2$

$0 = x\left(1 - \frac{x}{50}\right)$ implies $x = 50$.

39. $x(t) = t^2 - 3t + 2 = (t - 2)(t - 1)$

(a) $v(t) = x'(t) = 2t - 3$

$a(t) = v'(t) = 2$

(c) $v(t) = 0$ for $t = \frac{3}{2}$.

$x = \left(\frac{3}{2} - 2\right)\left(\frac{3}{2} - 1\right) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$

41. $f(x) = (3x^2 + 7)(x^2 - 2x + 3)$

$f'(x) = (3x^2 + 7)(2x - 2) + (x^2 - 2x + 3)(6x)$

$= 2(6x^3 - 9x^2 + 16x - 7)$

45. $f(x) = 2x - x^{-2}$

$f'(x) = 2 + 2x^{-3} = 2\left(1 + \frac{1}{x^3}\right)$

$= \frac{2(x^3 + 1)}{x^3}$

49. $f(x) = (4 - 3x^2)^{-1}$

$f'(x) = -(4 - 3x^2)^{-2}(-6x) = \frac{6x}{(4 - 3x^2)^2}$

53. $y = 3x^2 \sec x$

$y' = 3x^2 \sec x \tan x + 6x \sec x$

35. $s(t) = -16t^2 + s_0$

$s(9.2) = -16(9.2)^2 + s_0 = 0$

$s_0 = 1354.24$

The building is approximately 1354 feet high (or 415 m).

(c) Ball reaches maximum height when $x = 25$.

(d) $y = x - 0.02x^2$

$y' = 1 - 0.04x$

$y'(0) = 1$

$y'(10) = 0.6$

$y'(25) = 0$

$y'(30) = -0.2$

$y'(50) = -1$

(e) $y'(25) = 0$

(b) $v(t) < 0$ for $t < \frac{3}{2}$.

(d) $x(t) = 0$ for $t = 1, 2$.

$|v(1)| = |2(1) - 3| = 1$

$|v(2)| = |2(2) - 3| = 1$

The speed is 1 when the position is 0.

43. $h(x) = \sqrt{x} \sin x = x^{1/2} \sin x$

$$h'(x) = \frac{1}{2\sqrt{x}} \sin x + \sqrt{x} \cos x$$

47. $f(x) = \frac{x^2 + x - 1}{x^2 - 1}$

$f'(x) = \frac{(x^2 - 1)(2x + 1) - (x^2 + x - 1)(2x)}{(x^2 - 1)^2}$

$= \frac{-(x^2 + 1)}{(x^2 - 1)^2}$

51. $y = \frac{x^2}{\cos x}$

$y' = \frac{\cos x\,(2x) - x^2(-\sin x)}{\cos^2 x} = \frac{2x \cos x + x^2 \sin x}{\cos^2 x}$

55. $y = -x \tan x$

$y' = -x \sec^2 x - \tan x$

57. $y = x\cos x - \sin x$

$y' = -x\sin x + \cos x - \cos x = -x\sin x$

59. $g(t) = t^3 - 3t + 2$

$g'(t) = 3t^2 - 3$

$g''(t) = 6t$

61. $f(\theta) = 3\tan\theta$

$f'(\theta) = 3\sec^2\theta$

$f''(\theta) = 6\sec\theta(\sec\theta\tan\theta) = 6\sec^2\theta\tan\theta$

63. $y = 2\sin x + 3\cos x$

$y' = 2\cos x - 3\sin x$

$y'' = -2\sin x - 3\cos x$

$y'' + y = -(2\sin x + 3\cos x) + (2\sin x + 3\cos x)$

$= 0$

65. $f(x) = (1 - x^3)^{1/2}$

$f'(x) = \frac{1}{2}(1 - x^3)^{-1/2}(-3x^2)$

$= -\frac{3x^2}{2\sqrt{1 - x^3}}$

67. $h(x) = \left(\frac{x - 3}{x^2 + 1}\right)^2$

$h'(x) = 2\left(\frac{x - 3}{x^2 + 1}\right)\left(\frac{(x^2 + 1)(1) - (x - 3)(2x)}{(x^2 + 1)^2}\right)$

$= \frac{2(x - 3)(-x^2 + 6x + 1)}{(x^2 + 1)^3}$

69. $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$

$f'(s) = (s^2 - 1)^{5/2}(3s^2) + (s^3 + 5)\left(\frac{5}{2}\right)(s^2 - 1)^{3/2}(2s)$

$= s(s^2 - 1)^{3/2}[3s(s^2 - 1) + 5(s^3 + 5)]$

$= s(s^2 - 1)^{3/2}(8s^3 - 3s + 25)$

71. $y = 3\cos(3x + 1)$

$y' = -9\sin(3x + 1)$

73. $y = \frac{1}{2}\csc 2x$

$y' = \frac{1}{2}(-\csc 2x\cot 2x)(2)$

$= -\csc 2x\cot 2x$

75. $y = \frac{x}{2} - \frac{\sin 2x}{4}$

$y' = \frac{1}{2} - \frac{1}{4}\cos 2x(2)$

$= \frac{1}{2}(1 - \cos 2x) = \sin^2 x$

77. $y = \frac{2}{3}\sin^{3/2}x - \frac{2}{7}\sin^{7/2}x$

$y' = \sin^{1/2}x\cos x - \sin^{5/2}x\cos x$

$= (\cos x)\sqrt{\sin x}(1 - \sin^2 x)$

$= (\cos^3 x)\sqrt{\sin x}$

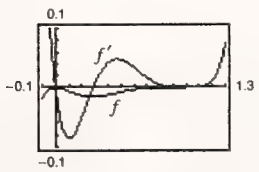
79. $y = \frac{\sin \pi x}{x + 2}$

$y' = \frac{(x + 2)\pi\cos \pi x - \sin \pi x}{(x + 2)^2}$

81. $f(t) = t^2(t - 1)^5$

$f'(t) = t(t - 1)^4(7t - 2)$

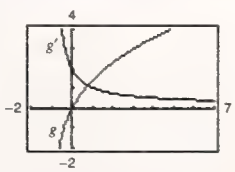
The zeros of f' correspond to the points on the graph of f where the tangent line is horizontal.



83. $g(x) = 2x(x + 1)^{-1/2}$

$g'(x) = \frac{x + 2}{(x + 1)^{3/2}}$

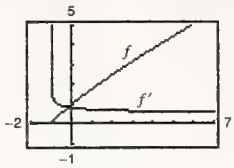
g' does not equal zero for any value of x in the domain. The graph of g has no horizontal tangent lines.



85. $f(t) = (t+1)^{1/2}(t+1)^{1/3} = (t+1)^{5/6}$

$$f'(t) = \frac{5}{6(t+1)^{1/6}}$$

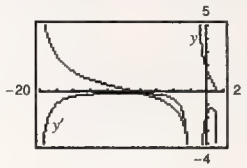
f' does not equal zero for any x in the domain. The graph of f has no horizontal tangent lines.



87. $y = \tan\sqrt{1-x}$

$$y' = -\frac{\sec^2\sqrt{1-x}}{2\sqrt{1-x}}$$

y' does not equal zero for any x in the domain. The graph has no horizontal tangent lines.



89. $y = 2x^2 + \sin 2x$

$y' = 4x + 2\cos 2x$

$y'' = 4 - 4\sin 2x$

91. $f(x) = \cot x$

$f'(x) = -\csc^2 x$

$f'' = -2\csc x(-\csc x \cdot \cot x)$

$= 2\csc^2 x \cot x$

93. $f(t) = \dfrac{t}{(1-t)^2}$

$$f'(t) = \frac{t+1}{(1-t)^3}$$

$$f''(t) = \frac{2(t+2)}{(1-t)^4}$$

95. $g(\theta) = \tan 3\theta - \sin(\theta - 1)$

$g'(\theta) = 3\sec^2 3\theta - \cos(\theta - 1)$

$g''(\theta) = 18\sec^2 3\theta \tan 3\theta + \sin(\theta - 1)$

97. $T = 700(t^2 + 4t + 10)^{-1}$

$$T' = \frac{-1400(t+2)}{(t^2+4t+10)^2}$$

(a) When $t = 1$,

$$T' = \frac{-1400(1+2)}{(1+4+10)^2} \approx -18.667 \text{ deg/hr.}$$

(b) When $t = 3$,

$$T' = \frac{-1400(3+2)}{(9+12+10)^2} \approx -7.284 \text{ deg/hr.}$$

(c) When $t = 5$,

$$T' = \frac{-1400(5+2)}{(25+30+10)^2} \approx -3.240 \text{ deg/hr.}$$

(d) When $t = 10$,

$$T' = \frac{-1400(10+2)}{(100+40+10)^2} \approx -0.747 \text{ deg/hr.}$$

99. $x^2 + 3xy + y^3 = 10$

$2x + 3xy' + 3y + 3y^2y' = 0$

$3(x + y^2)y' = -(2x + 3y)$

$$y' = \frac{-(2x+3y)}{3(x+y^2)}$$

101. $y\sqrt{x} - x\sqrt{y} = 16$

$$y\left(\frac{1}{2}x^{-1/2}\right) + x^{1/2}y' - x\left(\frac{1}{2}y^{-1/2}y'\right) - y^{1/2} = 0$$

$$\left(\sqrt{x} - \frac{x}{2\sqrt{y}}\right)y' = \sqrt{y} - \frac{y}{2\sqrt{x}}$$

$$\frac{2\sqrt{xy} - x}{2\sqrt{y}}y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}}$$

$$y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}} \cdot \frac{2\sqrt{y}}{2\sqrt{xy} - x} = \frac{2y\sqrt{x} - y\sqrt{y}}{2x\sqrt{y} - x\sqrt{x}}$$

103. $x \sin y = y \cos x$

$(x \cos y)y' + \sin y = -y \sin x + y' \cos x$

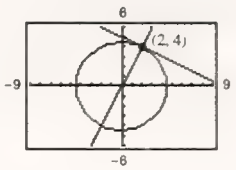
$y'(x \cos y - \cos x) = -y \sin x - \sin y$

$$y' = \frac{y \sin x + \sin y}{\cos x - x \cos y}$$

105. $x^2 + y^2 = 20$

$2x + 2yy' = 0$

$$y' = -\frac{x}{y}$$



At $(2, 4)$: $y' = -\frac{1}{2}$

Tangent line: $y - 4 = -\frac{1}{2}(x - 2)$

$x + 2y - 10 = 0$

Normal line: $y - 4 = 2(x - 2)$

$2x - y = 0$

107. $y = \sqrt{x}$

$\frac{dy}{dt} = 2$ units/sec

$$\frac{dy}{dt} = \frac{1}{2\sqrt{x}}\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 2\sqrt{x}\frac{dy}{dt} = 4\sqrt{x}$$

(a) When $x = \frac{1}{2}$, $\frac{dx}{dt} = 2\sqrt{2}$ units/sec.

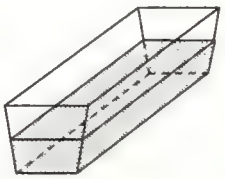
(b) When $x = 1$, $\frac{dx}{dt} = 4$ units/sec.

(c) When $x = 4$, $\frac{dx}{dt} = 8$ units/sec.

109. $\frac{s}{h} = \frac{1/2}{2}$

$$s = \frac{1}{4}h$$

$$\frac{dV}{dt} = 1$$



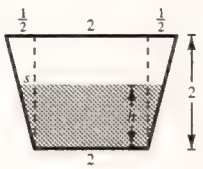
Width of water at depth h:

$$w = 2 + 2s = 2 + 2\left(\frac{1}{4}h\right) = \frac{4 + h}{2}$$

$$V = \frac{5}{2}\left(2 + \frac{4 + h}{2}\right)h = \frac{5}{4}(8 + h)h$$

$$\frac{dV}{dt} = \frac{5}{2}(4 + h)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2(dV/dt)}{5(4 + h)}$$



When $h = 1$, $\frac{dh}{dt} = \frac{2}{25}$ m/min.

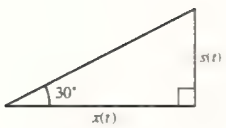
111. $s(t) = 60 - 4.9t^2$

$s'(t) = -9.8t$

$s = 35 = 60 - 4.9t^2$

$4.9t^2 = 25$

$$t = \frac{5}{\sqrt{4.9}}$$



$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{s(t)}{x(t)}$$

$x(t) = \sqrt{3}s(t)$

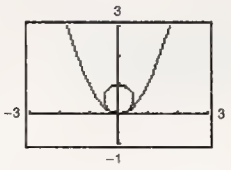
$$\frac{dx}{dt} = \sqrt{3}\frac{ds}{dt} = \sqrt{3}(-9.8)\frac{5}{\sqrt{4.9}}$$

≈ -38.34 m/sec

Problem Solving for Chapter 2

1. (a) $x^2 + (y - r)^2 = r^2$ Circle

$x^2 = y$ Parabola



Substituting,

$(y - r)^2 = r^2 - y$

$y^2 - 2ry + r^2 = r^2 - y$

$y^2 - 2ry + y = 0$

$y(y - 2r + 1) = 0$

Since you want only one solution, let $1 - 2r = 0 \Rightarrow r = \frac{1}{2}$

Graph $y = x^2$ and $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$

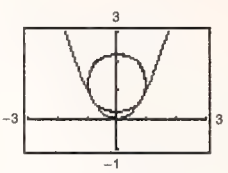
(b) Let (x, y) be a point of tangency: $x^2 + (y - b)^2 = 1 \Rightarrow 2x + 2(y - b)y' = 0 \Rightarrow y' = \dfrac{x}{b - y}$ (circle).

$y = x^2 \Rightarrow y' = 2x$ (parabola). Equating,

$$2x = \frac{x}{b - y}$$

$$2(b - y) = 1$$

$$b - y = \frac{1}{2} \Rightarrow b = y + \frac{1}{2}$$



Also, $x^2 + (y - b)^2 = 1$ and $y = x^2$ imply

$$y + (y - b)^2 = 1 \Rightarrow y + \left[y - \left(y + \frac{1}{2}\right)\right] = 1 \Rightarrow y - \frac{1}{2} = 1 \Rightarrow y = \frac{3}{4} \text{ and } b = \frac{5}{4}.$$

Center: $\left(0, \dfrac{5}{4}\right)$

Graph $y = x^2$ and $x^2 + \left(y - \dfrac{5}{4}\right)^2 = 1$

3. (a) $f(x) = \cos x$ $\quad P_1(x) = a_0 + a_1x$

$f(0) = 1$ $\quad P_1(0) = a_0 \Rightarrow a_0 = 1$

$f'(0) = 0$ $\quad P'_1(0) = a_1 \Rightarrow a_1 = 0$

$P_1(x) = 1$

(b) $f(x) = \cos x$ $\quad P_2(x) = a_0 + a_1x + a_2x^2$

$f(0) = 1$ $\quad P_2(0) = a_0 \Rightarrow a_0 = 1$

$f'(0) = 0$ $\quad P'_2(0) = a_1 \Rightarrow a_1 = 0$

$f''(0) = -1$ $\quad P''_2(0) = 2a_2 \Rightarrow a_2 = -\frac{1}{2}$

$P_2(x) = 1 - \frac{1}{2}x^2$

(c)

x	-1.0	-0.1	-0.001	0	0.001	0.1	1.0
$\cos x$	0.5403	0.9950	≈ 1	1	≈ 1	0.9950	0.5403
$P_2(x)$	0.5	0.9950	≈ 1	1	≈ 1	0.9950	0.5

$P_2(x)$ is a good approximation of $f(x) = \cos x$ when x is near 0.

(d) $f(x) = \sin x$ $\quad P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

$f(0) = 0$ $\quad P_3(0) = a_0 \Rightarrow a_0 = 0$

$f'(0) = 1$ $\quad P'_3(0) = a_1 \Rightarrow a_1 = 1$

$f''(0) = 0$ $\quad P''_3(0) = 2a_2 \Rightarrow a_2 = 0$

$f'''(0) = -1$ $\quad P'''_3(0) = 6a_3 \Rightarrow a_3 = -\frac{1}{6}$

$P_3(x) = x - \frac{1}{6}x^3$

5. Let $p(x) = Ax^3 + Bx^2 + Cx + D$

$p'(x) = 3Ax^2 + 2Bx + C$

At (1, 1): $A + B + C + D = 1$ Equation 1

$3A + 2B + C = 14$ Equation 2

At (−1, −3): $-A + B - C + D = -3$ Equation 3

$3A - 2B + C = -2$ Equation 4

Adding Equations 1 and 3: $2B + 2D = -2$

Subtracting Equations 1 and 3: $2A + 2C = 4$

Adding Equations 2 and 4: $6A + 2C = 12$

Subtracting Equations 2 and 4: $4B = 16$

Hence, $B = 4$ and $D = \frac{1}{2}(-2 - 2B) = -5$

Subtracting $2A + 2C = 4$ and $6A + 2C = 12$, you obtain $4A = 8 \Rightarrow A = 2$. Finally, $C = \frac{1}{2}(4 - 2A) = 0$

Thus, $p(x) = 2x^3 + 4x^2 - 5$.

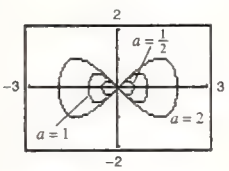
7. (a) $x^4 = a^2x^2 - a^2y^2$

$a^2y^2 = a^2x^2 - x^4$

$y = \dfrac{\pm\sqrt{a^2x^2 - x^4}}{a}$

Graph: $y_1 = \dfrac{\sqrt{a^2x^2 - x^4}}{a}$ and $y_2 = -\dfrac{\sqrt{a^2x^2 - x^4}}{a}$

(b)



$(\pm a, 0)$ are the x-intercepts, along with (0, 0).

(c) Differentiating implicitly,

$4x^3 = 2a^2x - 2a^2yy'$

$y' = \dfrac{2a^2x - 4x^3}{2a^2y} = \dfrac{x(a^2 - 2x^2)}{a^2y} = 0 \Rightarrow 2x^2 = a^2 \Rightarrow x = \dfrac{\pm a}{\sqrt{2}}.$

$\left(\dfrac{a^2}{2}\right)^2 = a^2\left(\dfrac{a^2}{2}\right) - a^2y^2$

$\dfrac{a^4}{4} = \dfrac{a^4}{2} - a^2y^2$

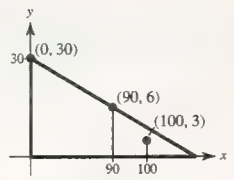
$a^2y^2 = \dfrac{a^4}{4}$

$y^2 = \dfrac{a^2}{4}$

$y = \pm\dfrac{a}{2}$

Four points: $\left(\dfrac{a}{\sqrt{2}}, \dfrac{a}{2}\right), \left(\dfrac{a}{\sqrt{2}}, -\dfrac{a}{2}\right), \left(-\dfrac{a}{\sqrt{2}}, \dfrac{a}{2}\right), \left(\dfrac{-a}{\sqrt{2}}, -\dfrac{a}{2}\right)$

9. (a)



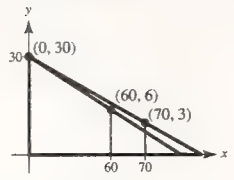
Not drawn to scale

Line determined by (0, 30) and (90, 6):

$$y - 30 = \frac{30 - 6}{0 - 90}(x - 0) = -\frac{24}{90}x = -\frac{4}{15}x \Rightarrow y = -\frac{4}{15}x + 30$$

When $x = 100$, $y = \frac{-4}{15}(100) + 30 = \frac{10}{3} > 3 \Rightarrow$ Shadow determined by man.

(b)



Not drawn to scale

Line determined by (0, 30) and (60, 6):

$$y - 30 = \frac{30 - 6}{0 - 60}(x - 0) = -\frac{2}{5}x \Rightarrow y = -\frac{2}{5}x + 30$$

When $x = 70$, $y = \frac{-2}{5}(70) + 30 = 2 < 3 \Rightarrow$ Shadow determined by child.

(c) Need (0, 30), $(d, 6)$, $(d + 10, 3)$ collinear.

$$\frac{30 - 6}{0 - d} = \frac{6 - 3}{d - (d + 10)} \Rightarrow \frac{24}{d} = \frac{3}{10} \Rightarrow d = 80 \text{ feet}$$

(d) Let y be the length of the street light to the tip of the shadow. We know that $\frac{dx}{dt} = -5$.

For $x > 80$, the shadow is determined by the man.

$$\frac{y}{30} = \frac{y - x}{6} \Rightarrow y = \frac{5}{4}x \text{ and } \frac{dy}{dt} = \frac{5}{4}\frac{dx}{dt} = \frac{-25}{4}.$$

For $x < 80$, the shadow is determined by the child.

$$\frac{y}{30} = \frac{y - x - 10}{3} \Rightarrow y = \frac{10}{9}x + \frac{100}{9} \text{ and } \frac{dy}{dt} = \frac{10}{9}\frac{dx}{dt} = \frac{-50}{9}.$$

Therefore,

$$\frac{dy}{dt} = \begin{cases} \frac{-25}{4} & x > 80 \\ \frac{-50}{9} & 0 < x < 80 \end{cases}$$

$\frac{dy}{dt}$ is not continuous at $x = 80$.

11. $L'(x) = \lim_{\Delta x \to 0} \frac{L(x + \Delta x) - L(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{L(x) + L(\Delta x) - L(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{L(\Delta x)}{\Delta x}$$

Also, $L'(0) = \lim_{\Delta x \to 0} \frac{L(\Delta x) - L(0)}{\Delta x}$

But, $L(0) = 0$ because $L(0) = L(0 + 0) = L(0) + L(0) \Rightarrow L(0) = 0$.

Thus, $L'(x) = L'(0)$, for all x.

The graph of L is a line through the origin of slope $L'(0)$.

13. (a)

z (degrees)	0.1	0.01	0.0001
$\dfrac{\sin z}{z}$	0.0174524	0.0174533	0.0174533

(b) $\lim_{z \to 0} \dfrac{\sin z}{z} \approx 0.0174533$

In fact, $\lim_{z \to 0} \dfrac{\sin z}{z} = \dfrac{\pi}{180}$

(c) $$\frac{d}{dz}(\sin z) = \lim_{\Delta z \to 0} \frac{\sin(z + \Delta z) - \sin z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\sin z \cdot \cos \Delta z + \sin \Delta z \cdot \cos z - \sin z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left[\sin z \left(\frac{\cos \Delta z - 1}{\Delta z} \right) \right] + \lim_{\Delta z \to 0} \left[\cos z \left(\frac{\sin \Delta z}{\Delta z} \right) \right]$$

$$= \sin z(0) + \cos z \left(\frac{\pi}{180} \right) = \frac{\pi}{180} \cos z$$

(d) $S(90) = \sin\left(\dfrac{\pi}{180} 90\right) = \sin \dfrac{\pi}{2} = 1$; $C(180) = \cos\left(\dfrac{\pi}{180} 180\right) = -1$

$$\frac{d}{dz} S(z) = \frac{d}{dz} \sin(cz) = c \cdot \cos(cz) = \frac{\pi}{180} C(z)$$

(e) The formulas for the derivatives are more complicated in degrees.

15. $j(t) = a'(t)$

(a) $j(t)$ is the rate of change of the acceleration.

(b) From Exercise 102 in Section 2.3,

$s(t) = -8.25t^2 + 66t$

$v(t) = -16.5t + 66$

$a(t) = -16.5$

$a'(t) = j(t) = 0$

CHAPTER 3
Applications of Differentiation

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CHAPTER 3
Applications of Differentiation

Section 3.1 Extrema on an Interval

Solutions to Odd-Numbered Exercises

1. $f(x) = \dfrac{x^2}{x^2 + 4}$

$f'(x) = \dfrac{(x^2 + 4)(2x) - (x^2)(2x)}{(x^2 + 4)^2} = \dfrac{8x}{(x^2 + 4)^2}$

$f'(0) = 0$

3. $f(x) = x + \dfrac{27}{2x^2} = x + \dfrac{27}{x}x^{-2}$

$f'(x) = 1 - 27x^{-3} = 1 - \dfrac{27}{x^3}$

$f'(3) = 1 - \dfrac{27}{3^3} = 1 - 1 = 0$

5. $f(x) = (x + 2)^{2/3}$

$f'(x) = \dfrac{2}{3}(x + 2)^{-1/3}$

$f'(-2)$ is undefined.

7. Critical numbers: $x = 2$

$x = 2$: absolute maximum

9. Critical numbers: $x = 1, 2, 3$

$x = 1, 3$: absolute maximum

$x = 2$: absolute minimum

11. $f(x) = x^2(x - 3) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

Critical numbers: $x = 0, x = 2$

13. $g(t) = t\sqrt{4 - t},\ t < 3$

$g'(t) = t\left[\dfrac{1}{2}(4 - t)^{-1/2}(-1)\right] + (4 - t)^{1/2}$

$= \dfrac{1}{2}(4 - t)^{-1/2}[-t + 2(4 - t)]$

$= \dfrac{8 - 3t}{2\sqrt{4 - t}}$

Critical number is $t = \dfrac{8}{3}$.

15. $h(x) = \sin^2 x + \cos x,\ 0 < x < 2\pi$

$h'(x) = 2 \sin x \cos x - \sin x = \sin x(2 \cos x - 1)$

On $(0, 2\pi)$, critical numbers: $x = \dfrac{\pi}{3}, x = \pi, x = \dfrac{5\pi}{3}$

17. $f(x) = 2(3 - x), [-1, 2]$

$f'(x) = -2 \Rightarrow$ No critical numbers

Left endpoint: $(-1, 8)$ Maximum

Right endpoint: $(2, 2)$ Minimum

19. $f(x) = -x^2 + 3x, [0, 3]$

$f'(x) = -2x + 3$

Left endpoint: $(0, 0)$ Minimum

Critical number: $\left(\frac{3}{2}, \frac{9}{4}\right)$ Maximum

Right endpoint: $(3, 0)$ Minimum

21. $f(x) = x^3 - \frac{3}{2}x^2, [-1, 2]$

$f'(x) = 3x^2 - 3x = 3x(x - 1)$

Left endpoint: $\left(-1, -\frac{5}{2}\right)$ Minimum

Right endpoint: $(2, 2)$ Maximum

Critical number: $(0, 0)$

Critical number: $\left(1, -\frac{1}{2}\right)$

23. $f(x) = 3x^{2/3} - 2x, [-1, 1]$

$f'(x) = 2x^{-1/3} - 2 = \frac{2\left(1 - \sqrt[3]{x}\right)}{\sqrt[3]{x}}$

Left endpoint: $(-1, 5)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $(1, 1)$

25. $g(t) = \frac{t^2}{t^2 + 3}, [-1, 1]$

$g'(t) = \frac{6t}{(t^2 + 3)^2}$

Left endpoint: $\left(-1, \frac{1}{4}\right)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $\left(1, \frac{1}{4}\right)$ Maximum

27. $h(s) = \frac{1}{s - 2}, [0, 1]$

$h'(s) = \frac{-1}{(s - 2)^2}$

Left endpoint: $\left(0, -\frac{1}{2}\right)$ Maximum

Right endpoint: $(1, -1)$ Minimum

29. $f(x) = \cos \pi x, \left[0, \frac{1}{6}\right]$

$f'(x) = -\pi \sin \pi x$

Left endpoint: $(0, 1)$ Maximum

Right endpoint: $\left(\frac{1}{6}, \frac{\sqrt{3}}{2}\right)$ Minimum

31. $y = \frac{4}{x} + \tan\frac{\pi x}{8}, [1, 2]$

$y' = \frac{-4}{x^2} + \frac{\pi}{8}\sec^2\frac{\pi x}{8} = 0$

$\frac{\pi}{8}\sec^2\frac{\pi x}{8} = \frac{4}{x^2}$

On the interval $[1, 2]$, this equation has no solutions. Thus, there are no critical numbers.

Left endpoint: $\left(1, \sqrt{2} + 3\right) \approx (1, 4.4142)$ Maximum

Right endpoint: $(2, 3)$ Minimum

33. (a) Minimum: $(0, -3)$

Maximum: $(2, 1)$

(b) Minimum: $(0, -3)$

(c) Maximum: $(2, 1)$

(d) No extrema

35. $f(x) = x^2 - 2x$

(a) Minimum: $(1, -1)$

Maximum: $(-1, 3)$

(b) Maximum: $(3, 3)$

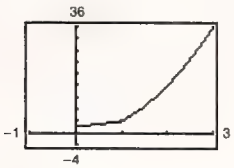
(c) Minimum: $(1, -1)$

(d) Minimum: $(1, -1)$

37. $f(x) = \begin{cases} 2x + 2, & 0 \le x \le 1 \\ 4x^2, & 1 < x \le 3 \end{cases}$

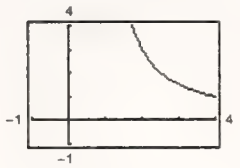
Left endpoint: (0, 2) Minimum

Right endpoint: (3, 36) Maximum

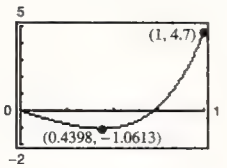


39. $f(x) = \dfrac{3}{x-1}, (1, 4]$

Right endpoint: (4, 1) Minimum



41. (a)



Maximum: (1, 4.7) (endpoint)

Minimum: (0.4398, −1.0613)

(b) $f(x) = 3.2x^5 + 5x^3 - 3.5x, [0, 1]$

$$f'(x) = 16x^4 + 15x^2 - 3.5$$

$$16x^4 + 15x^2 - 3.5 = 0$$

$$x^2 = \frac{-15 \pm \sqrt{(15)^2 - 4(16)(-3.5)}}{2(16)}$$

$$= \frac{-15 \pm \sqrt{449}}{32}$$

$$x = \sqrt{\frac{-15 + \sqrt{449}}{32}} \approx 0.4398$$

$$f(0) = 0$$

$$f(1) = 4.7 \text{ Maximum (endpoint)}$$

$$f\left(\sqrt{\frac{-15 + \sqrt{449}}{32}}\right) \approx -1.0613$$

Minimum: (0.4398, −1.0613)

43. $f(x) = (1 + x^3)^{1/2}, [0, 2]$

$$f'(x) = \frac{3}{2}x^2(1 + x^3)^{-1/2}$$

$$f''(x) = \frac{3}{4}(x^4 + 4x)(1 + x^3)^{-3/2}$$

$$f'''(x) = -\frac{3}{8}(x^6 + 20x^3 - 8)(1 + x^3)^{-5/2}$$

Setting $f''' = 0$, we have $x^6 + 20x^3 - 8 = 0$.

$$x^3 = \frac{-20 \pm \sqrt{400 - 4(1)(-8)}}{2}$$

$$x = \sqrt[3]{-10 \pm \sqrt{108}} = \sqrt{3} - 1$$

In the interval [0, 2], choose

$$x = \sqrt[3]{-10 \pm \sqrt{108}} = \sqrt{3} - 1 \approx 0.732.$$

$\left|f''\left(\sqrt[3]{-10 + \sqrt{108}}\right)\right| \approx 1.47$ is the maximum value.

45. $f(x) = (x + 1)^{2/3}, [0, 2]$

$$f'(x) = \frac{2}{3}(x + 1)^{-1/3}$$

$$f''(x) = -\frac{2}{9}(x + 1)^{-4/3}$$

$$f'''(x) = \frac{8}{27}(x + 1)^{-7/3}$$

$$f^{(4)}(x) = -\frac{56}{81}(x + 1)^{-10/3}$$

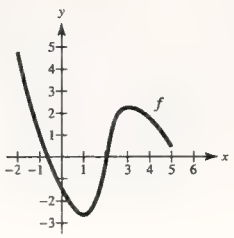
$$f^{(5)}(x) = \frac{560}{243}(x + 1)^{-13/3}$$

$|f^{(4)}(0)| = \dfrac{56}{81}$ is the maximum value.

47. $f(x) = \tan x$

f is continuous on $[0, \pi/4]$ but not on $[0, \pi]$. $\lim_{x \to \pi/2^-} \tan x = \infty$.

49.



51. (a) Yes

(b) No

53. (a) No

(b) Yes

55. $P = VI - RI^2 = 12I - 0.5I^2, 0 \le I \le 15$

$P = 0$ when $I = 0$.

$P = 67.5$ when $I = 15$.

$P' = 12 - I = 0$

Critical number: $I = 12$ amps

When $I = 12$ amps, $P = 72$, the maximum output.

No, a 20-amp fuse would not increase the power output. P is decreasing for $I > 12$.

57. $S = 6hs + \frac{3s^2}{2}\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right), \frac{\pi}{6} \le \theta \le \frac{\pi}{2}$

$$\frac{dS}{d\theta} = \frac{3s^2}{2}(-\sqrt{3}\csc\theta\cot\theta + \csc^2\theta)$$

$$= \frac{3s^2}{2}\csc\theta(-\sqrt{3}\cot\theta + \csc\theta) = 0$$

$$\csc\theta = \sqrt{3}\cot\theta$$

$$\sec\theta = \sqrt{3}$$

$$\theta = \operatorname{arcsec}\sqrt{3} \approx 0.9553 \text{ radians}$$

$$S\left(\frac{\pi}{6}\right) = 6hs + \frac{3s^2}{2}(\sqrt{3})$$

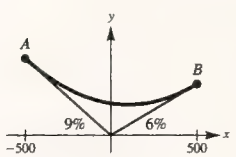
$$S\left(\frac{\pi}{2}\right) = 6hs + \frac{3s^2}{2}(\sqrt{3})$$

$$S(\operatorname{arcsec}\sqrt{3}) = 6hs + \frac{3s^2}{2}(\sqrt{2})$$

S is minimum when $\theta = \operatorname{arcsec}\sqrt{3} \approx 0.9553$ radians.

59. (a) $y = ax^2 + bx + c$

$y' = 2ax + b$



The coordinates of B are $(500, 30)$, and those of A are $(-500, 45)$. From the slopes at A and B,

$$-1000a + b = -0.09$$

$$1000a + b = 0.06.$$

Solving these two equations, you obtain $a = 3/40000$ and $b = -3/200$. From the points $(500, 30)$ and $(-500, 45)$, you obtain

$$30 = \frac{3}{40000}500^2 + 500\left(\frac{-3}{200}\right) + c$$

$$45 = \frac{3}{40000}500^2 - 500\left(\frac{-3}{200}\right) + c.$$

In both cases, $c = 18.75 = \frac{75}{4}$. Thus,

$$y = \frac{3}{40000}x^2 - \frac{3}{200}x + \frac{75}{4}.$$

—CONTINUED—

59. —CONTINUED—

(b)

x	-500	-400	-300	-200	-100	0	100	200	300	400	500
d	0	.75	3	6.75	12	18.75	12	6.75	3	.75	0

For $-500 \le x \le 0$, $d = (ax^2 + bx + c) - (-0.09x)$.

For $0 \le x \le 500$, $d = (ax^2 + bx + c) - (0.06x)$.

(c) The lowest point on the highway is (100, 18), which is not directly over the point where the two hillsides come together.

61. True. See Exercise 25.

63. True.

Section 3.2 Rolle's Theorem and the Mean Value Theorem

1. Rolle's Theorem does not apply to $f(x) = 1 - |x - 1|$ over $[0, 2]$ since f is not differentiable at $x = 1$.

3. $f(x) = x^2 - x - 2 = (x - 2)(x + 1)$

x-intercepts: $(-1, 0), (2, 0)$

$f'(x) = 2x - 1 = 0$ at $x = \frac{1}{2}$.

5. $f(x) = x\sqrt{x + 4}$

x-intercepts: $(-4, 0), (0, 0)$

$$f'(x) = x\frac{1}{2}(x + 4)^{-1/2} + (x + 4)^{1/2}$$

$$= (x + 4)^{-1/2}\left(\frac{x}{2} + (x + 4)\right)$$

$$f'(x) = \left(\frac{3}{2}x + 4\right)(x + 4)^{-1/2} = 0 \text{ at } x = -\frac{8}{3}$$

7. $f(x) = x^2 - 2x, [0, 2]$

$f(0) = f(2) = 0$

f is continuous on $[0, 2]$. f is differentiable on $(0, 2)$. Rolle's Theorem applies.

$$f'(x) = 2x - 2$$

$$2x - 2 = 0 \Rightarrow x = 1$$

c value: 1

9. $f(x) = (x - 1)(x - 2)(x - 3), [1, 3]$

$f(1) = f(3) = 0$

f is continuous on $[1, 3]$. f is differentiable on $(1, 3)$. Rolle's Theorem applies.

$$f(x) = x^3 - 6x^2 + 11x - 6$$

$$f'(x) = 3x^2 - 12x + 11$$

$$3x^2 - 12x + 11 = 0 \Rightarrow x = \frac{6 \pm \sqrt{3}}{3}$$

$$c = \frac{6 - \sqrt{3}}{3}, c = \frac{6 + \sqrt{3}}{3}$$

11. $f(x) = x^{2/3} - 1, [-8, 8]$

$f(-8) = f(8) = 3$

f is continuous on $[-8, 8]$. f is not differentiable on $(-8, 8)$ since $f'(0)$ does not exist. Rolle's Theorem does not apply.

13. $f(x) = \dfrac{x^2 - 2x - 3}{x + 2}$, $[-1, 3]$

$f(-1) = f(3) = 0$

f is continuous on $[-1, 3]$. **(Note:** The discontinuity, $x = -2$, is not in the interval.) f is differentiable on $(-1, 3)$. Rolle's Theorem applies.

$$f'(x) = \frac{(x + 2)(2x - 2) - (x^2 - 2x - 3)(1)}{(x + 2)^2} = 0$$

$$\frac{x^2 + 4x - 1}{(x + 2)^2} = 0$$

$$x = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$$

c value: $-2+\sqrt{5}$

15. $f(x) = \sin x$, $[0, 2\pi]$

$f(0) = f(2\pi) = 0$

f is continuous on $[0, 2\pi]$. f is differentiable on $(0, 2\pi)$. Rolle's Theorem applies.

$f'(x) = \cos x$

c values: $\dfrac{\pi}{2}, \dfrac{3\pi}{2}$

17. $f(x) = \dfrac{6x}{\pi} - 4\sin^2 x$, $\left[0, \dfrac{\pi}{6}\right]$

$f(0) = f\left(\dfrac{\pi}{6}\right) = 0$

f is continuous on $[0, \pi/6]$. f is differentiable on $(0, \pi/6)$. Rolle's Theorem applies.

$$f'(x) = \frac{6}{\pi} - 8\sin x \cos x = 0$$

$$\frac{6}{\pi} = 8\sin x \cos x$$

$$\frac{3}{4\pi} = \frac{1}{2}\sin 2x$$

$$\frac{3}{2\pi} = \sin 2x$$

$$\frac{1}{2}\arcsin\left(\frac{3}{2\pi}\right) = x$$

$$x \approx 0.2489$$

c value: 0.2489

19. $f(x) = \tan x$, $[0, \pi]$

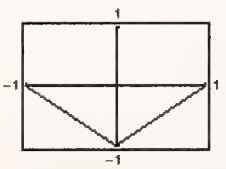
$f(0) = f(\pi) = 0$

f is not continuous on $[0, \pi]$ since $f(\pi/2)$ does not exist. Rolle's Theorem does not apply.

21. $f(x) = |x| - 1$, $[-1, 1]$

$f(-1) = f(1) = 0$

f is continuous on $[-1, 1]$. f is not differentiable on $(-1, 1)$ since $f'(0)$ does not exist. Rolle's Theorem does not apply.



23. $f(x) = 4x - \tan \pi x, \left[-\frac{1}{4}, \frac{1}{4}\right]$

$f\left(-\frac{1}{4}\right) = f\left(\frac{1}{4}\right) = 0$

f is continuous on $[-1/4, 1/4]$. f is differentiable on $(-1/4, 1/4)$. Rolle's Theorem applies.

$$f'(x) = 4 - \pi \sec^2 \pi x = 0$$

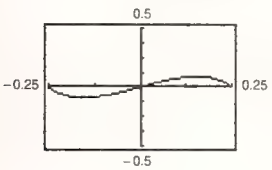
$$\sec^2 \pi x = \frac{4}{\pi}$$

$$\sec \pi x = \pm\frac{2}{\sqrt{\pi}}$$

$$x = \pm\frac{1}{\pi} \operatorname{arcsec} \frac{2}{\sqrt{\pi}} = \pm\frac{1}{\pi} \arccos \frac{\sqrt{\pi}}{2}$$

$$\approx \pm 0.1533 \text{ radian}$$

c values: ± 0.1533 radian



25. $f(t) = -16t^2 + 48t + 32$

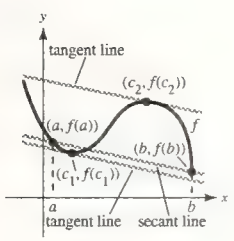
(a) $f(1) = f(2) = 64$

(b) $v = f'(t)$ must be 0 at some time in $(1, 2)$.

$$f'(t) = -32t + 48 = 0$$

$$t = \frac{3}{2} \text{ seconds}$$

27.



29. $f(x) = \frac{1}{x - 3}, [0, 6]$

f has a discontinuity at $x = 3$.

31. $f(x) = x^2$ is continuous on $[-2, 1]$ and differentiable on $(-2, 1)$.

$$\frac{f(1) - f(-2)}{1 - (-2)} = \frac{1 - 4}{3} = -1$$

$f'(x) = 2x = -1$ when $x = -\frac{1}{2}$. Therefore,

$$c = -\frac{1}{2}.$$

33. $f(x) = x^{2/3}$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

$$\frac{f(1) - f(0)}{1 - 0} = 1$$

$$f'(x) = \frac{2}{3}x^{-1/3} = 1$$

$$x = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

$$c = \frac{8}{27}$$

35. $f(x) = \sqrt{2 - x}$ is continuous on $[-7, 2]$ and differentiable on $(-7, 2)$.

$$\frac{f(2) - f(-7)}{2 - (-7)} = \frac{0 - 3}{9} = -\frac{1}{3}$$

$$f'(x) = \frac{-1}{2\sqrt{2 - x}} = -\frac{1}{3}$$

$$2\sqrt{2 - x} = 3$$

$$\sqrt{2 - x} = \frac{3}{2}$$

$$2 - x = \frac{9}{4}$$

$$x = -\frac{1}{4}$$

$$c = -\frac{1}{4}$$

37. $f(x) = \sin x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

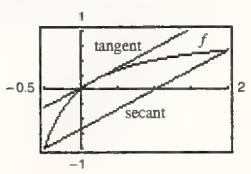
$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

$$f'(x) = \cos x = 0$$

$$c = \frac{\pi}{2}$$

39. $f(x) = \dfrac{x}{x + 1}$ on $\left[-\dfrac{1}{2}, 2\right]$.

(a)



(b) Secant line:

$$\text{slope} = \frac{f(2) - f(-1/2)}{2 - (-1/2)} = \frac{2/3 - (-1)}{5/2} = \frac{2}{3}$$

$$y - \frac{2}{3} = \frac{2}{3}(x - 2)$$

$$3y - 2 = 2x - 4$$

$$3y - 2x + 2 = 0$$

(c) $f'(x) = \dfrac{1}{(x + 1)^2} = \dfrac{2}{3}$

$$(x + 1)^2 = \frac{3}{2}$$

$$x = -1 \pm \sqrt{\frac{3}{2}} = -1 \pm \frac{\sqrt{6}}{2}$$

In the interval $[-1/2, 2]$, $c = -1 + (\sqrt{6}/2)$.

$$f(c) = \frac{-1 + (\sqrt{6}/2)}{[-1 + (\sqrt{6}/2)] + 1} = \frac{-2 + \sqrt{6}}{\sqrt{6}} = \frac{-2}{\sqrt{6}} + 1$$

Tangent line: $y - 1 + \dfrac{2}{\sqrt{6}} = \dfrac{2}{3}\left(x - \dfrac{\sqrt{6}}{2} + 1\right)$

$$y - 1 + \frac{\sqrt{6}}{3} = \frac{2}{3}x - \frac{\sqrt{6}}{3} + \frac{2}{3}$$

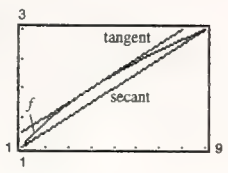
$$3y - 2x - 5 + 2\sqrt{6} = 0$$

41. $f(x) = \sqrt{x}$, $[1, 9]$

$(1, 1), (9, 3)$

$$m = \frac{3 - 1}{9 - 1} = \frac{1}{4}$$

(a)



(b) Secant line: $y - 1 = \frac{1}{4}(x - 1)$

$$y = \frac{1}{4}x + \frac{3}{4}$$

$$0 = x - 4y + 3$$

(c) $$f'(x) = \frac{1}{2\sqrt{x}}$$

$$\frac{f(9) - f(1)}{9 - 1} = \frac{1}{4}$$

$$\frac{1}{2\sqrt{c}} = \frac{1}{4}$$

$$\sqrt{c} = 2$$

$$c = 4$$

$$(c, f(c)) = (4, 2)$$

$$m = f'(4) = \frac{1}{4}$$

Tangent line: $y - 2 = \frac{1}{4}(x - 4)$

$$y = \frac{1}{4}x + 1$$

$$0 = x - 4y + 4$$

43. $s(t) = -4.9t^2 + 500$

(a) $V_{\text{avg}} = \dfrac{s(3) - s(0)}{3 - 0} = \dfrac{455.9 - 500}{3} = -14.7$ m/sec

(b) $s(t)$ is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Therefore, the Mean Value Theorem applies.

$$v(t) = s'(t) = -9.8t = -14.7 \text{ m/sec}$$

$$t = \frac{-14.7}{-9.8} = 1.5 \text{ seconds}$$

45. No. Let $f(x) = x^2$ on $[-1, 2]$.

$$f'(x) = 2x$$

$f'(0) = 0$ and zero is in the interval $(-1, 2)$ but $f(-1) \neq f(2)$.

47. Let $S(t)$ be the position function of the plane. If $t = 0$ corresponds to 2 P.M., $S(0) = 0$, $S(5.5) = 2500$ and the Mean Value Theorem says that there exists a time t_0, $0 < t_0 < 5.5$, such that

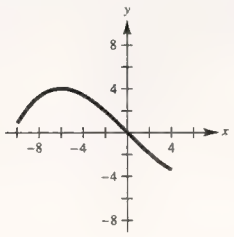
$$S'(t_0) = v(t_0) = \frac{2500 - 0}{5.5 - 0} \approx 454.54.$$

Applying the Intermediate Value Theorem to the velocity function on the intervals $[0, t_0]$ and $[t_0, 5.5]$, you see that there are at least two times during the flight when the speed was 400 miles per hour. $(0 < 400 < 454.54)$

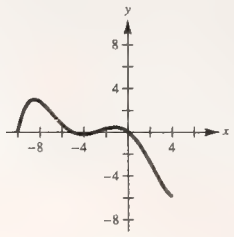
49. (a) f is continuous on $[-10, 4]$ and changes sign, $(f(-8) > 0, f(3) < 0)$. By the Intermediate Value Theorem, there exists at least one value of x in $[-10, 4]$ satisfying $f(x) = 0$.

(b) There exist real numbers a and b such that $-10 < a < b < 4$ and $f(a) = f(b) = 2$. Therefore, by Rolle's Theorem there exists at least one number c in $(-10, 4)$ such that $f'(c) = 0$. This is called a critical number.

(c)

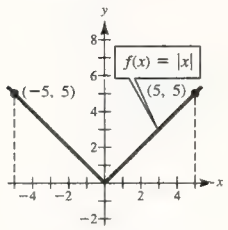


(d)



(e) No, f' did not have to be continuous on $[-10, 4]$.

51. f is continuous on $[-5, 5]$ and does not satisfy the conditions of the Mean Value Theorem.
$\Rightarrow f$ is not differentiable on $(-5, 5)$.
Example: $f(x) = |x|$



53. False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

55. True. A polynomial is continuous and differentiable everywhere.

57. Suppose that $p(x) = x^{2n+1} + ax + b$ has two real roots x_1 and x_2. Then by Rolle's Theorem, since $p(x_1) = p(x_2) = 0$, there exists c in (x_1, x_2) such that $p'(c) = 0$. But $p'(x) = (2n + 1)x^{2n} + a \neq 0$, since $n > 0, a > 0$. Therefore, $p(x)$ cannot have two real roots.

59. If $p(x) = Ax^2 + Bx + C$, then

$$p'(x) = 2Ax + B = \frac{f(b) - f(a)}{b - a} = \frac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b - a}$$

$$= \frac{A(b^2 - a^2) + B(b - a)}{b - a}$$

$$= \frac{(b - a)[A(b + a) + B]}{b - a}$$

$$= A(b + a) + B.$$

Thus, $2Ax = A(b + a)$ and $x = (b + a)/2$ which is the midpoint of $[a, b]$.

61. $f(x) = \frac{1}{2}\cos x$ differentiable on $(-\infty, \infty)$.

$f'(x) = -\frac{1}{2}\sin x$

$-\frac{1}{2} \leq f'(x) \leq \frac{1}{2} \Rightarrow f'(x) < 1$ for all real numbers.

Thus, from Exercise 60, f has, at most, one fixed point. $(x \approx 0.4502)$

Section 3.3 Increasing and Decreasing Functions and the First Derivative Test

1. $f(x) = x^2 - 6x + 8$

Increasing on: $(3, \infty)$

Decreasing on: $(-\infty, 3)$

3. $y = \frac{x^3}{4} - 3x$

Increasing on: $(-\infty, -2), (2, \infty)$

Decreasing on: $(-2, 2)$

5. $f(x) = \frac{1}{x^2} = x^{-2}$

$f'(x) = \frac{-2}{x^3}$

Discontinuity: $x = 0$

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on $(-\infty, 0)$

Decreasing on $(0, \infty)$

7. $g(x) = x^2 - 2x - 8$

$g'(x) = 2x - 2$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $g'(x)$:	$g' < 0$	$g' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(1, \infty)$

Decreasing on: $(-\infty, 1)$

9. $y = x\sqrt{16 - x^2}$ Domain: $[-4, 4]$

$y' = \frac{-2(x^2 - 8)}{\sqrt{16 - x^2}} = \frac{-2}{\sqrt{16 - x^2}}(x - 2\sqrt{2})(x + 2\sqrt{2})$

Critical numbers: $x = \pm 2\sqrt{2}$

Test intervals:	$-4 < x < -2\sqrt{2}$	$-2\sqrt{2} < x < 2\sqrt{2}$	$2\sqrt{2} < x < 4$
Sign of y':	$y' < 0$	$y' > 0$	$y' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on $(-2\sqrt{2}, 2\sqrt{2})$

Decreasing on $(-4, -2\sqrt{2}), (2\sqrt{2}, 4)$

11. $f(x) = x^2 - 6x$

$f'(x) = 2x - 6 = 0$

Critical number: $x = 3$

Test intervals:	$-\infty < x < 3$	$3 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(3, \infty)$

Decreasing on: $(-\infty, 3)$

Relative minimum: $(3, -9)$

13. $f(x) = -2x^2 + 4x + 3$

$f'(x) = -4x + 4 = 0$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, 1)$

Decreasing on: $(1, \infty)$

Relative maximum: $(1, 5)$

15. $f(x) = 2x^3 + 3x^2 - 12x$

$f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1) = 0$

Critical numbers: $x = -2, 1$

Test intervals:	$-\infty < x < -2$	$-2 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -2), (1, \infty)$

Decreasing on: $(-2, 1)$

Relative maximum: $(-2, 20)$

Relative minimum: $(1, -7)$

17. $f(x) = x^2(3 - x) = 3x^2 - x^3$

$f'(x) = 6x - 3x^2 = 3x(2 - x)$

Critical numbers: $x = 0, 2$

Test intervals:	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(0, 2)$

Decreasing on: $(-\infty, 0), (2, \infty)$

Relative maximum: $(2, 4)$

Relative minimum: $(0, 0)$

19. $f(x) = \dfrac{x^5 - 5x}{5}$

$f'(x) = x^4 - 1$

Critical numbers: $x = -1, 1$

Test intervals:	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -1), (1, \infty)$

Decreasing on: $(-1, 1)$

Relative maximum: $\left(-1, \frac{4}{5}\right)$

Relative minimum: $\left(1, -\frac{4}{5}\right)$

21. $f(x) = x^{1/3} + 1$

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

Critical number: $x = 0$

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, \infty)$

No relative extrema

23. $f(x) = (x - 1)^{2/3}$

$f'(x) = \frac{2}{3(x - 1)^{1/3}}$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(1, \infty)$

Decreasing on: $(-\infty, 1)$

Relative minimum: $(1, 0)$

25. $f(x) = 5 - |x - 5|$

$f'(x) = -\frac{x - 5}{|x - 5|} = \begin{cases} 1, & x < 5 \\ -1, & x > 5 \end{cases}$

Critical number: $x = 5$

Test intervals:	$-\infty < x < 5$	$5 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, 5)$

Decreasing on: $(5, \infty)$

Relative maximum: $(5, 5)$

27. $f(x) = x + \frac{1}{x}$

$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$

Critical numbers: $x = -1, 1$

Discontinuity: $x = 0$

Test intervals:	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on: $(-\infty, -1), (1, \infty)$

Decreasing on: $(-1, 0), (0, 1)$

Relative maximum: $(-1, -2)$

Relative minimum: $(1, 2)$

29. $f(x) = \dfrac{x^2}{x^2 - 9}$

$$f'(x) = \frac{(x^2 - 9)(2x) - (x^2)(2x)}{(x^2 - 9)^2} = \frac{-18x}{(x^2 - 9)^2}$$

Critical number: $x = 0$

Discontinuities: $x = -3, 3$

Test intervals:	$-\infty < x < -3$	$-3 < x < 0$	$0 < x < 3$	$3 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$	$f' < 0$	$f' < 0$
Conclusion:	Increasing	Increasing	Decreasing	Decreasing

Increasing on: $(-\infty, -3), (-3, 0)$

Decreasing on: $(0, 3), (3, \infty)$

Relative maximum: $(0, 0)$

31. $f(x) = \dfrac{x^2 - 2x + 1}{x + 1}$

$$f'(x) = \frac{(x + 1)(2x - 2) - (x^2 - 2x + 1)(1)}{(x + 1)^2} = \frac{x^2 + 2x - 3}{(x + 1)^2} = \frac{(x + 3)(x - 1)}{(x + 1)^2}$$

Critical numbers: $x = -3, 1$

Discontinuity: $x = -1$

Test intervals:	$-\infty < x < -3$	$-3 < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on: $(-\infty, -3), (1, \infty)$

Decreasing on: $(-3, -1), (-1, 1)$

Relative maximum: $(-3, -8)$

Relative minimum: $(1, 0)$

33. $f(x) = \dfrac{x}{2} + \cos x, 0 < x < 2\pi$

$$f'(x) = \frac{1}{2} - \sin x = 0$$

Critical numbers: $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

Test intervals:	$0 < x < \dfrac{\pi}{6}$	$\dfrac{\pi}{6} < x < \dfrac{5\pi}{6}$	$\dfrac{5\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \dfrac{\pi}{6}\right), \left(\dfrac{5\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{6}, \dfrac{5\pi}{6}\right)$

Relative maximum: $\left(\dfrac{\pi}{6}, \dfrac{\pi + 6\sqrt{3}}{12}\right)$

Relative minimum: $\left(\dfrac{5\pi}{6}, \dfrac{5\pi - 6\sqrt{3}}{12}\right)$

35. $f(x) = \sin^2 x + \sin x, 0 < x < 2\pi$

$f'(x) = 2 \sin x \cos x + \cos x = \cos x(2 \sin x + 1) = 0$

Critical numbers: $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{7\pi}{6}$	$\frac{7\pi}{6} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < \frac{11\pi}{6}$	$\frac{11\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \frac{\pi}{2}\right), \left(\frac{7\pi}{6}, \frac{3\pi}{2}\right), \left(\frac{11\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\frac{\pi}{2}, \frac{7\pi}{6}\right), \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)$

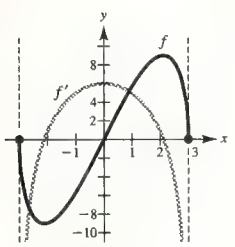
Relative minima: $\left(\frac{7\pi}{6}, -\frac{1}{4}\right), \left(\frac{11\pi}{6}, -\frac{1}{4}\right)$

Relative maxima: $\left(\frac{\pi}{2}, 2\right), \left(\frac{3\pi}{2}, 0\right)$

37. $f(x) = 2x\sqrt{9 - x^2}, [-3, 3]$

(a) $f'(x) = \frac{2(9 - 2x^2)}{\sqrt{9 - x^2}}$

(b)



(c) $\frac{2(9 - 2x^2)}{\sqrt{9 - x^2}} = 0$

Critical numbers: $x = \pm\frac{3}{\sqrt{2}} = \pm\frac{3\sqrt{2}}{2}$

(d) Intervals:

$\left(-3, -\frac{3\sqrt{2}}{2}\right)$	$\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$	$\left(\frac{3\sqrt{2}}{2}, 3\right)$
$f'(x) < 0$	$f'(x) > 0$	$f'(x) < 0$
Decreasing	Increasing	Decreasing

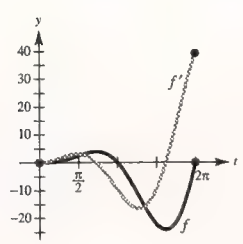
f is increasing when f' is positive and decreasing when f' is negative.

39. $f(t) = t^2 \sin t, [0, 2\pi]$

(a) $f'(t) = t^2 \cos t + 2t \sin t$

$= t(t \cos t + 2 \sin t)$

(b)



(c) $t(t \cos t + 2 \sin t) = 0$

$t = 0$ or $t = -2 \tan t$

$t \cot t = -2$

$t \approx 2.2889, 5.0870$ (graphing utility)

Critical numbers: $t = 2.2889, t = 5.0870$

(d) Intervals:

$(0, 2.2889)$	$(2.2889, 5.0870)$	$(5.0870, 2\pi)$
$f'(t) > 0$	$f'(t) < 0$	$f'(t) > 0$
Increasing	Decreasing	Increasing

f is increasing when f' is positive and decreasing when f' is negative.

41. $f(x) = \dfrac{x^5 - 4x^3 + 3x}{x^2 - 1} = \dfrac{(x^2 - 1)(x^3 - 3x)}{x^2 - 1} = x^3 - 3x,\ x \neq \pm 1$

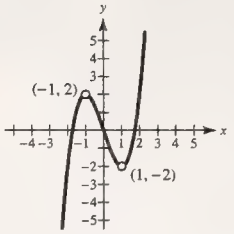
$f(x) = g(x) = x^3 - 3x$ for all $x \neq \pm 1$.

$f'(x) = 3x^2 - 3 = 3(x^2 - 1), x \neq \pm 1 \quad f'(x) \neq 0$

f symmetric about origin

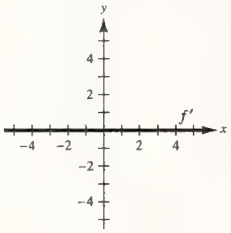
zeros of f: $(0, 0), (\pm\sqrt{3}, 0)$

No relative extrema

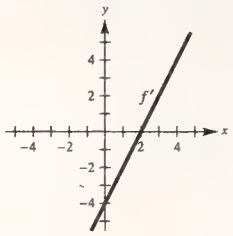


Holes at $(-1, 2)$ and $(1, -2)$

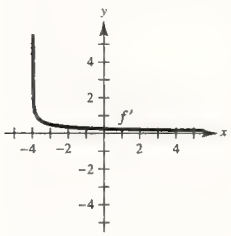
43. $f(x) = c$ is constant $\Rightarrow f'(x) = 0$



45. f is quadratic $\Rightarrow f'$ is a line.



47. f has positive, but decreasing slope



In Exercises 49–53, $f'(x) > 0$ on $(-\infty, -4)$, $f'(x) < 0$ on $(-4, 6)$ and $f'(x) > 0$ on $(6, \infty)$.

49. $g(x) = f(x) + 5$

$g'(x) = f'(x)$

$g'(0) = f'(0) < 0$

51. $g(x) = -f(x)$

$g'(x) = -f'(x)$

$g'(-6) = -f'(-6) < 0$

53. $g(x) = f(x - 10)$

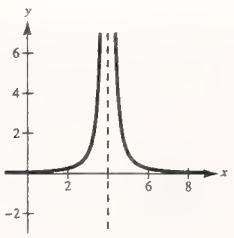
$g'(x) = f'(x - 10)$

$g'(0) = f'(-10) > 0$

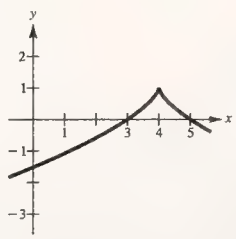
55. $f'(x) = \begin{cases} > 0, & x < 4 \Rightarrow f \text{ is increasing on } (-\infty, 4). \\ \text{undefined}, & x = 4 \\ < 0, & x > 4 \Rightarrow f \text{ is decreasing on } (4, \infty). \end{cases}$

Two possibilities for $f(x)$ are given below.

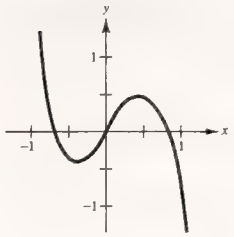
(a)



(b)



57. The critical numbers are in intervals $(-0.50, -0.25)$ and $(0.25, 0.50)$ since the sign of f' changes in these intervals. f is decreasing on approximately $(-1, -0.40)$, $(0.48, 1)$, and increasing on $(-0.40, 0.48)$.



Relative minimum when $x \approx -0.40$.

Relative maximum when $x \approx 0.48$.

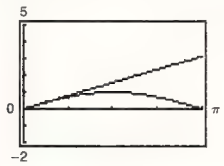
59. $f(x) = x$, $g(x) = \sin x$, $0 < x < \pi$

(a)

x	0.5	1	1.5	2	2.5	3
$f(x)$	0.5	1	1.5	2	2.5	3
$g(x)$	0.479	0.841	0.997	0.909	0.598	0.141

$f(x)$ seems greater than $g(x)$ on $(0, \pi)$.

(b)



$x > \sin x$ on $(0, \pi)$

(c) Let $h(x) = f(x) - g(x) = x - \sin x$

$h'(x) = 1 - \cos x > 0$ on $(0, \pi)$.

Therefore, $h(x)$ is increasing on $(0, \pi)$. Since $h(0) = 0$, $h(x) > 0$ on $(0, \pi)$. Thus,

$$x - \sin x > 0$$

$$x > \sin x$$

$$f(x) > g(x) \text{ on } (0, \pi).$$

61. $v = k(R - r)r^2 = k(Rr^2 - r^3)$

$v' = k(2Rr - 3r^2)$

$= kr(2R - 3r) = 0$

$r = 0$ or $\frac{2}{3}R$

Maximum when $r = \frac{2}{3}R$.

63. $P = \dfrac{vR_1R_2}{(R_1 + R_2)^2}$, v and R_1 are constant

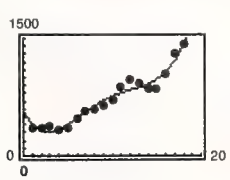
$$\frac{dP}{dR_2} = \frac{(R_1 + R_2)^2(vR_1) - vR_1R_2[2(R_1 + R_2)(1)]}{(R_1 + R_2)^4}$$

$$= \frac{vR_1(R_1 - R_2)}{(R_1 + R_2)^3} = 0 \Rightarrow R_2 = R_1$$

Maximum when $R_1 = R_2$.

65. (a) $B = 0.1198t^4 - 4.4879t^3 + 56.9909t^2 - 223.0222t + 579.9541$

(b)



(c) $B' = 0$ for $t \approx 2.78$, or 1983, (311.1 thousand bankruptcies)
Actual minimum: 1984 (344.3 thousand bankruptcies)

67. (a) Use a cubic polynomial
$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.

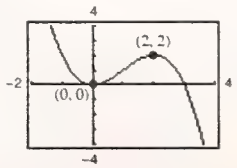
(b) $f'(x) = 3a_3x^2 + 2a_2x + a_1$.

$(0, 0)$:	$0 = a_0$	$(f(0) = 0)$
	$0 = a_1$	$(f'(0) = 0)$
$(2, 2)$:	$2 = 8a_3 + 4a_2$	$(f(2) = 2)$
	$0 = 12a_3 + 4a_2$	$(f'(2) = 0)$

(c) The solution is $a_0 = a_1 = 0$, $a_2 = \frac{3}{2}$, $a_3 = -\frac{1}{2}$:

$$f(x) = -\frac{1}{2}x^3 + \frac{3}{2}x^2.$$

(d)



69. (a) Use a fourth degree polynomial $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

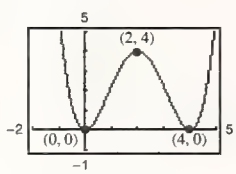
(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

$$\begin{aligned}
(0, 0): \quad & 0 = a_0 && (f(0) = 0)\\
& 0 = a_1 && (f'(0) = 0)\\
(4, 0): \quad & 0 = 256a_4 + 64a_3 + 16a_2 && (f(4) = 0)\\
& 0 = 256a_4 + 48a_3 + 8a_2 && (f'(4) = 0)\\
(2, 4): \quad & 4 = 16a_4 + 8a_3 + 4a_2 && (f(2) = 4)\\
& 0 = 32a_4 + 12a_3 + 4a_2 && (f'(2) = 0)
\end{aligned}$$

(c) The solution is $a_0 = a_1 = 0$, $a_2 = 4$, $a_3 = -2$, $a_4 = \frac{1}{4}$.

$$f(x) = \frac{1}{4}x^4 - 2x^3 + 4x^2$$

(d)



71. True

Let $h(x) = f(x) + g(x)$ where f and g are increasing. Then $h'(x) = f'(x) + g'(x) > 0$ since $f'(x) > 0$ and $g'(x) > 0$.

73. False

Let $f(x) = x^3$, then $f'(x) = 3x^2$ and f only has one critical number. Or, let $f(x) = x^3 + 3x + 1$, then $f'(x) = 3(x^2 + 1)$ has no critical numbers.

75. False. For example, $f(x) = x^3$ does not have a relative extrema at the critical number $x = 0$.

77. Assume that $f'(x) < 0$ for all x in the interval (a, b) and let $x_1 < x_2$ be any two points in the interval. By the Mean Value Theorem, we know there exists a number c such that $x_1 < c < x_2$, and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(c) < 0$ and $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) < 0$, which implies that $f(x_2) < f(x_1)$. Thus, f is decreasing on the interval.

79. Let $f(x) = (1 + x)^n - nx - 1$. Then

$$\begin{aligned}
f'(x) &= n(1 + x)^{n-1} - n\\
&= n[(1 + x)^{n-1} - 1] > 0 \text{ since } x > 0 \text{ and } n > 1.
\end{aligned}$$

Thus, $f(x)$ is increasing on $(0, \infty)$. Since $f(0) = 0 \Rightarrow f(x) > 0$ on $(0, \infty)$

$$(1 + x)^n - nx - 1 > 0 \Rightarrow (1 + x)^n > 1 + nx.$$

Section 3.4 Concavity and the Second Derivative Test

1. $y = x^2 - x - 2, y'' = 2$

Concave upward: $(-\infty, \infty)$

3. $f(x) = \dfrac{24}{x^2 + 12}, y'' = \dfrac{-144(4 - x^2)}{(x^2 + 12)^3}$

Concave upward: $(-\infty, -2), (2, \infty)$

Concave downward: $(-2, 2)$

5. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}, y'' = \dfrac{4(3x^2 + 1)}{(x^2 - 1)^3}$

Concave upward: $(-\infty, -1), (1, \infty)$

Concave downward: $(-1, 1)$

7. $f(x) = 3x^2 - x^3$

$f'(x) = 6x - 3x^2$

$f''(x) = 6 - 6x$

Concave upward: $(-\infty, 1)$

Concave downward: $(1, \infty)$

9. $y = 2x - \tan x, \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$y' = 2 - \sec^2 x$

$y'' = -2 \sec^2 x \tan x$

Concave upward: $\left(-\dfrac{\pi}{2}, 0\right)$

Concave downward: $\left(0, \dfrac{\pi}{2}\right)$

11. $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12$

$f''(x) = 6(x - 2) = 0$ when $x = 2$.

The concavity changes at $x = 2$. $(2, 8)$ is a point of inflection.

Concave upward: $(2, \infty)$

Concave downward: $(-\infty, 2)$

13. $f(x) = \dfrac{1}{4}x^4 - 2x^2$

$f'(x) = x^3 - 4x$

$f''(x) = 3x^2 - 4$

$f''(x) = 3x^2 - 4 = 0$ when $x = \pm\dfrac{2}{\sqrt{3}}$.

Test interval:	$-\infty < x < -\dfrac{2}{\sqrt{3}}$	$-\dfrac{2}{\sqrt{3}} < x < \dfrac{2}{\sqrt{3}}$	$\dfrac{2}{\sqrt{3}} < x < \infty$
Sign of $f''(x)$:	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Points of inflection: $\left(\pm\dfrac{2}{\sqrt{3}}, -\dfrac{20}{9}\right)$

15. $f(x) = x(x-4)^3$

$f'(x) = x[3(x-4)^2] + (x-4)^3$

$= (x-4)^2(4x-4)$

$f''(x) = 4(x-1)[2(x-4)] + 4(x-4)^2$

$= 4(x-4)[2(x-1) + (x-4)]$

$= 4(x-4)(3x-6) = 12(x-4)(x-2)$

$f''(x) = 12(x-4)(x-2) = 0$ when $x = 2, 4$.

Test interval:	$-\infty < x < 2$	$2 < x < 4$	$4 < x < \infty$
Sign of $f''(x)$:	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Points of inflection: $(2, -16)$, $(4, 0)$

17. $f(x) = x\sqrt{x+3}$, Domain: $[-3, \infty)$

$$f'(x) = x\left(\frac{1}{2}\right)(x+3)^{-1/2} + \sqrt{x+3} = \frac{3(x+2)}{2\sqrt{x+3}}$$

$$f''(x) = \frac{6\sqrt{x+3} - 3(x+2)(x+3)^{-1/2}}{4(x+3)} = \frac{3(x+4)}{4(x+3)^{3/2}}$$

$f''(x) > 0$ on the entire domain of f (except for $x = -3$, for which $f''(x)$ is undefined). There are no points of inflection. Concave upward on $(-3, \infty)$

19. $f(x) = \dfrac{x}{x^2+1}$

$$f'(x) = \frac{1-x^2}{(x^2+1)^2}$$

$$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3} = 0 \text{ when } x = 0, \pm\sqrt{3}$$

Test intervals:	$-\infty < x < -\sqrt{3}$	$-\sqrt{3} < x < 0$	$0 < x < \sqrt{3}$	$\sqrt{3} < x < \infty$
Sign of $f'(x)$:	$f'' < 0$	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward	Concave downward	Concave upward

Points of inflection: $\left(-\sqrt{3}, -\dfrac{\sqrt{3}}{4}\right)$, $(0, 0)$, $\left(\sqrt{3}, \dfrac{\sqrt{3}}{4}\right)$

21. $f(x) = \sin\left(\dfrac{x}{2}\right)$, $0 \le x \le 4\pi$

$$f'(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right)$$

$$f''(x) = -\frac{1}{4}\sin\left(\frac{x}{2}\right)$$

$f''(x) = 0$ when $x = 0, 2\pi, 4\pi$.

Point of inflection: $(2\pi, 0)$

Test interval:	$0 < x < 2\pi$	$2\pi < x < 4\pi$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward

23. $f(x) = \sec\left(x - \frac{\pi}{2}\right), 0 < x < 4\pi$

$f'(x) = \sec\left(x - \frac{\pi}{2}\right)\tan\left(x - \frac{\pi}{2}\right)$

$f''(x) = \sec^3\left(x - \frac{\pi}{2}\right) + \sec\left(x - \frac{\pi}{2}\right)\tan^2\left(x - \frac{\pi}{2}\right) \neq 0$ for any x in the domain of f.

Concave upward: $(0, \pi)$, $(2\pi, 3\pi)$

Concave downward: $(\pi, 2\pi)$, $(3\pi, 4\pi)$

No points of inflection

25. $f(x) = 2\sin x + \sin 2x, 0 \leq x \leq 2\pi$

$f'(x) = 2\cos x + 2\cos 2x$

$f''(x) = -2\sin x - 4\sin 2x = -2\sin x(1 + 4\cos x)$

$f''(x) = 0$ when $x = 0, 1.823, \pi, 4.460$.

Test interval:	$0 < x < 1.823$	$1.823 < x < \pi$	$\pi < x < 4.460$	$4.460 < x < 2\pi$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward	Concave downward	Concave upward

Points of inflection: $(1.823, 1.452)$, $(\pi, 0)$, $(4.46, -1.452)$

27. $f(x) = x^4 - 4x^3 + 2$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$

$f''(x) = 12x^2 - 24x = 12x(x - 2)$

Critical numbers: $x = 0, x = 3$

However, $f''(0) = 0$, so we must use the First Derivative Test. $f'(x) < 0$ on the intervals $(-\infty, 0)$ and $(0, 3)$; hence, $(0, 2)$ is not an extremum. $f''(3) > 0$ so $(3, -25)$ is a relative minimum.

29. $f(x) = (x - 5)^2$

$f'(x) = 2(x - 5)$

$f''(x) = 2$

Critical number: $x = 5$

$f''(5) > 0$

Therefore, $(5, 0)$ is a relative minimum.

31. $f(x) = x^3 - 3x^2 + 3$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

$f''(x) = 6x - 6 = 6(x - 1)$

Critical numbers: $x = 0, x = 2$

$f''(0) = -6 < 0$

Therefore, $(0, 3)$ is a relative maximum.

$f''(2) = 6 > 0$

Therefore, $(2, -1)$ is a relative minimum.

33. $g(x) = x^2(6 - x)^3$

$g'(x) = x(x - 6)^2(12 - 5x)$

$g''(x) = 4(6 - x)(5x^2 - 24x + 18)$

Critical numbers: $x = 0, \frac{12}{5}, 6$

$g''(0) = 432 > 0$

Therefore, $(0, 0)$ is a relative minimum.

$g''\left(\frac{12}{5}\right) = -155.52 < 0$

Therefore, $\left(\frac{12}{5}, 268.7\right)$ is a relative minimum.

$g''(6) = 0$

Test fails by the First Derivative Test, $(6, 0)$ is not an extremum.

35. $f(x) = x^{2/3} - 3$

$$f'(x) = \frac{2}{3x^{1/3}}$$

$$f''(x) = \frac{-2}{9x^{4/3}}$$

Critical number: $x = 0$

However, $f''(0)$ is undefined, so we must use the First Derivative Test. Since $f'(x) < 0$ on $(-\infty, 0)$ and $f'(x) > 0$ on $(0, \infty)$, $(0, -3)$ is a relative minimum.

37. $f(x) = x + \dfrac{4}{x}$

$$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2}$$

$$f''(x) = \frac{8}{x^3}$$

Critical numbers: $x = \pm 2$

$f''(-2) < 0$

Therefore, $(-2, -4)$ is a relative maximum.

$f''(2) > 0$

Therefore, $(2, 4)$ is a relative minimum.

39. $f(x) = \cos x - x, \ 0 \le x \le 4\pi$

$f'(x) = -\sin x - 1 \le 0$

Therefore, f is non-increasing and there are no relative extrema.

41. $f(x) = 0.2x^2(x - 3)^3, [-1, 4]$

(a) $f'(x) = 0.2x(5x - 6)(x - 3)^2$

$f''(x) = (x - 3)(4x^2 - 9.6x + 3.6)$

$= 0.4(x - 3)(10x^2 - 24x + 9)$

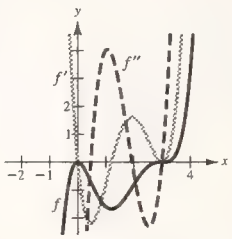
(b) $f''(0) < 0 \Rightarrow (0, 0)$ is a relative maximum.

$f''\left(\frac{6}{5}\right) > 0 \Rightarrow (1.2, -1.6796)$ is a relative minimum.

Points of inflection:

$(3, 0), (0.4652, -0.7049), (1.9348, -0.9049)$

(c)



f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

43. $f(x) = \sin x - \dfrac{1}{3}\sin 3x + \dfrac{1}{5}\sin 5x, [0, \pi]$

(a) $f'(x) = \cos x - \cos 3x + \cos 5x$

$f'(x) = 0$ when $x = \dfrac{\pi}{6}, x = \dfrac{\pi}{2}, x = \dfrac{5\pi}{6}$.

$f''(x) = -\sin x + 3\sin 3x - 5\sin 5x$

$f''(x) = 0$ when $x = \dfrac{\pi}{6}, x = \dfrac{5\pi}{6}, x \approx 1.1731, x \approx 1.9685$

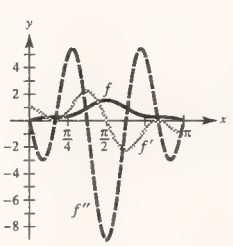
(b) $f''\left(\dfrac{\pi}{2}\right) < 0 \Rightarrow \left(\dfrac{\pi}{2}, 1.53333\right)$ is a relative maximum.

Points of inflection: $\left(\dfrac{\pi}{6}, 0.2667\right), (1.1731, 0.9638),$

$(1.9685, 0.9637), \left(\dfrac{5\pi}{6}, 0.2667\right)$

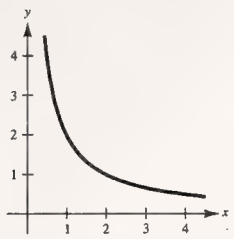
Note: $(0, 0)$ and $(\pi, 0)$ are not points of inflection since they are endpoints.

(c)



The graph of f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

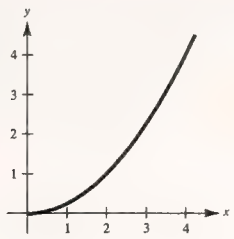
45. (a)



$f' < 0$ means f decreasing

f' increasing means concave upward

(b)



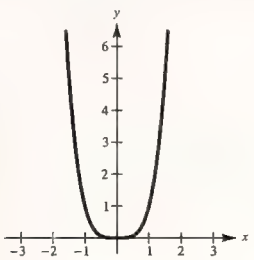
$f' > 0$ means f increasing

f' increasing means concave upward

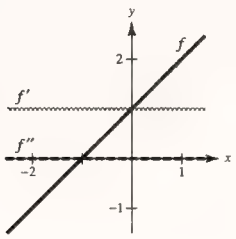
47. Let $f(x) = x^4$.

$f''(x) = 12x^2$

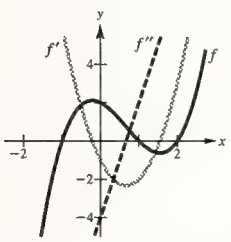
$f''(0) = 0$, but $(0, 0)$ is not a point of inflection.



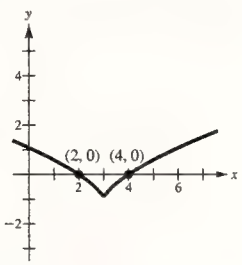
49.



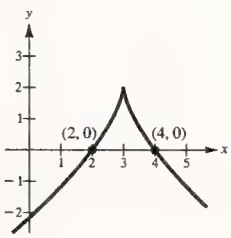
51.



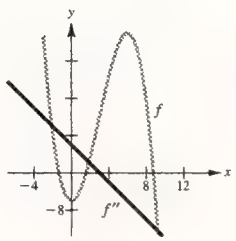
53.



55.



57.



f'' is linear.

f' is quadratic.

f is cubic.

f concave upwards on $(-\infty, 3)$, downward on $(3, \infty)$.

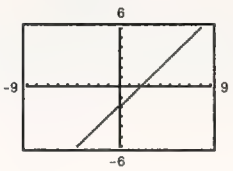
59. (a) $n = 1$:

$f(x) = x - 2$

$f'(x) = 1$

$f''(x) = 0$

No inflection points



$n = 2$:

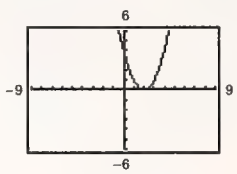
$f(x) = (x - 2)^2$

$f'(x) = 2(x - 2)$

$f''(x) = 2$

No inflection points

Relative minimum: (2, 0)



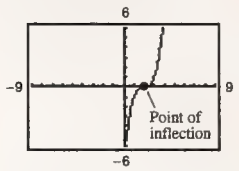
$n = 3$:

$f(x) = (x - 2)^3$

$f'(x) = 3(x - 2)^2$

$f''(x) = 6(x - 2)$

Inflection point: (2, 0)



$n = 4$:

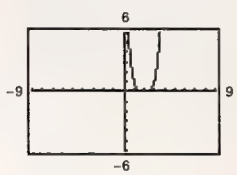
$f(x) = (x - 2)^4$

$f'(x) = 4(x - 2)^3$

$f''(x) = 12(x - 2)^2$

No inflection points:

Relative minimum: (2, 0)



Conclusion: If $n \geq 3$ and n is odd, then (2, 0) is an inflection point. If $n \geq 2$ and n is even, then (2, 0) is a relative minimum.

(b) Let $f(x) = (x - 2)^n$, $f'(x) = n(x - 2)^{n-1}$, $f''(x) = n(n - 1)(x - 2)^{n-2}$.

For $n \geq 3$ and odd, $n - 2$ is also odd and the concavity changes at $x = 2$.

For $n \geq 4$ and even, $n - 2$ is also even and the concavity does not change at $x = 2$.

Thus, $x = 2$ is an inflection point if and only if $n \geq 3$ is odd.

61. $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: (3, 3)

Relative minimum: (5, 1)

Point of inflection: (4, 2)

$f'(x) = 3ax^2 + 2bx + c, f''(x) = 6ax + 2b$

$$\left.\begin{aligned} f(3) &= 27a + 9b + 3c + d = 3 \\ f(5) &= 125a + 25b + 5c + d = 1 \end{aligned}\right\} 98a + 16b + 2c = -2 \Rightarrow 49a + 8b + c = -1$$

$f'(3) = 27a + 6b + c = 0$, $f''(4) = 24a + 2b = 0$

$$\begin{array}{rrrr} 49a + 8b + c = -1 & & 24a + 2b = 0 & \\ \underline{27a + 6b + c = 0} & & \underline{22a + 2b = -1} & \\ 22a + 2b = -1 & & 2a = 1 & \end{array}$$

$a = \frac{1}{2}, b = -6, c = \frac{45}{2}, d = -24$

$f(x) = \frac{1}{2}x^3 - 6x^2 + \frac{45}{2}x - 24$

63. $f(x) = ax^3 + bx^2 + cx + d$

Maximum: $(-4, 1)$

Minimum: $(0, 0)$

(a) $f'(x) = 3ax^2 + 2bx + c, \quad f''(x) = 6ax + 2b$

$f(0) = 0 \Rightarrow d = 0$

$f(-4) = 1 \Rightarrow -64a + 16b - 4c = 1$

$f'(-4) = 0 \Rightarrow 48a - 8b + c = 0$

$f'(0) = 0 \Rightarrow c = 0$

Solving this system yields $a = \frac{1}{32}$ and $b = 6a = \frac{3}{16}$.

$f(x) = \frac{1}{32}x^3 + \frac{3}{16}x^2$

(b) The plane would be descending at the greatest rate at the point of inflection.

$$f''(x) = 6ax + 2b = \frac{3}{16}x + \frac{3}{8} = 0 \Rightarrow x = -2.$$

Two miles from touchdown.

65. $D = 2x^4 - 5Lx^3 + 3L^2x^2$

$D' = 8x^3 - 15Lx^2 + 6L^2x = x(8x^2 - 15Lx + 6L^2) = 0$

$$x = 0 \text{ or } x = \frac{15L \pm \sqrt{33}L}{16} = \left(\frac{15 \pm \sqrt{33}}{16}\right)L$$

By the Second Derivative Test, the deflection is maximum when

$$x = \left(\frac{15 - \sqrt{33}}{16}\right)L \approx 0.578L.$$

67. $C = 0.5x^2 + 15x + 5000$

$$\overline{C} = \frac{C}{x} = 0.5x + 15 + \frac{5000}{x}$$

$\overline{C}$ = average cost per unit

$$\frac{d\overline{C}}{dx} = 0.5 - \frac{5000}{x^2} = 0 \text{ when } x = 100$$

By the First Derivative Test, $\overline{C}$ is minimized when $x = 100$ units.

69. $$S = \frac{5000t^2}{8 + t^2}$$

$$S'(t) = \frac{80{,}000t}{(8 + t^2)^2}$$

$$S''(t) = \frac{80{,}000(8 - 3t^2)}{(8 + t^2)^3}$$

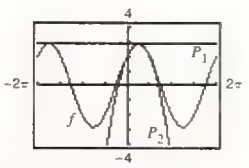
$S''(t) = 0$ for $t = \sqrt{8/3} \approx 1.633$.

Sales are increasing at the greatest rate at $t = 1.633$ years.

71. $f(x) = 2(\sin x + \cos x), \quad f\left(\frac{\pi}{4}\right) = 2\sqrt{2}$

$f'(x) = 2(\cos x - \sin x), \quad f'\left(\frac{\pi}{4}\right) = 0$

$f''(x) = 2(-\sin x - \cos x), \quad f''\left(\frac{\pi}{4}\right) = -2\sqrt{2}$



$P_1(x) = 2\sqrt{2} + 0\left(x - \frac{\pi}{4}\right) = 2\sqrt{2}$

$P_1'(x) = 0$

$P_2(x) = 2\sqrt{2} + 0\left(x - \frac{\pi}{4}\right) + \frac{1}{2}\left(-2\sqrt{2}\right)\left(x - \frac{\pi}{4}\right)^2 = 2\sqrt{2} - \sqrt{2}\left(x - \frac{\pi}{4}\right)^2$

$P_2'(x) = -2\sqrt{2}\left(x - \frac{\pi}{4}\right)$

$P_2''(x) = -2\sqrt{2}$

The values of f, P_1, P_2, and their first derivatives are equal at $x = \pi/4$. The values of the second derivatives of f and P_2 are equal at $x = \pi/4$. The approximations worsen as you move away from $x = \pi/4$.

73. $f(x) = \sqrt{1-x}$, $\quad f(0) = 1$

$$f'(x) = -\frac{1}{2\sqrt{1-x}}, \qquad f'(0) = -\frac{1}{2}$$

$$f''(x) = -\frac{1}{4(1-x)^{3/2}}, \qquad f''(0) = -\frac{1}{4}$$

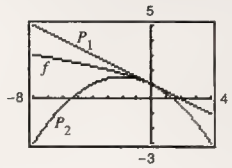
$$P_1(x) = 1 + \left(-\frac{1}{2}\right)(x-0) = 1 - \frac{x}{2}$$

$$P_1'(x) = -\frac{1}{2}$$

$$P_2(x) = 1 + \left(-\frac{1}{2}\right)(x-0) + \frac{1}{2}\left(-\frac{1}{4}\right)(x-0)^2 = 1 - \frac{x}{2} - \frac{x^2}{8}$$

$$P_2'(x) = -\frac{1}{2} - \frac{x}{4}$$

$$P_2''(x) = -\frac{1}{4}$$



The values of f, P_1, P_2, and their first derivatives are equal at $x = 0$. The values of the second derivatives of f and P_2 are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

75. $f(x) = x\sin\left(\frac{1}{x}\right)$

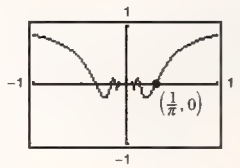
$$f'(x) = x\left[-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)\right] + \sin\left(\frac{1}{x}\right) = -\frac{1}{x}\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$$

$$f''(x) = -\frac{1}{x}\left[\frac{1}{x^2}\sin\left(\frac{1}{x}\right)\right] + \frac{1}{x^2}\cos\left(\frac{1}{x}\right) - \frac{1}{x^2}\cos\left(\frac{1}{x}\right) = -\frac{1}{x^3}\sin\left(\frac{1}{x}\right) = 0$$

$$x = \frac{1}{\pi}$$

Point of inflection: $\left(\frac{1}{\pi}, 0\right)$

When $x > 1/\pi, f'' < 0$, so the graph is concave downward.



77. Assume the zeros of f are all real. Then express the function as $f(x) = a(x - r_1)(x - r_2)(x - r_3)$ where r_1, r_2, and r_3 are the distinct zeros of f. From the Product Rule for a function involving three factors, we have

$$f'(x) = a[(x - r_1)(x - r_2) + (x - r_1)(x - r_3) + (x - r_2)(x - r_3)]$$

$$f''(x) = a[(x - r_1) + (x - r_2) + (x - r_1) + (x - r_3) + (x - r_2) + (x - r_3)]$$

$$= a[6x - 2(r_1 + r_2 + r_3)].$$

Consequently, $f''(x) = 0$ if

$$x = \frac{2(r_1 + r_2 + r_3)}{6} = \frac{r_1 + r_2 + r_3}{3} = (\text{Average of } r_1, r_2, \text{ and } r_3).$$

79. True. Let $y = ax^3 + bx^2 + cx + d$, $a \neq 0$. Then $y'' = 6ax + 2b = 0$ when $x = -(b/3a)$, and the concavity changes at this point.

81. False.

$f(x) = 3\sin x + 2\cos x$

$f'(x) = 3\cos x - 2\sin x$

$3\cos x - 2\sin x = 0$

$3\cos x = 2\sin x$

$\frac{3}{2} = \tan x$

Critical number: $x = \tan^{-1}\left(\frac{3}{2}\right)$

$f\left(\tan^{-1}\frac{3}{2}\right) \approx 3.60555$ is the maximum value of y.

83. False. Concavity is determined by f''.

Section 3.5 Limits at Infinity

1. $f(x) = \dfrac{3x^2}{x^2 + 2}$

No vertical asymptotes

Horizontal asymptote: $y = 3$

Matches (f)

3. $f(x) = \dfrac{x}{x^2 + 2}$

No vertical asymptotes

Horizontal asymptote: $y = 0$

Matches (d)

5. $f(x) = \dfrac{4\sin x}{x^2 + 1}$

No vertical asymptotes

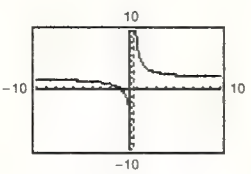
Horizontal asymptotes: $y = 0$

Matches (b)

7. $f(x) = \dfrac{4x + 3}{2x - 1}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	7	2.26	2.025	2.0025	2.0003	2	2

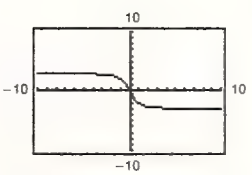
$\lim_{x\to\infty} f(x) = 2$



9. $f(x) = \dfrac{-6x}{\sqrt{4x^2 + 5}}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	-2	-2.98	-2.9998	-3	-3	-3	-3

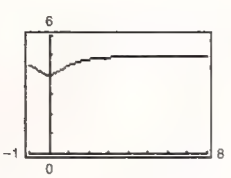
$\lim_{x\to\infty} f(x) = -3$



11. $f(x) = 5 - \dfrac{1}{x^2 + 1}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	4.5	4.99	4.9999	4.999999	5	5	5

$\lim_{x\to\infty} f(x) = 5$



13. (a) $h(x) = \dfrac{f(x)}{x^2} = \dfrac{5x^3 - 3x^2 + 10}{x^2} = 5x - 3 + \dfrac{10}{x^2}$

$\lim_{x\to\infty} h(x) = \infty$ (Limit does not exist)

(b) $h(x) = \dfrac{f(x)}{x^3} = \dfrac{5x^3 - 3x^2 + 10}{x^3} = 5 - \dfrac{3}{x} + \dfrac{10}{x^3}$

$\lim_{x\to\infty} h(x) = 5$

(c) $h(x) = \dfrac{f(x)}{x^4} = \dfrac{5x^3 - 3x^2 + 10}{x^4} = \dfrac{5}{x} - \dfrac{3}{x^2} + \dfrac{10}{x^4}$

$\lim_{x\to\infty} h(x) = 0$

15. (a) $\lim_{x\to\infty} \dfrac{x^2 + 2}{x^3 - 1} = 0$

(b) $\lim_{x\to\infty} \dfrac{x^2 + 2}{x^2 - 1} = 1$

(c) $\lim_{x\to\infty} \dfrac{x^2 + 2}{x - 1} = \infty$ (Limit does not exist)

17. (a) $\lim_{x\to\infty} \dfrac{5 - 2x^{3/2}}{3x^2 - 4} = 0$

(b) $\lim_{x\to\infty} \dfrac{5 - 2x^{3/2}}{3x^{3/2} - 4} = -\dfrac{2}{3}$

(c) $\lim_{x\to\infty} \dfrac{5 - 2x^{3/2}}{3x - 4} = -\infty$ (Limit does not exist)

19. $\lim_{x\to\infty} \dfrac{2x - 1}{3x + 2} = \lim_{x\to\infty} \dfrac{2 - (1/x)}{3 + (2/x)} = \dfrac{2 - 0}{3 + 0} = \dfrac{2}{3}$

21. $\lim_{x\to\infty} \dfrac{x}{x^2 - 1} = \lim_{x\to\infty} \dfrac{1/x}{1 - (1/x^2)} = \dfrac{0}{1} = 0$

23. $\lim_{x\to-\infty} \dfrac{5x^2}{x + 3} = \lim_{x\to-\infty} \dfrac{5x}{1 + (3/x)} = -\infty$

Limit does not exist.

25. $\lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2 - x}} = \lim_{x\to-\infty} \dfrac{1}{\dfrac{\sqrt{x^2 - x}}{-\sqrt{x^2}}}$, $\left(\text{for } x < 0 \text{ we have } x = -\sqrt{x^2}\right)$

$= \lim_{x\to-\infty} \dfrac{-1}{\sqrt{1 - (1/x)}} = -1$

27. $\lim_{x\to-\infty} \dfrac{2x + 1}{\sqrt{x^2 - x}} = \lim_{x\to-\infty} \dfrac{2 + \dfrac{1}{x}}{\left(\dfrac{\sqrt{x^2 - x}}{-\sqrt{x^2}}\right)}$ $\left(\text{for } x < 0, x = -\sqrt{x^2}\right)$

$= \lim_{x\to\infty} \dfrac{-2 - (1/x)}{\sqrt{x + (1/x)}} = -2$

29. Since $(-1/x) \le (\sin(2x))/x \le (1/x)$ for all $x \neq 0$, we have by the Squeeze Theorem,

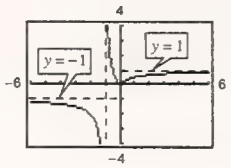
$\lim_{x\to\infty} -\dfrac{1}{x} \le \lim_{x\to\infty} \dfrac{\sin(2x)}{x} \le \lim_{x\to\infty} \dfrac{1}{x}$

$0 \le \lim_{x\to\infty} \dfrac{\sin(2x)}{x} \le 0.$

Therefore, $\lim_{x\to\infty} \dfrac{\sin(2x)}{x} = 0.$

31. $\lim_{x\to\infty} \dfrac{1}{2x + \sin x} = 0$

33. (a) $f(x) = \dfrac{|x|}{x+1}$



$$\lim_{x\to\infty} \frac{|x|}{x+1} = 1$$

$$\lim_{x\to-\infty} \frac{|x|}{x+1} = -1$$

Therefore, $y = 1$ and $y = -1$ are both horizontal asymptotes.

35. $\lim\limits_{x\to\infty} x \sin\dfrac{1}{x} = \lim\limits_{t\to0^+} \dfrac{\sin t}{t} = 1$

(Let $x = 1/t$.)

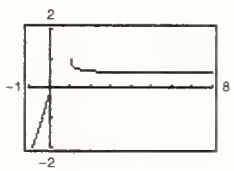
37. $\lim\limits_{x\to-\infty} \left(x + \sqrt{x^2+3}\right) = \lim\limits_{x\to-\infty} \left[\left(x + \sqrt{x^2+3}\right) \cdot \dfrac{x - \sqrt{x^2+3}}{x - \sqrt{x^2+3}}\right] = \lim\limits_{x\to-\infty} \dfrac{-3}{x - \sqrt{x^2+3}} = 0$

39. $\lim\limits_{x\to\infty} \left(x - \sqrt{x^2+x}\right) = \lim\limits_{x\to\infty} \left[\left(x - \sqrt{x^2+x}\right) \cdot \dfrac{x + \sqrt{x^2+x}}{x + \sqrt{x^2+x}}\right]$

$$= \lim_{x\to\infty} \frac{-x}{x + \sqrt{x^2+x}} = \lim_{x\to\infty} \frac{-1}{1 + \sqrt{1 + (1/x)}} = -\frac{1}{2}$$

41.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1	0.513	0.501	0.500	0.500	0.500	0.500



$$\lim_{x\to\infty} \left(x - \sqrt{x(x-1)}\right) = \lim_{x\to\infty} \frac{x - \sqrt{x^2-x}}{1} \cdot \frac{x + \sqrt{x^2-x}}{x + \sqrt{x^2-x}}$$

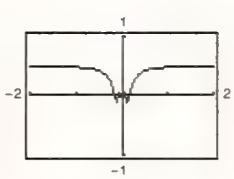
$$= \lim_{x\to\infty} \frac{x}{x + \sqrt{x^2-x}}$$

$$= \lim_{x\to\infty} \frac{1}{1 + \sqrt{1 - (1/x)}}$$

$$= \frac{1}{2}$$

43.

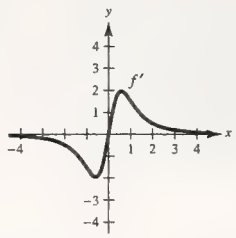
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	0.479	0.500	0.500	0.500	0.500	0.500	0.500



Let $x = 1/t$.

$$\lim_{x\to\infty} x \sin\left(\frac{1}{2x}\right) = \lim_{t\to0^+} \frac{\sin(t/2)}{t} = \lim_{t\to0^+} \frac{1}{2}\frac{\sin(t/2)}{t/2} = \frac{1}{2}$$

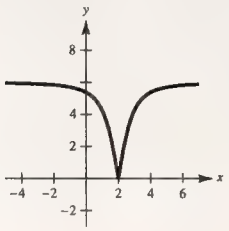
45. (a)



(b) $\lim_{x \to \infty} f(x) = 3 \qquad \lim_{x \to \infty} f'(x) = 0$

(c) Since $\lim_{x \to \infty} f(x) = 3$, the graph approaches that of a horizontal line, $\lim_{x \to \infty} f'(x) = 0$.

47. Yes. For example, let $f(x) = \dfrac{6|x-2|}{\sqrt{(x-2)^2+1}}$.



49. $y = \dfrac{2+x}{1-x}$

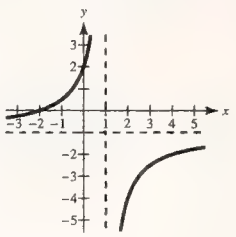
Intercepts: $(-2, 0), (0, 2)$

Symmetry: none

Horizontal asymptote: $y = -1$ since

$$\lim_{x \to -\infty} \frac{2+x}{1-x} = -1 = \lim_{x \to \infty} \frac{2+x}{1-x}.$$

Discontinuity: $x = 1$ (Vertical asymptote)



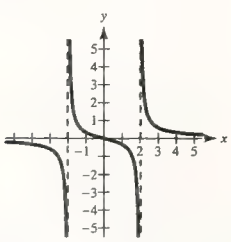
51. $y = \dfrac{x}{x^2-4}$

Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = \pm 2$



53. $y = \dfrac{x^2}{x^2+9}$

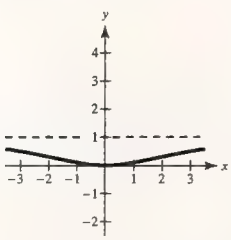
Intercept: $(0, 0)$

Symmetry: y-axis

Horizontal asymptote: $y = 1$ since

$$\lim_{x \to -\infty} \frac{x^2}{x^2+9} = 1 = \lim_{x \to \infty} \frac{x^2}{x^2+9}.$$

Relative minimum: $(0, 0)$



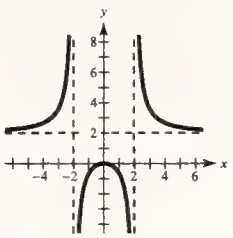
55. $y = \dfrac{2x^2}{x^2 - 4}$

Intercept: $(0, 0)$

Symmetry: y-axis

Horizontal asymptote: $y = 2$

Vertical asymptote: $x = \pm 2$



57. $xy^2 = 4$

Domain: $x > 0$

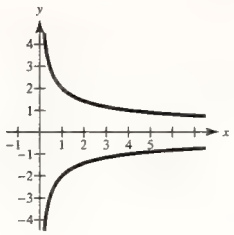
Intercepts: none

Symmetry: x-axis

Horizontal asymptote: $y = 0$ since

$$\lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0 = \lim_{x \to \infty} -\frac{2}{\sqrt{x}}.$$

Discontinuity: $x = 0$ (Vertical asymptote)



59. $y = \dfrac{2x}{1 - x}$

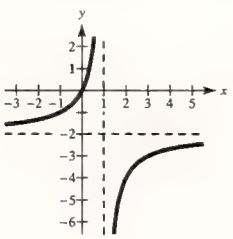
Intercept: $(0, 0)$

Symmetry: none

Horizontal asymptote: $y = -2$ since

$$\lim_{x \to -\infty} \frac{2x}{1 - x} = -2 = \lim_{x \to \infty} \frac{2x}{1 - x}.$$

Discontinuity: $x = 1$ (Vertical asymptote)



61. $y = 2 - \dfrac{3}{x^2}$

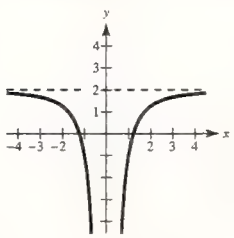
Intercepts: $\left(\pm\sqrt{3/2}, 0\right)$

Symmetry: y-axis

Horizontal asymptote: $y = 2$ since

$$\lim_{x \to -\infty} \left(2 - \frac{3}{x^2}\right) = 2 = \lim_{x \to \infty} \left(2 - \frac{3}{x^2}\right).$$

Discontinuity: $x = 0$ (Vertical asymptote)



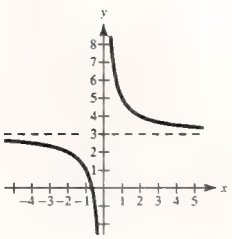
63. $y = 3 + \dfrac{2}{x}$

Intercept: $y = 0 = 3 + \dfrac{2}{x} \Rightarrow \dfrac{2}{x} = -3 \Rightarrow x = -\dfrac{2}{3}\left(-\dfrac{2}{3}, 0\right)$

Symmetry: none

Horizontal asymptote: $y = 3$

Vertical asymptote: $x = 0$



65. $y = \dfrac{x^3}{\sqrt{x^2 - 4}}$

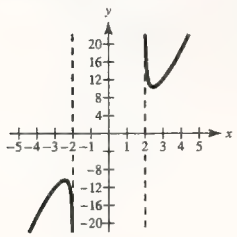
Domain: $(-\infty, -2), (2, \infty)$

Intercepts: none

Symmetry: origin

Horizontal asymptote: none

Vertical asymptotes: $x = \pm 2$ (discontinuities)



67. $f(x) = 5 - \dfrac{1}{x^2} = \dfrac{5x^2 - 1}{x^2}$

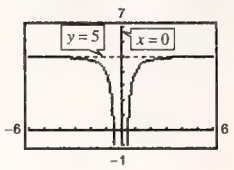
Domain: $(-\infty, 0), (0, \infty)$

$f'(x) = \dfrac{2}{x^3} \Rightarrow$ No relative extrema

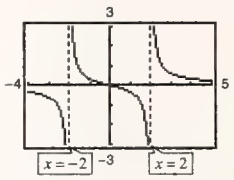
$f''(x) = -\dfrac{6}{x^4} \Rightarrow$ No points of inflection

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 5$



69. $f(x) = \dfrac{x}{x^2 - 4}$



$$f'(x) = \frac{(x^2 - 4) - x(2x)}{(x^2 - 4)^2}$$

$$= \frac{-(x^2 + 4)}{(x^2 - 4)^2} \neq 0 \text{ for any } x \text{ in the domain of } f.$$

$$f''(x) = \frac{(x^2 - 4)^2(-2x) + (x^2 + 4)(2)(x^2 - 4)(2x)}{(x^2 - 4)^2}$$

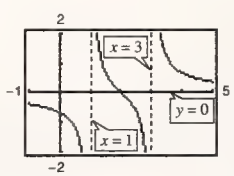
$$= \frac{2x(x^2 + 12)}{(x^2 - 4)^3} = 0 \text{ when } x = 0.$$

Since $f''(x) > 0$ on $(-2, 0)$ and $f''(x) < 0$ on $(0, 2)$, then $(0, 0)$ is a point of inflection.

Vertical asymptotes: $x = \pm 2$

Horizontal asymptote: $y = 0$

71. $f(x) = \dfrac{x - 2}{x^2 - 4x + 3} = \dfrac{x - 2}{(x - 1)(x - 3)}$



$$f'(x) = \frac{(x^2 - 4x + 3) - (x - 2)(2x - 4)}{(x^2 - 4x + 3)^2} = \frac{-x^2 + 4x - 5}{(x^2 - 4x + 3)^2} \neq 0$$

$$f''(x) = \frac{(x^2 - 4x + 3)^2(-2x + 4) - (-x^2 + 4x - 5)(2)(x^2 - 4x + 3)(2x - 4)}{(x^2 - 4x + 3)^4}$$

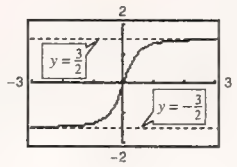
$$= \frac{2(x^3 - 6x^2 + 15x - 14)}{(x^2 - 4x + 3)^3} = 0 \text{ when } x = 2.$$

Since $f''(x) > 0$ on $(1, 2)$ and $f''(x) < 0$ on $(2, 3)$, then $(2, 0)$ is a point of inflection.

Vertical asymptote: $x = 1, x = 3$

Horizontal asymptote: $y = 0$

73. $f(x) = \dfrac{3x}{\sqrt{4x^2 + 1}}$



$f'(x) = \dfrac{3}{(4x^2 + 1)^{3/2}} \Rightarrow$ No relative extrema

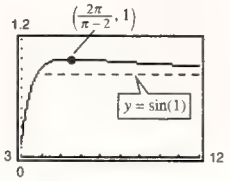
$f''(x) = \dfrac{-36x}{(4x^2 + 1)^{5/2}} = 0$ when $x = 0$.

Point of inflection: $(0, 0)$

Horizontal asymptotes: $y = \pm\dfrac{3}{2}$

No vertical asymptotes

75. $g(x) = \sin\left(\dfrac{x}{x - 2}\right),\ 3 < x < \infty$



$g'(x) = \dfrac{-2\cos\left(\dfrac{x}{x - 2}\right)}{(x - 2)^2}$

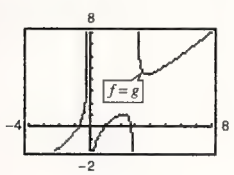
Horizontal asymptote: $y = 1$

Relative maximum: $\dfrac{x}{x - 2} = \dfrac{\pi}{2} \Rightarrow x = \dfrac{2\pi}{\pi - 2} \approx 5.5039$

No vertical asymptotes

77. $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x - 3)}$, $g(x) = x + \dfrac{2}{x(x - 3)}$

(a)

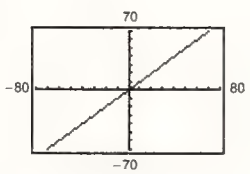


(b) $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x - 3)}$

$= \dfrac{x^2(x - 3)}{x(x - 3)} + \dfrac{2}{x(x - 3)}$

$= x + \dfrac{2}{x(x - 3)} = g(x)$

(c)



The graph appears as the slant asymptote $y = x$.

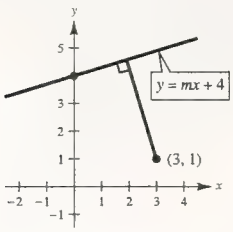
79. $C = 0.5x + 500$

$\overline{C} = \dfrac{C}{x}$

$\overline{C} = 0.5 + \dfrac{500}{x}$

$\lim\limits_{x \to \infty} \left(0.5 + \dfrac{500}{x}\right) = 0.5$

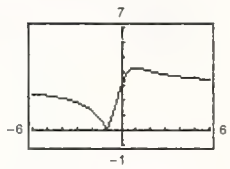
81. line: $mx - y + 4 = 0$



(a) $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \dfrac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}}$

$= \dfrac{|3m + 3|}{\sqrt{m^2 + 1}}$

(b)

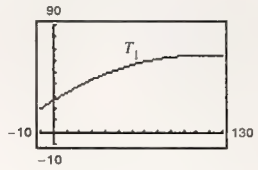


(c) $\lim_{m \to \infty} d(m) = 3 = \lim_{m \to -\infty} d(m)$

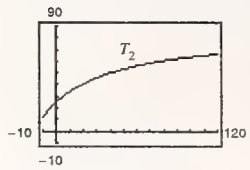
The line approaches the vertical line $x = 0$. Hence, the distance approaches 3.

83. (a) $T_1(t) = -0.003t^2 + 0.677t + 26.564$

(b)



(c)



$T_2 = \dfrac{1451 + 86t}{58 + t}$

(d) $T_1(0) \approx 26.6$

$T_2(0) \approx 25.0$

(e) $\lim_{t \to \infty} T_2 = \dfrac{86}{1} = 86$

(f) The limiting temperature is 86.
T_1 has no horizontal asymptote.

85. Answers will vary. See page 195.

87. False. Let $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$. (See Exercise 2.)

Section 3.6 A Summary of Curve Sketching

1. f has constant negative slope. Matches (D)

3. The slope is periodic, and zero at $x = 0$. Matches (A)

5. (a) $f'(x) = 0$ for $x = -2$ and $x = 2$

f' is negative for $-2 < x < 2$ (decreasing function).

f' is positive for $x > 2$ and $x < -2$
(increasing function).

(b) $f''(x) = 0$ at $x = 0$ (Inflection point).

f'' is positive for $x > 0$ (Concave upwards).

f'' is negative for $x < 0$ (Concave downward).

(c) f' is increasing on $(0, \infty)$. $(f'' > 0)$

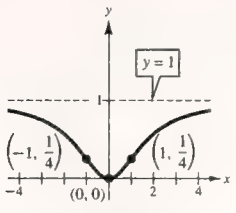
(d) $f'(x)$ is minimum at $x = 0$. The rate of change of f at $x = 0$ is less than the rate of change of f for all other values of x.

7. $y = \dfrac{x^2}{x^2 + 3}$

$y' = \dfrac{6x}{(x^2 + 3)^2} = 0$ when $x = 0$.

$y'' = \dfrac{18(1 - x^2)}{(x^2 + 3)^3} = 0$ when $x = \pm 1$.

Horizontal asymptote: $y = 1$



	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$-$	Decreasing, concave down
$x = -1$	$\frac{1}{4}$	$-$	0	Point of inflection
$-1 < x < 0$		$-$	$+$	Decreasing, concave up
$x = 0$	0	0	$+$	Relative minimum
$0 < x < 1$		$+$	$+$	Increasing, concave up
$x = 1$	$\frac{1}{4}$	$+$	0	Point of inflection
$1 < x < \infty$		$+$	$-$	Increasing, concave down

9. $y = \dfrac{1}{x - 2} - 3$

$y' = -\dfrac{1}{(x - 2)^2} < 0$ when $x \neq 2$.

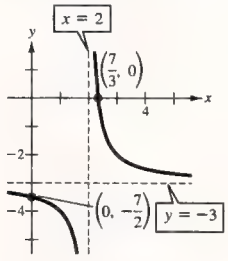
$y'' = \dfrac{2}{(x - 2)^3}$

No relative extrema, no points of inflection

Intercepts: $\left(\frac{7}{3}, 0\right), \left(0, -\frac{7}{2}\right)$

Vertical asymptote: $x = 2$

Horizontal asymptote: $y = -3$



11. $y = \dfrac{2x}{x^2 - 1}$

$y' = \dfrac{-2(x^2 + 1)}{(x^2 - 1)^2} < 0$ if $x \neq \pm 1$.

$y'' = \dfrac{4x(x^2 + 3)}{(x^2 - 1)^3} = 0$ if $x = 0$.

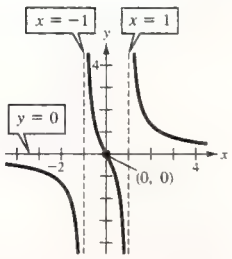
Inflection point: $(0, 0)$

Intercept: $(0, 0)$

Vertical asymptote: $x = \pm 1$

Horizontal asymptote: $y = 0$

Symmetry with respect to the origin



13. $g(x) = x + \dfrac{4}{x^2 + 1}$

$g'(x) = 1 - \dfrac{8x}{(x^2 + 1)^2} = \dfrac{x^4 + 2x^2 - 8x + 1}{(x^2 + 1)^2} = 0$ when $x \approx 0.1292, 1.6085$

$g''(x) = \dfrac{8(3x^2 - 1)}{(x^2 + 1)^3} = 0$ when $x = \pm\dfrac{\sqrt{3}}{3}$

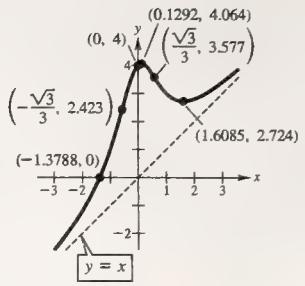
$g''(0.1292) < 0$, therefore, $(0.1292, 4.064)$ is relative maximum.

$g''(1.6085) > 0$, therefore, $(1.6085, 2.724)$ is a relative minimum.

Points of inflection: $\left(-\dfrac{\sqrt{3}}{3}, 2.423\right), \left(\dfrac{\sqrt{3}}{3}, 3.577\right)$

Intercepts: $(0, 4), (-1.3788, 0)$

Slant asymptote: $y = x$



15. $f(x) = \dfrac{x^2 + 1}{x} = x + \dfrac{1}{x}$

$f'(x) = 1 - \dfrac{1}{x^2} = 0$ when $x = \pm 1$.

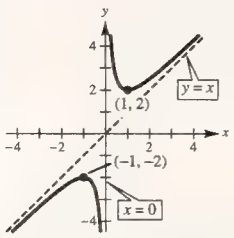
$f''(x) = \dfrac{2}{x^3} \neq 0$

Relative maximum: $(-1, -2)$

Relative minimum: $(1, 2)$

Vertical asymptote: $x = 0$

Slant asymptote: $y = x$



17. $y = \dfrac{x^2 - 6x + 12}{x - 4} = x - 2 + \dfrac{4}{x - 4}$

$y' = 1 - \dfrac{4}{(x - 4)^2}$

$= \dfrac{(x - 2)(x - 6)}{(x - 4)^2} = 0$ when $x = 2, 6$.

$y'' = \dfrac{8}{(x - 4)^3}$

$y'' < 0$ when $x = 2$.

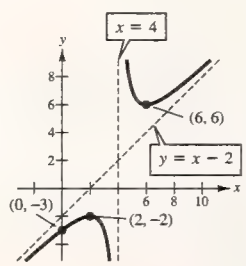
Therefore, $(2, -2)$ is a relative maximum.

$y'' > 0$ when $x = 6$.

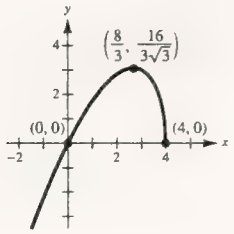
Therefore, $(6, 6)$ is a relative minimum.

Vertical asymptote: $x = 4$

Slant asymptote: $y = x - 2$



19. $y = x\sqrt{x-4}$,



Domain: $(-\infty, 4]$

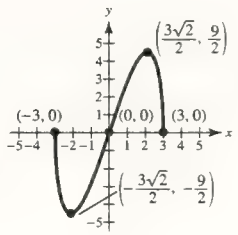
$y' = \dfrac{8-3x}{2\sqrt{4-x}} = 0$ when $x = \dfrac{8}{3}$ and undefined when $x = 4$.

$y'' = \dfrac{3x-16}{4(4-x)^{3/2}} = 0$ when $x = \dfrac{16}{3}$ and undefined when $x = 4$.

Note: $x = \frac{16}{3}$ is not in the domain.

	y	y'	y''	Conclusion
$-\infty < x < \frac{8}{3}$		+	−	Increasing, concave down
$x = \frac{8}{3}$	$\frac{16}{3\sqrt{3}}$	0	−	Relative maximum
$\frac{8}{3} < x < 4$		−	−	Decreasing, concave down
$x = 4$	0	Undefined	Undefined	Endpoint

21. $h(x) = x\sqrt{9-x^2}$ Domain: $-3 \le x \le 3$



$h'(x) = \dfrac{9-2x^2}{\sqrt{9-x^2}} = 0$ when $x = \pm\dfrac{3}{\sqrt{2}} = \pm\dfrac{3\sqrt{2}}{2}$

$h''(x) = \dfrac{x(2x^2-27)}{(9-x^2)^{3/2}} = 0$ when $x = 0$

Relative maximum: $\left(\dfrac{3\sqrt{2}}{2}, \dfrac{9}{2}\right)$

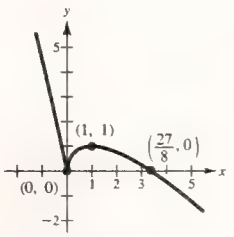
Relative minimum: $\left(-\dfrac{3\sqrt{2}}{2}, -\dfrac{9}{2}\right)$

Intercepts: $(0, 0), (\pm 3, 0)$

Symmetric with respect to the origin

Point of inflection: $(0, 0)$

23. $y = 3x^{2/3} - 2x$



$y' = 2x^{-1/3} - 2 = \dfrac{2(1-x^{1/3})}{x^{1/3}}$

$= 0$ when $x = 1$ and undefined when $x = 0$.

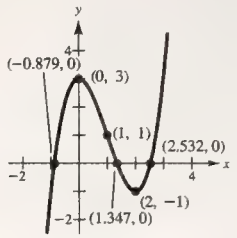
$y'' = \dfrac{-2}{3x^{4/3}} < 0$ when $x \ne 0$.

	y	y'	y''	Conclusion
$-\infty < x < 0$		−	−	Decreasing, concave down
$x = 0$	0	Undefined	Undefined	Relative minimum
$0 < x < 1$		+	−	Increasing, concave down
$x = 1$	1	0	−	Relative maximum
$1 < x < \infty$		−	−	Decreasing, concave down

25. $y = x^3 - 3x^2 + 3$

$y' = 3x^2 - 6x = 3x(x - 2) = 0$ when $x = 0, x = 2$

$y'' = 6x - 6 = 6(x - 1) = 0$ when $x = 1$



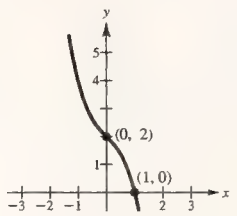
	y	y'	y''	Conclusion
$-\infty < x < 0$		+	−	Increasing, concave down
$x = 0$	3	0	−	Relative maximum
$0 < x < 1$		−	−	Decreasing, concave down
$x = 1$	1	−	0	Point of inflection
$1 < x < 2$		−	+	Decreasing, concave up
$x = 2$	−1	0	+	Relative minimum
$2 < x < \infty$		+	+	Increasing, concave up

27. $y = 2 - x - x^3$

$y' = -1 - 3x^2$

No critical numbers

$y'' = -6x = 0$ when $x = 0$.

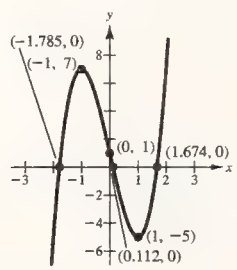


	y	y'	y''	Conclusion
$-\infty < x < 0$		−	+	Decreasing, concave up
$x = 0$	2	−	0	Point of inflection
$0 < x < \infty$		−	−	Decreasing, concave down

29. $f(x) = 3x^3 - 9x + 1$

$f'(x) = 9x^2 - 9 = 9(x^2 - 1) = 0$ when $x = \pm 1$

$f''(x) = 18x = 0$ when $x = 0$

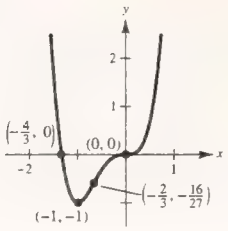


	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < -1$		+	−	Increasing, concave down
$x = -1$	7	0	−	Relative maximum
$-1 < x < 0$		−	−	Decreasing, concave down
$x = 0$	1	−	0	Point of inflection
$0 < x < 1$		−	+	Decreasing, concave up
$x = 1$	−5	0	+	Relative minimum
$1 < x < \infty$		+	+	Increasing, concave up

31. $y = 3x^4 + 4x^3$

$y' = 12x^3 + 12x^2 = 12x^2(x + 1) = 0$ when $x = 0, x = -1$.

$y'' = 36x^2 + 24x = 12x(3x + 2) = 0$ when $x = 0, x = -\frac{2}{3}$.

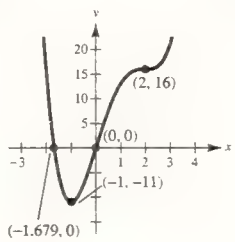


	y	y'	y''	Conclusion
$-\infty < x < -1$		−	+	Decreasing, concave up
$x = -1$	-1	0	+	Relative minimum
$-1 < x < -\frac{2}{3}$		+	+	Increasing, concave up
$x = -\frac{2}{3}$	$-\frac{16}{27}$	+	0	Point of inflection
$-\frac{2}{3} < x < 0$		+	−	Increasing, concave down
$x = 0$	0	0	0	Point of inflection
$0 < x < \infty$		+	+	Increasing, concave up

33. $f(x) = x^4 - 4x^3 + 16x$

$f'(x) = 4x^3 - 12x^2 + 16 = 4(x + 1)(x - 2)^2 = 0$ when $x = -1, x = 2$.

$f''(x) = 12x^2 - 24x = 12x(x - 2) = 0$ when $x = 0, x = 2$.

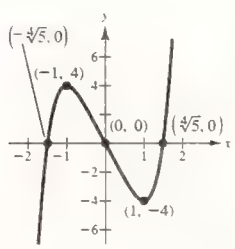


	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < -1$		−	+	Decreasing, concave up
$x = -1$	-11	0	+	Relative minimum
$-1 < x < 0$		+	+	Increasing, concave up
$x = 0$	0	+	0	Point of inflection
$0 < x < 2$		+	−	Increasing, concave down
$x = 2$	16	0	0	Point of inflection
$2 < x < \infty$		+	+	Increasing, concave up

35. $y = x^5 - 5x$

$y' = 5x^4 - 5 = 5(x^4 - 1) = 0$ when $x = \pm 1$.

$y'' = 20x^3 = 0$ when $x = 0$.

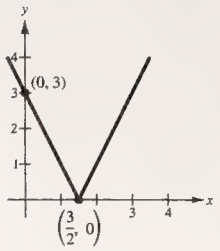


	y	y'	y''	Conclusion
$-\infty < x < -1$		+	−	Increasing, concave down
$x = -1$	4	0	−	Relative maximum
$-1 < x < 0$		−	−	Decreasing, concave down
$x = 0$	0	−	0	Point of inflection
$0 < x < 1$		−	+	Decreasing, concave up
$x = 1$	-4	0	+	Relative minimum
$1 < x < \infty$		+	+	Increasing, concave up

37. $y = |2x - 3|$

$y' = \dfrac{2(2x-3)}{|2x-3|}$ undefined at $x = \dfrac{3}{2}$.

$y'' = 0$



	y	y'	Conclusion
$-\infty < x < \frac{3}{2}$		$-$	Decreasing
$x = \frac{3}{2}$	0	Undefined	Relative minimum
$\frac{3}{2} < x < \infty$		$+$	Increasing

39. $y = \sin x - \dfrac{1}{18}\sin 3x,\ 0 \le x \le 2\pi$

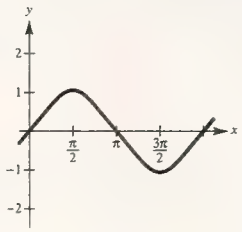
$y' = \cos x - \dfrac{1}{6}\cos 3x = 0$ when $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

$y'' = -\sin x + \dfrac{1}{2}\sin 3x = 0$ when $x = 0, \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \pi, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$.

Relative maximum: $\left(\dfrac{\pi}{2}, \dfrac{19}{18}\right)$

Relative minimum: $\left(\dfrac{3\pi}{2}, -\dfrac{19}{18}\right)$

Inflection points: $\left(\dfrac{\pi}{6}, \dfrac{4}{9}\right), \left(\dfrac{5\pi}{6}, \dfrac{4}{9}\right), (\pi, 0), \left(\dfrac{7\pi}{6}, -\dfrac{4}{9}\right), \left(\dfrac{11\pi}{6}, -\dfrac{4}{9}\right)$



41. $y = 2x - \tan x,\ -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$y' = 2 - \sec^2 x = 0$ when $x = \pm\dfrac{\pi}{4}$.

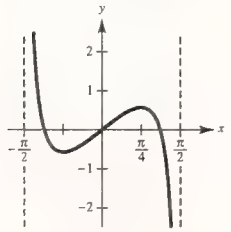
$y'' = -2\sec^2 x \tan x = 0$ when $x = 0$.

Relative maximum: $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2} - 1\right)$

Relative minimum: $\left(-\dfrac{\pi}{4}, 1 - \dfrac{\pi}{2}\right)$

Inflection point: (0, 0)

Vertical asymptotes: $x = \pm\dfrac{\pi}{2}$

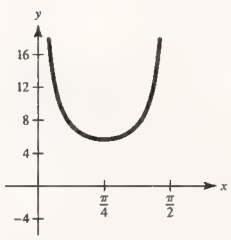


43. $y = 2(\csc x + \sec x),\ 0 < x < \dfrac{\pi}{2}$

$y' = 2(\sec x \tan x - \csc x \cot x) = 0 \implies x = \pi/4$

Relative minimum: $\left(\dfrac{\pi}{4}, 4\sqrt{2}\right)$

Vertical asymptotes: $x = 0, x = \dfrac{\pi}{2}$



45. $g(x) = x \tan x, \quad -\dfrac{3\pi}{2} < x < \dfrac{3\pi}{2}$

$g'(x) = \dfrac{x + \sin x \cos x}{\cos^2 x} = 0$ when $x = 0$

$g''(x) = \dfrac{2(\cos x + x \sin x)}{\cos^3 x}$

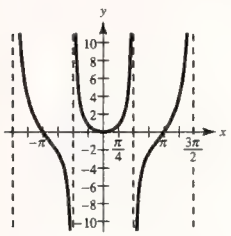
Vertical asymptotes: $x = -\dfrac{3\pi}{2}, -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

Intercepts: $(-\pi, 0), (0, 0), (\pi, 0)$

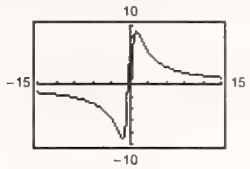
Symmetric with respect to y-axis.

Increasing on $\left(0, \dfrac{\pi}{2}\right)$ and $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

Points of inflection: $(\pm 2.80, 0)$



47. $f(x) = \dfrac{20x}{x^2 + 1} - \dfrac{1}{x} = \dfrac{19x^2 - 1}{x(x^2 + 1)}$



$x = 0$ vertical asymptote

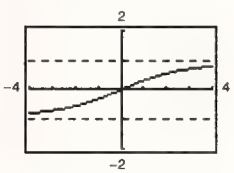
$y = 0$ horizontal asymptote

Minimum: $(-1.10, -9.05)$

Maximum: $(1.10, 9.05)$

Points of inflection: $(-1.84, -7.86), (1.84, 7.86)$

49. $y = \dfrac{x}{\sqrt{x^2 + 7}}$



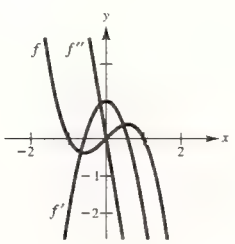
$(0, 0)$ point of inflection

$y = \pm 1$ horizontal asymptotes

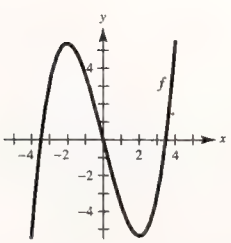
51. f is cubic.

f' is quadratic.

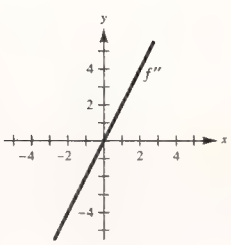
f'' is linear.



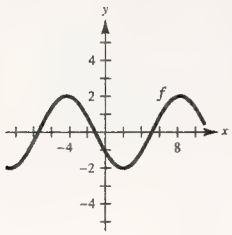
53.

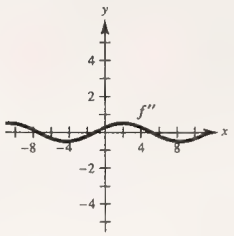


(any vertical translate of f will do)



55.





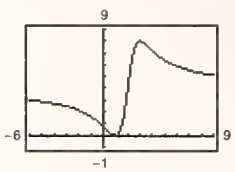
(any vertical translate of f will do)

57. Since the slope is negative, the function is decreasing on $(2, 8)$, and hence $f(3) > f(5)$.

59. $f(x) = \dfrac{4(x-1)^2}{x^2 - 4x + 5}$

Vertical asymptote: none

Horizontal asymptote: $y = 4$

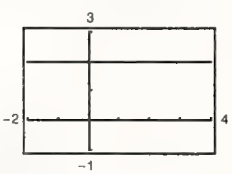


The graph crosses the horizontal asymptote $y = 4$. If a function has a vertical asymptote at $x = c$, the graph would not cross it since $f(c)$ is undefined.

61. $h(x) = \dfrac{6 - 2x}{3 - x}$

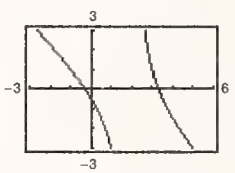
$$= \frac{2(3-x)}{3-x} = \begin{cases} 2, & \text{if } x \neq 3 \\ \text{Undefined}, & \text{if } x = 3 \end{cases}$$

The rational function is not reduced to lowest terms.



hole at $(3, 2)$

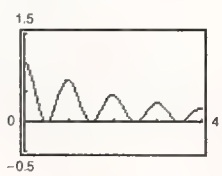
63. $f(x) = -\dfrac{x^2 - 3x - 1}{x - 2} = -x + 1 + \dfrac{3}{x - 2}$



The graph appears to approach the slant asymptote $y = -x + 1$.

65. $f(x) = \dfrac{\cos^2 \pi x}{\sqrt{x^2 + 1}}$, $(0, 4)$

(a)



On $(0, 4)$ there seem to be 7 critical numbers:

0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5

(b) $f'(x) = \dfrac{-\cos \pi x(x \cos \pi x + 2\pi(x^2 + 1)\sin \pi x)}{(x^2 + 1)^{3/2}} = 0$

Critical numbers $\approx \dfrac{1}{2}, 0.97, \dfrac{3}{2}, 1.98, \dfrac{5}{2}, 2.98, \dfrac{7}{2}$.

The critical numbers where maxima occur appear to be integers in part (a), but approximating them using f' shows that they are not integers.

67. Vertical asymptote: $x = 5$

Horizontal asymptote: $y = 0$

$$y = \frac{1}{x - 5}$$

69. Vertical asymptote: $x = 5$

Slant asymptote: $y = 3x + 2$

$$y = 3x + 2 + \frac{1}{x - 5} = \frac{3x^2 - 13x - 9}{x - 5}$$

71. $f(x) = \dfrac{ax}{(x - b)^2}$

(a) The graph has a vertical asymptote at $x = b$. If $a > 0$, the graph approaches ∞ as $x \to b$. If $a < 0$, the graph approaches $-\infty$ as $x \to b$. The graph approaches its vertical asymptote faster as $|a| \to 0$.

(b) As b varies, the position of the vertical asymptote changes: $x = b$. Also, the coordinates of the minimum ($a > 0$) or maximum ($a < 0$) are changed.

73. $f(x) = \dfrac{3x^n}{x^4 + 1}$

(a) For n even, f is symmetric about the y-axis. For n odd, f is symmetric about the origin.

(b) The x-axis will be the horizontal asymptote if the degree of the numerator is less than 4. That is, $n = 0, 1, 2, 3$.

(c) $n = 4$ gives $y = 3$ as the horizontal asymptote.

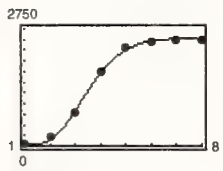
(d) There is a slant asymptote $y = 3x$ if $n = 5$:

$$\frac{3x^5}{x^4 + 1} = 3x - \frac{3x}{x^4 + 1}.$$

(e)

n	0	1	2	3	4	5
M	1	2	3	2	1	0
N	2	3	4	5	2	3

75. (a)



(b) When $t = 10$, $N(10) \approx 2434$ bacteria.

(c) N is a maximum when $t \approx 7.2$ (seventh day).

(d) $N'(t) = 0$ for $t \approx 3.2$

(e) $\lim_{t \to \infty} N(t) = \dfrac{13{,}250}{7} \approx 1892.86$

Section 3.7 Optimization Problems

1. (a)

First Number, x	Second Number	Product, P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$

—CONTINUED—

1. —CONTINUED—

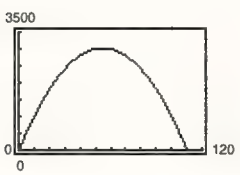
(b)

First Number, x	Second Number	Product, P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$
70	$110 - 70$	$70(110 - 70) = 2800$
80	$110 - 80$	$80(110 - 80) = 2400$
90	$110 - 90$	$90(110 - 90) = 1800$
100	$110 - 100$	$100(110 - 100) = 1000$

The maximum is attained near $x = 50$ and 60.

(c) $P = x(110 - x) = 110x - x^2$

(d)



The solution appears to be $x = 55$.

(e) $\dfrac{dP}{dx} = 110 - 2x = 0$ when $x = 55$.

$\dfrac{d^2P}{dx^2} = -2 < 0$

P is a maximum when $x = 110 - x = 55$.

The two numbers are 55 and 55.

3. Let x and y be two positive numbers such that $xy = 192$.

$$S = x + y = x + \frac{192}{x}$$

$\dfrac{dS}{dx} = 1 - \dfrac{192}{x^2} = 0$ when $x = \sqrt{192}$.

$\dfrac{d^2S}{dx^2} = \dfrac{384}{x^3} > 0$ when $x = \sqrt{192}$.

S is a minimum when $x = y = \sqrt{192}$.

5. Let x be a positive number.

$$S = x + \frac{1}{x}$$

$\dfrac{dS}{dx} = 1 - \dfrac{1}{x^2} = 0$ when $x = 1$.

$\dfrac{d^2S}{dx^2} = \dfrac{2}{x^3} > 0$ when $x = 1$.

The sum is a minimum when $x = 1$ and $1/x = 1$.

7. Let x be the length and y the width of the rectangle.

$2x + 2y = 100$

$y = 50 - x$

$A = xy = x(50 - x)$

$\dfrac{dA}{dx} = 50 - 2x = 0$ when $x = 25$.

$\dfrac{d^2A}{dx^2} = -2 < 0$ when $x = 25$.

A is maximum when $x = y = 25$ meters.

9. Let x be the length and y the width of the rectangle.

$xy = 64$

$y = \dfrac{64}{x}$

$$P = 2x + 2y = 2x + 2\left(\frac{64}{x}\right) = 2x + \frac{128}{x}$$

$\dfrac{dP}{dx} = 2 - \dfrac{128}{x^2} = 0$ when $x = 8$.

$\dfrac{d^2P}{dx^2} = \dfrac{256}{x^3} > 0$ when $x = 8$.

P is minimum when $x = y = 8$ feet.

11. $d = \sqrt{(x-4)^2 + (\sqrt{x}-0)^2}$

$= \sqrt{x^2 - 7x + 16}$

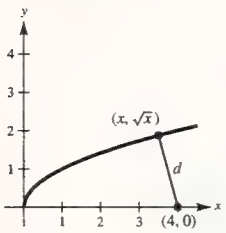
Since d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$f(x) = x^2 - 7x + 16.$

$f'(x) = 2x - 7 = 0$

$x = \frac{7}{2}$

By the First Derivative Test, the point nearest to (4, 0) is $(7/2, \sqrt{7/2})$.



13. $d = \sqrt{(x-2)^2 + [x^2 - (1/2)]^2}$

$= \sqrt{x^4 - 4x + (17/4)}$

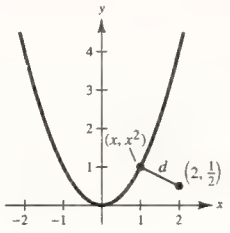
Since d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$f(x) = x^4 - 4x + \frac{17}{4}.$

$f'(x) = 4x^3 - 4 = 0$

$x = 1$

By the First Derivative Test, the point nearest to $\left(2, \frac{1}{2}\right)$ is (1, 1).



15. $\frac{dQ}{dx} = kx(Q_0 - x) = kQ_0x - kx^2$

$\frac{d^2Q}{dx^2} = kQ_0 - 2kx$

$= k(Q_0 - 2x) = 0$ when $x = \frac{Q_0}{2}$.

$\frac{d^3Q}{dx^3} = -2k < 0$ when $x = \frac{Q_0}{2}$.

dQ/dx is maximum when $x = Q_0/2$.

17. $xy = 180{,}000$ (see figure)

$S = x + 2y = \left(x + \frac{360{,}000}{x}\right)$ where S is the length of fence needed.

$\frac{dS}{dx} = 1 - \frac{360{,}000}{x^2} = 0$ when $x = 600$.

$\frac{d^2S}{dx^2} = \frac{720{,}000}{x^3} > 0$ when $x = 600$.

S is a minimum when $x = 600$ meters and $y = 300$ meters.



19. (a) $A = 4$(area of side) $+ 2$(area of Top)

(a) $A = 4(3)(11) + 2(3)(3) = 150$ square inches

(b) $A = 4(5)(5) + 2(5)(5) = 150$ square inches

(c) $A = 4(3.25)(6) + 2(6)(6) = 150$ square inches

(b) $V =$ (length)(width)(height)

(a) $V = (3)(3)(11) = 99$ cubic inches

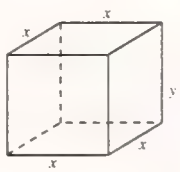
(b) $V = (5)(5)(5) = 125$ cubic inches

(c) $V = (6)(6)(3.25) = 117$ cubic inches

(c) $S = 4xy + 2x^2 = 150 \Rightarrow y = \frac{150 - 2x^2}{4x}$

$V = x^2y = x^2\left(\frac{150 - 2x^2}{4x}\right) = \frac{75}{2}x - \frac{1}{2}x^3$

$V' = \frac{75}{2} - \frac{3}{2}x^2 = 0 \Rightarrow x = \pm 5$



By the First Derivative Test, $x = 5$ yields the maximum volume. Dimensions: $5 \times 5 \times 5$. (A cube!)

21. (a) $V = x(s - 2x)^2, 0 < x < \frac{s}{2}$

$$\frac{dV}{dx} = 2x(s - 2x)(-2) + (s - 2x)^2$$

$$= (s - 2x)(s - 6x) = 0 \text{ when } x = \frac{s}{2}, \frac{s}{6} \ (s/2 \text{ is not in the domain}).$$

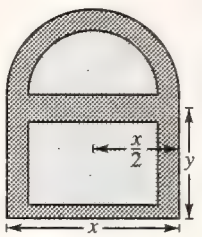
$$\frac{d^2V}{dx^2} = 24x - 8s$$

$$\frac{d^2V}{dx^2} < 0 \text{ when } x = \frac{s}{6}.$$

$$V = \frac{2s^3}{27} \text{ is maximum when } x = \frac{5}{6}.$$

(b) If the length is doubled, $V = \frac{2}{27}(2s)^3 = 8\left(\frac{2}{27}s^3\right)$. Volume is increased by a factor of 8.

23. $16 = 2y + x + \pi\left(\frac{x}{2}\right)$



$$32 = 4y + 2x + \pi x$$

$$y = \frac{32 - 2x - \pi x}{4}$$

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2 = \left(\frac{32 - 2x - \pi x}{4}\right)x + \frac{\pi x^2}{8}$$

$$= 8x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2$$

$$\frac{dA}{dx} = 8 - x - \frac{\pi}{2}x + \frac{\pi}{4}x = 8 - x\left(1 + \frac{\pi}{4}\right)$$

$$= 0 \text{ when } x = \frac{8}{1 + (\pi/4)} = \frac{32}{4 + \pi}.$$

$$\frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{4}\right) < 0 \text{ when } x = \frac{32}{4 + \pi}$$

$$y = \frac{32 - 2[32/(4 + \pi)] - \pi[32/(4 + \pi)]}{4} = \frac{16}{4 + \pi}$$

The area is maximum when $y = \frac{16}{4 + \pi}$ feet and $x = \frac{32}{4 + \pi}$ feet.

25. (a) $\frac{y - 2}{0 - 1} = \frac{0 - 2}{x - 1}$

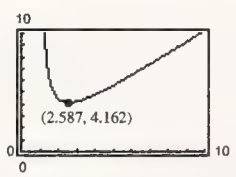
$$y = 2 + \frac{2}{x - 1}$$

$$L = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(2 + \frac{2}{x - 1}\right)^2}$$

$$= \sqrt{x^2 + 4 + \frac{8}{x - 1} + \frac{4}{(x - 1)^2}}, \quad x > 1$$

—CONTINUED—

(b)



L is minimum when $x \approx 2.587$ and $L \approx 4.162$.

25. —CONTINUED—

(c) Area $= A(x) = \frac{1}{2}xy = \frac{1}{2}x\left(2 + \frac{2}{x-1}\right) = x + \frac{x}{x-1}$

$A'(x) = 1 + \frac{(x-1) - x}{(x-1)^2} = 1 - \frac{1}{(x-1)^2} = 0$

$(x-1)^2 = 1$

$x - 1 = \pm 1$

$x = 0, 2$ (select $x = 2$)

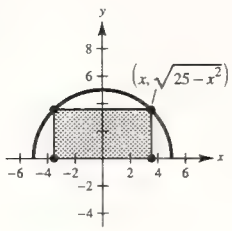
Then $y = 4$ and $A = 4$.

Vertices: $(0, 0)$, $(2, 0)$, $(0, 4)$

27. $A = 2xy = 2x\sqrt{25 - x^2}$ (see figure)

$\frac{dA}{dx} = 2x\left(\frac{1}{2}\right)\left(\frac{-2x}{\sqrt{25 - x^2}}\right) + 2\sqrt{25 - x^2}$

$= 2\left(\frac{25 - 2x^2}{\sqrt{25 - x^2}}\right) = 0$ when $x = y = \frac{5\sqrt{2}}{2} \approx 3.54$.



By the First Derivative Test, the inscribed rectangle of maximum area has vertices

$\left(\pm\frac{5\sqrt{2}}{2}, 0\right), \left(\pm\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right)$.

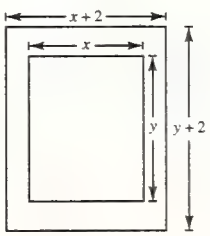
Width: $\frac{5\sqrt{2}}{2}$; Length: $5\sqrt{2}$

29. $xy = 30 \Rightarrow y = \frac{30}{x}$

$A = (x + 2)\left(\frac{30}{x} + 2\right)$ (see figure)

$\frac{dA}{dx} = (x + 2)\left(\frac{-30}{x^2}\right) + \left(\frac{30}{x} + 2\right) = \frac{2(x^2 - 30)}{x^2} = 0$ when $x = \sqrt{30}$.

$y = \frac{30}{\sqrt{30}} = \sqrt{30}$



By the First Derivative Test, the dimensions $(x + 2)$ by $(y + 2)$ are $\left(2 + \sqrt{30}\right)$ by $\left(2 + \sqrt{30}\right)$ (approximately 7.477 by 7.477). These dimensions yield a minimum area.

31. $V = \pi r^2 h = 22$ cubic inches or $h = \frac{22}{\pi r^2}$

(a)

Radius, r	Height	Surface Area
0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \frac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \frac{22}{\pi(0.4)^2}\right] \approx 111.0$
0.6	$\frac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \frac{22}{\pi(0.6)^2}\right] \approx 75.6$
0.8	$\frac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \frac{22}{\pi(0.8)^2}\right] \approx 59.0$

—CONTINUED—

31. —CONTINUED—

(b)

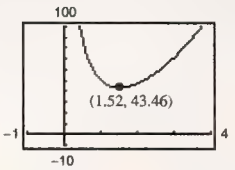
Radius, r	Height	Surface Area
0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \frac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \frac{22}{\pi(0.4)^2}\right] \approx 111.0$
0.6	$\frac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \frac{22}{\pi(0.6)^2}\right] \approx 75.6$
0.8	$\frac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \frac{22}{\pi(0.8)^2}\right] \approx 59.0$
1.0	$\frac{22}{\pi(1.0)^2}$	$2\pi(1.0)\left[1.0 + \frac{22}{\pi(1.0)^2}\right] \approx 50.3$
1.2	$\frac{22}{\pi(1.2)^2}$	$2\pi(1.2)\left[1.2 + \frac{22}{\pi(1.2)^2}\right] \approx 45.7$
1.4	$\frac{22}{\pi(1.4)^2}$	$2\pi(1.4)\left[1.4 + \frac{22}{\pi(1.4)^2}\right] \approx 43.7$
1.6	$\frac{22}{\pi(1.6)^2}$	$2\pi(1.6)\left[1.6 + \frac{22}{\pi(1.6)^2}\right] \approx 43.6$
1.8	$\frac{22}{\pi(1.8)^2}$	$2\pi(1.8)\left[1.8 + \frac{22}{\pi(1.8)^2}\right] \approx 44.8$
2.0	$\frac{22}{\pi(2.0)^2}$	$2\pi(2.0)\left[2.0 + \frac{22}{\pi(2.0)^2}\right] \approx 47.1$

The minimum seems to be about 43.6 for $r = 1.6$.

(c) $S = 2\pi r^2 + 2\pi rh$

$$= 2\pi r(r + h) = 2\pi r\left[r + \frac{22}{\pi r^2}\right] = 2\pi r^2 + \frac{44}{r}$$

(d)



The minimum seems to be 43.46 for $r \approx 1.52$.

(e) $\frac{dS}{dr} = 4\pi r - \frac{44}{r^2} = 0$ when $r = \sqrt[3]{11/\pi} \approx 1.52$ in.

$h = \frac{22}{\pi r^2} \approx 3.04$ in.

Note: Notice that

$$h = \frac{22}{\pi r^2} = \frac{22}{\pi(11/\pi)^{2/3}} = 2\left(\frac{11^{1/3}}{\pi^{1/3}}\right) = 2r.$$

33. Let x be the sides of the square ends and y the length of the package.

$P = 4x + y = 108 \Rightarrow y = 108 - 4x$

$V = x^2y = x^2(108 - 4x) = 108x^2 - 4x^3$

$\frac{dV}{dx} = 216x - 12x^2$

$= 12x(18 - x) = 0$ when $x = 18$.

$\frac{d^2V}{dx^2} = 216 - 24x = -216 < 0$ when $x = 18$.

The volume is maximum when $x = 18$ inches and $y = 108 - 4(18) = 36$ inches.

35. $V = \frac{1}{3}\pi x^2 h = \frac{1}{3}\pi x^2\left(r + \sqrt{r^2 - x^2}\right)$ (see figure)

$$\frac{dV}{dx} = \frac{1}{3}\pi\left[\frac{-x^3}{\sqrt{r^2 - x^2}} + 2x\left(r + \sqrt{r^2 - x^2}\right)\right] = \frac{\pi x}{3\sqrt{r^2 - x^2}}\left(2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2\right) = 0$$

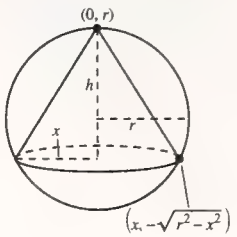
$$2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2 = 0$$

$$2r\sqrt{r^2 - x^2} = 3x^2 - 2r^2$$

$$4r^2(r^2 - x^2) = 9x^4 - 12x^2r^2 + 4r^4$$

$$0 = 9x^4 - 8x^2r^2 = x^2(9x^2 - 8r^2)$$

$$x = 0, \frac{2\sqrt{2}r}{3}$$



By the First Derivative Test, the volume is a maximum when

$$x = \frac{2\sqrt{2}r}{3} \text{ and } h = r + \sqrt{r^2 - x^2} = \frac{4r}{3}.$$

Thus, the maximum volume is

$$V = \frac{1}{3}\pi\left(\frac{8r^2}{9}\right)\left(\frac{4r}{3}\right) = \frac{32\pi r^3}{81} \text{ cubic units.}$$

37. No, there is no minimum area. If the sides are x and y, then $2x + 2y = 20 \Rightarrow y = 10 - x$. The area is $A(x) = x(10 - x) = 10x - x^2$. This can be made arbitrarily small by selecting $x \approx 0$.

39. $V = 12 = \frac{4}{3}\pi r^3 + \pi r^2 h$

$$h = \frac{12 - (4/3)\pi r^3}{\pi r^2} = \frac{12}{\pi r^2} - \frac{4}{3}r$$

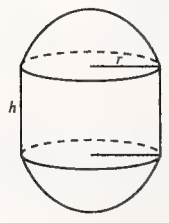
$$S = 4\pi r^2 + 2\pi rh = 4\pi r^2 + 2\pi r\left(\frac{12}{\pi r^2} - \frac{4}{3}r\right)$$

$$= 4\pi r^2 + \frac{24}{r} - \frac{8}{3}\pi r^2 = \frac{4}{3}\pi r^2 + \frac{24}{r}$$

$\frac{dS}{dr} = \frac{8}{3}\pi r - \frac{24}{r^2} = 0$ when $r = \sqrt[3]{9/\pi} \approx 1.42$ cm.

$\frac{d^2S}{dr^2} = \frac{8}{3}\pi + \frac{48}{r^3} > 0$ when $r = \sqrt[3]{9/\pi}$ cm.

The surface area is minimum when $r = \sqrt[3]{9/\pi}$ cm and $h = 0$. The resulting solid is a sphere of radius $r \approx 1.42$ cm.



41. Let x be the length of a side of the square and y the length of a side of the triangle.

$$4x + 3y = 10$$

$$A = x^2 + \frac{1}{2}y\left(\frac{\sqrt{3}}{2}y\right)$$

$$= \frac{(10 - 3y)^2}{16} + \frac{\sqrt{3}}{4}y^2$$

$$\frac{dA}{dy} = \frac{1}{8}(10 - 3y)(-3) + \frac{\sqrt{3}}{2}y = 0$$

$$-30 + 9y + 4\sqrt{3}y = 0$$

$$y = \frac{30}{9 + 4\sqrt{3}}$$

$$\frac{d^2A}{dy^2} = \frac{9 + 4\sqrt{3}}{8} > 0$$

A is minimum when

$$y = \frac{30}{9 + 4\sqrt{3}} \text{ and } x = \frac{10\sqrt{3}}{9 + 4\sqrt{3}}.$$

43. Let S be the strength and k the constant of proportionality. Given $h^2 + w^2 = 24^2$, $h^2 = 24^2 - w^2$,

$$S = kwh^2$$

$$S = kw(576 - w^2) = k(576w - w^3)$$

$$\frac{dS}{dw} = k(576 - 3w^3) = 0 \text{ when } w = 8\sqrt{3}, h = 8\sqrt{6}.$$

$$\frac{d^2S}{dw^2} = -6kw < 0 \text{ when } w = 8\sqrt{3}.$$

These values yield a maximum.

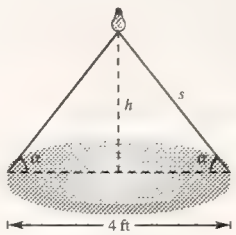
45. $R = \dfrac{v_0^2}{g}\sin 2\theta$

$$\frac{dR}{d\theta} = \frac{2v_0^2}{g}\cos 2\theta = 0 \text{ when } \theta = \frac{\pi}{4}, \frac{3\pi}{4}.$$

$$\frac{d^2R}{d\theta^2} = -\frac{4v_0^2}{g}\sin 2\theta < 0 \text{ when } \theta = \frac{\pi}{4}.$$

By the Second Derivative Test, R is maximum when $\theta = \pi/4$.

47. $\sin\alpha = \dfrac{h}{s} \Rightarrow s = \dfrac{h}{\sin\alpha}$, $0 < \alpha < \dfrac{\pi}{2}$



$$\tan\alpha = \frac{h}{2} \Rightarrow h = 2\tan\alpha \Rightarrow s = \frac{2\tan\alpha}{\sin\alpha} = 2\sec\alpha$$

$$I = \frac{k\sin\alpha}{s^2} = \frac{k\sin\alpha}{4\sec^2\alpha} = \frac{k}{4}\sin\alpha\cos^2\alpha$$

$$\frac{dI}{d\alpha} = \frac{k}{4}[\sin\alpha(-2\sin\alpha\cos\alpha) + \cos^2\alpha(\cos\alpha)]$$

$$= \frac{k}{4}\cos\alpha[\cos^2\alpha - 2\sin^2\alpha]$$

$$= \frac{k}{4}\cos\alpha[1 - 3\sin^2\alpha]$$

$$= 0 \text{ when } \alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \text{ or when } \sin\alpha = \pm\frac{1}{\sqrt{3}}.$$

Since α is acute, we have

$$\sin\alpha = \frac{1}{\sqrt{3}} \Rightarrow h = 2\tan\alpha = 2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2} \text{ feet.}$$

Since $(d^2I)/(d\alpha^2) = (k/4)\sin\alpha(9\sin^2\alpha - 7) < 0$ when $\sin\alpha = 1/\sqrt{3}$, this yields a maximum.

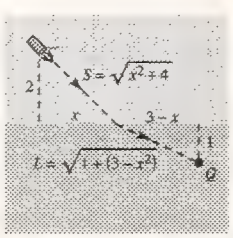
49.

$$S = \sqrt{x^2 + 4}, L = \sqrt{1 + (3 - x)^2}$$

$$\text{Time} = T = \frac{\sqrt{x^2 + 4}}{2} + \frac{\sqrt{x^2 - 6x + 10}}{4}$$

$$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2 + 4}} + \frac{x - 3}{4\sqrt{x^2 - 6x + 10}} = 0$$

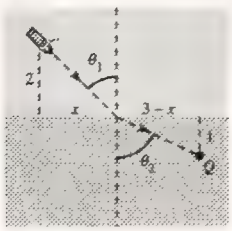
$$\frac{x^2}{x^2 + 4} = \frac{9 - 6x + x^2}{4(x^2 - 6x + 10)}$$



$$x^4 - 6x^3 + 9x^2 + 8x - 12 = 0$$

You need to find the roots of this equation in the interval $[0, 3]$. By using a computer or graphics calculator, you can determine that this equation has only one root in this interval ($x = 1$). Testing at this value and at the endpoints, you see that $x = 1$ yields the minimum time. Thus, the man should row to a point 1 mile from the nearest point on the coast.

51. $T = \dfrac{\sqrt{x^2 + 4}}{v_1} + \dfrac{\sqrt{x^2 - 6x + 10}}{v_2}$



$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2 + 4}} + \frac{x - 3}{v_2\sqrt{x^2 - 6x + 10}} = 0$$

Since

$$\frac{x}{\sqrt{x^2 + 4}} = \sin\theta_1 \text{ and } \frac{x - 3}{\sqrt{x^2 - 6x + 10}} = -\sin\theta_2$$

we have

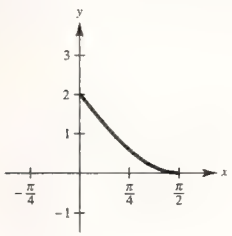
$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \Rightarrow \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

Since

$$\frac{d^2T}{dx^2} = \frac{4}{v_1(x^2 + 4)^{3/2}} + \frac{1}{v_2(x^2 - 6x + 10)^{3/2}} > 0$$

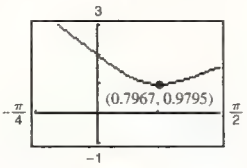
this condition yields a minimum time.

53. $f(x) = 2 - 2\sin x$



(a) Distance from origin to y-intercept is 2.
Distance from origin to x-intercept is $\pi/2 \approx 1.57$.

(b) $d = \sqrt{x^2 + y^2} = \sqrt{x^2 + (2 - 2\sin x)^2}$



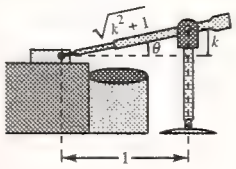
Minimum distance = 0.9795 at $x = 0.7967$.

(c) Let $f(x) = d^2(x) = x^2 + (2 - 2\sin x)^2$.

$f'(x) = 2x + 2(2 - 2\sin x)(-2\cos x)$

Setting $f'(x) = 0$, you obtain $x \approx 0.7967$, which corresponds to $d = 0.9795$.

55. $F\cos\theta = k(W - F\sin\theta)$



$$F = \frac{kW}{\cos\theta + k\sin\theta}$$

$$\frac{dF}{d\theta} = \frac{-kW(k\cos\theta - \sin\theta)}{(\cos\theta + k\sin\theta)^2} = 0$$

$k\cos\theta = \sin\theta \Rightarrow k = \tan\theta \Rightarrow \theta = \arctan k$

Since

$$\cos\theta + k\sin\theta = \frac{1}{\sqrt{k^2 + 1}} + \frac{k^2}{\sqrt{k^2 + 1}} = \sqrt{k^2 + 1},$$

the minimum force is

$$F = \frac{kW}{\cos\theta + k\sin\theta} = \frac{kW}{\sqrt{k^2 + 1}}.$$

57. (a)

Base 1	Base 2	Altitude	Area
8	$8 + 16\cos 10°$	$8\sin 10°$	≈ 22.1
8	$8 + 16\cos 20°$	$8\sin 20°$	≈ 42.5
8	$8 + 16\cos 30°$	$8\sin 30°$	≈ 59.7
8	$8 + 16\cos 40°$	$8\sin 40°$	≈ 72.7
8	$8 + 16\cos 50°$	$8\sin 50°$	≈ 80.5
8	$8 + 16\cos 60°$	$8\sin 60°$	≈ 83.1

(b)

Base 1	Base 2	Altitude	Area
8	$8 + 16\cos 10°$	$8\sin 10°$	≈ 22.1
8	$8 + 16\cos 20°$	$8\sin 20°$	≈ 42.5
8	$8 + 16\cos 30°$	$8\sin 30°$	≈ 59.7
8	$8 + 16\cos 40°$	$8\sin 40°$	≈ 72.7
8	$8 + 16\cos 50°$	$8\sin 50°$	≈ 80.5
8	$8 + 16\cos 60°$	$8\sin 60°$	≈ 83.1
8	$8 + 16\cos 70°$	$8\sin 70°$	≈ 80.7
8	$8 + 16\cos 80°$	$8\sin 80°$	≈ 74.0
8	$8 + 16\cos 90°$	$8\sin 90°$	≈ 64.0

The maximum cross-sectional area is approximately 83.1 square feet.

(c) $A = (a + b)\dfrac{h}{2}$

$= [8 + (8 + 16\cos\theta)]\dfrac{8\sin\theta}{2}$

$= 64(1 + \cos\theta)\sin\theta, 0° < \theta < 90°$

(d) $\dfrac{dA}{d\theta} = 64(1 + \cos\theta)\cos\theta + (-64\sin\theta)\sin\theta$

$= 64(\cos\theta + \cos^2\theta - \sin^2\theta)$

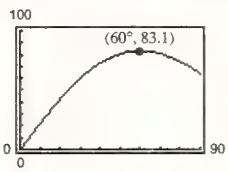
$= 64(2\cos^2\theta + \cos\theta - 1)$

$= 64(2\cos\theta - 1)(\cos\theta + 1)$

$= 0$ when $\theta = 60°, 180°, 300°$.

The maximum occurs when $\theta = 60°$.

(e)



59. $C = 100\left(\dfrac{200}{x^2} + \dfrac{x}{x + 30}\right), 1 \le x$

$C' = 100\left(-\dfrac{400}{x^3} + \dfrac{30}{(x + 30)^2}\right)$

Approximation: $x \approx 40.45$ units, or 4045 units

61. $S_1 = (4m - 1)^2 + (5m - 6)^2 + (10m - 3)^2$

$\dfrac{dS_1}{dm} = 2(4m - 1)(4) + 2(5m - 6)(5) + 2(10m - 3)(10) = 282m - 128 = 0$ when $m = \dfrac{64}{141}$.

Line: $y = \dfrac{64}{141}x$

$S = \left|4\left(\dfrac{64}{141}\right) - 1\right| + \left|5\left(\dfrac{64}{141}\right) - 6\right| + \left|10\left(\dfrac{64}{141}\right) - 3\right|$

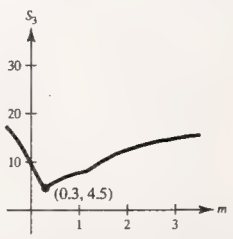
$= \left|\dfrac{256}{141} - 1\right| + \left|\dfrac{320}{141} - 6\right| + \left|\dfrac{640}{141} - 3\right| = \dfrac{858}{141} \approx 6.1$ mi

63. $S_3 = \dfrac{|4m - 1|}{\sqrt{m^2 + 1}} + \dfrac{|5m - 6|}{\sqrt{m^2 + 1}} + \dfrac{|10m - 3|}{\sqrt{m^2 + 1}}$

Using a graphing utility, you can see that the minimum occurs when $x \approx 0.3$.

Line: $y \approx 0.3x$

$S_3 = \dfrac{|4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3|}{\sqrt{(0.3)^2 + 1}} \approx 4.5$ mi.



Section 3.8 Newton's Method

1. $f(x) = x^2 - 3$

$f'(x) = 2x$

$x_1 = 1.7$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.7000	−0.1100	3.4000	−0.0324	1.7324
2	1.7324	0.0012	3.4648	0.0003	1.7321

3. $f(x) = \sin x$

$f'(x) = \cos x$

$x_1 = 3$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.0000	0.1411	−0.9900	−0.1425	3.1425
2	3.1425	−0.0009	−1.0000	0.0009	3.1416

5. $f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1$

Approximation of the zero of f is 0.682.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	−0.3750	1.7500	−0.2143	0.7143
2	0.7143	0.0788	2.5307	0.0311	0.6832
3	0.6832	0.0021	2.4003	0.0009	0.6823

7. $f(x) = 3\sqrt{x-1} - x$

$f'(x) = \dfrac{3}{2\sqrt{x-1}} - 1$

Approximation of the zero of f is 1.146.

Similarly, the other zero is approximately 7.854.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.2000	0.1416	2.3541	0.0602	1.1398
2	1.1398	−0.0181	3.0118	−0.0060	1.1458
3	1.1458	−0.0003	2.9284	−0.0001	1.1459

9. $f(x) = x^3 + 3$

$f'(x) = 3x^2$

Approximation of the zero of f is −1.442.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	−1.5000	−0.3750	6.7500	−0.0556	−1.4444
2	−1.4444	−0.0134	6.2589	−0.0021	−1.4423
3	−1.4423	−0.0003	6.2407	−0.0001	−1.4422

11. $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$

$f'(x) = 3x^2 - 7.8x + 4.79$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	−0.3360	1.6400	−0.2049	0.7049
2	0.7049	−0.0921	0.7824	−0.1177	0.8226
3	0.8226	−0.0231	0.4037	−0.0573	0.8799
4	0.8799	−0.0045	0.2495	−0.0181	0.8980
5	0.8980	−0.0004	0.2048	−0.0020	0.9000
6	0.9000	0.0000	0.2000	0.0000	0.9000

Approximation of the zero of f is 0.900.

—CONTINUED—

11. —CONTINUED—

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.1	0.0000	−0.1600	−0.0000	1.1000

Approximation of the zero of f is 1.100.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.9	0.0000	0.8000	0.0000	1.9000

Approximation of the zero of f is 1.900.

13. $f(x) = x + \sin(x + 1)$

$f'(x) = 1 + \cos(x + 1)$

Approximation of the zero of f is −0.489.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	−0.5000	−0.0206	1.8776	−0.0110	−0.4890
2	−0.4890	0.0000	1.8723	0.0000	−0.4890

15. $h(x) = f(x) - g(x) = 2x + 1 - \sqrt{x + 4}$

$h'(x) = 2 - \dfrac{1}{2\sqrt{x + 4}}$

Point of intersection of the graphs of f and g occurs when $x \approx 0.569$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	0.6000	0.0552	1.7669	0.0313	0.5687
2	0.5687	−0.0001	1.7661	0.0000	0.5687

17. $h(x) = f(x) - g(x) = x - \tan x$

$h'(x) = 1 - \sec^2 x$

Point of intersection of the graphs of f and g occurs when $x \approx 4.493$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	4.5000	−0.1373	−21.5048	0.0064	4.4936
2	4.4936	−0.0039	−20.2271	0.0002	4.4934

19. $f(x) = x^2 - a = 0$

$f'(x) = 2x$

$$x_{i+1} = x_i - \frac{x_i^2 - a}{2x_i}$$

$$= \frac{2x_i^2 - x_i^2 + a}{2x_i} = \frac{x_i^2 + a}{2x_i} = \frac{x_i}{2} + \frac{a}{2x_i}$$

21. $x_{i+1} = \dfrac{x_i^2 + 7}{2x_i}$

i	1	2	3	4	5
x_i	2.0000	2.7500	2.6477	2.6458	2.6458

$\sqrt{7} \approx 2.646$

23. $x_{i+1} = \dfrac{3x_i^4 + 6}{4x_i^3}$

i	1	2	3	4
x_i	1.5000	1.5694	1.5651	1.5651

$\sqrt[4]{6} \approx 1.565$

25. $f(x) = 1 + \cos x$

$f'(x) = -\sin x$

Approximation of the zero: 3.141

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.0000	0.0100	−0.1411	−0.0709	3.0709
2	3.0709	0.0025	−0.0706	−0.0354	3.1063
3	3.1063	0.0006	−0.0353	−0.0176	3.1239
4	3.1239	0.0002	−0.0177	−0.0088	3.1327
5	3.1327	0.0000	−0.0089	−0.0044	3.1371
6	3.1371	0.0000	−0.0045	−0.0022	3.1393
7	3.1393	0.0000	−0.0023	−0.0011	3.1404
8	3.1404	0.0000	−0.0012	−0.0006	3.1410

27. $y = 2x^3 - 6x^2 + 6x - 1 = f(x)$

$y' = 6x^2 - 12x + 6 = f'(x)$

$x_1 = 1$

$f'(x) = 0$; therefore, the method fails.

n	x_n	$f(x_n)$	$f'(x_n)$
1	1	1	0

29. $y = -x^3 + 6x^2 - 10x + 6 = f(x)$

$y' = -3x^2 + 12x - 10 = f'(x)$

$x_1 = 2$

$x_2 = 1$

$x_3 = 2$

$x_4 = 1$ and so on.

Fails to converge

31. Answers will vary. See page 222.

Newton's Method uses tangent lines to approximate c such that $f(c) = 0$.

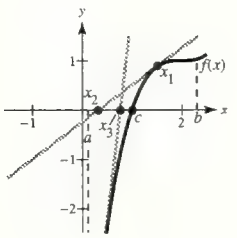
First, estimate an initial x_1 close to c (see graph).

Then determine x_2 by $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

Calculate a third estimate by $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$.

Continue this process until $|x_n - x_{n+1}|$ is within the desired accuracy.

Let x_{n+1} be the final approximation of c.



33. Let $g(x) = f(x) - x = \cos x - x$

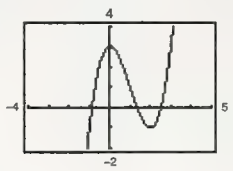
$g'(x) = -\sin x - 1$.

The fixed point is approximately 0.74.

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	1.0000	−0.4597	−1.8415	0.2496	0.7504
2	0.7504	−0.0190	−1.6819	0.0113	0.7391
3	0.7391	0.0000	−1.6736	0.0000	0.7391

35. $f(x) = x^3 - 3x^2 + 3$, $f'(x) = 3x^2 - 6x$

(a)



(b) $x_1 = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.333$$

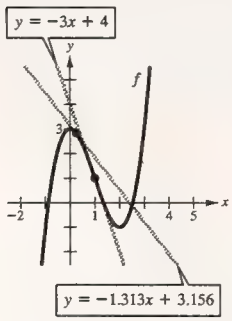
Continuing, the zero is 1.347.

(c) $x_1 = \frac{1}{4}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 2.405$$

Continuing, the zero is 2.532.

(d)



The x-intercepts correspond to the values resulting from the first iteration of Newton's Method.

(e) If the initial guess x_1 is not "close to" the desired zero of the function, the x-intercept of the tangent line may approximate another zero of the function.

37. $f(x) = \frac{1}{x} - a = 0$

$$f'(x) = -\frac{1}{x^2}$$

$$x_{n+1} = x_n - \frac{(1/x_n) - a}{-1/x_n^2} = x_n + x_n^2\left(\frac{1}{x_n} - a\right) = x_n + x_n - x_n^2 a = 2x_n - x_n^2 a = x_n(2 - ax_n)$$

39. $f(x) = x\cos x$, $(0, \pi)$

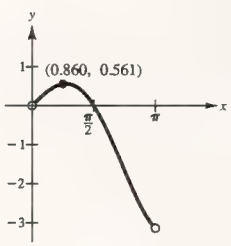
$f'(x) = -x\sin x + \cos x = 0$

Letting $F(x) = f'(x)$, we can use Newton's Method as follows.

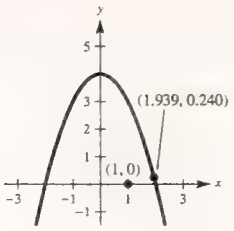
$[F'(x) = -2\sin x + x\cos x]$

n	x_n	$F(x_n)$	$F'(x_n)$	$\frac{F(x_n)}{F'(x_n)}$	$x_n - \frac{F(x_n)}{F'(x_n)}$
1	0.9000	−0.0834	−2.1261	0.0392	0.8608
2	0.8608	−0.0010	−2.0778	0.0005	0.8603

Approximation to the critical number: 0.860



41. $y = f(x) = 4 - x^2, (1, 0)$



$d = \sqrt{(x-1)^2 + (y-0)^2} = \sqrt{(x-1)^2 + (4-x^2)^2} = \sqrt{x^4 - 7x^2 - 2x + 17}$

d is minimized when $D = x^4 - 7x^2 - 2x + 17$ is a minimum.

$g(x) = D' = 4x^3 - 14x - 2$

$g'(x) = 12x^2 - 14$

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	2.0000	2.0000	34.0000	0.0588	1.9412
2	1.9412	0.0830	31.2191	0.0027	1.9385
3	1.9385	−0.0012	31.0934	0.0000	1.9385

$x \approx 1.939$

Point closest to (1, 0) is ≈ (1.939, 0.240).

43. Minimize: $T = \dfrac{\text{Distance rowed}}{\text{Rate rowed}} + \dfrac{\text{Distance walked}}{\text{Rate walked}}$

$$T = \frac{\sqrt{x^2+4}}{3} + \frac{\sqrt{x^2-6x+10}}{4}$$

$$T' = \frac{x}{3\sqrt{x^2+4}} + \frac{x-3}{4\sqrt{x^2-6x+10}} = 0$$

$$4x\sqrt{x^2-6x+10} = -3(x-3)\sqrt{x^2+4}$$

$$16x^2(x^2-6x+10) = 9(x-3)^2(x^2+4)$$

$7x^4 - 42x^3 + 43x^2 + 216x - 324 = 0$

Let $f(x) = 7x^4 - 42x^3 + 43x^2 + 216x - 324$ and $f'(x) = 28x^3 - 126x^2 + 86x + 216$. Since $f(1) = -100$ and $f(2) = 56$, the solution is in the interval (1, 2).

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.7000	19.5887	135.6240	0.1444	1.5556
2	1.5556	−1.0480	150.2780	−0.0070	1.5626
3	1.5626	0.0014	49.5591	0.0000	1.5626

Approximation: $x \approx 1.563$ miles

45. $2{,}500{,}000 = -76x^3 + 4830x^2 - 320{,}000$

$76x^3 - 4830x^2 + 2{,}820{,}000 = 0$

Let $f(x) = 76x^3 - 4830x^2 + 2{,}820{,}000$

$f'(x) = 228x^2 - 9660x.$

From the graph, choose $x_1 = 40$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	40.0000	−44000.0000	−21600.0000	2.0370	37.9630
2	37.9630	17157.6209	−38131.4039	−0.4500	38.4130
3	38.4130	780.0914	−34642.2263	−0.0225	38.4355
4	38.4355	2.6308	−34465.3435	−0.0001	38.4356

The zero occurs when $x \approx 38.4356$ which corresponds to $384,356.

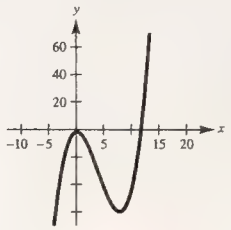
47. False. Let $f(x) = (x^2 - 1)/(x - 1)$. $x = 1$ is a discontinuity. It is not a zero of $f(x)$. This statement would be true if $f(x) = p(x)/q(x)$ is given in **reduced** form.

49. True

51. $f(x) = \frac{1}{4}x^3 - 3x^2 + \frac{3}{4}x - 2$

$f'(x) = \frac{3}{4}x^2 - 6x + \frac{3}{4}$

Let $x_1 = 12$.



n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	12.0000	7.0000	36.7500	0.1905	11.8095
2	11.8095	0.2151	34.4912	0.0062	11.8033
3	11.8033	0.0015	34.4186	0.0000	11.8033

Approximation: $x \approx 11.803$

Section 3.9 Differentials

1. $f(x) = x^2$

$f'(x) = 2x$

Tangent line at (2, 4): $y - f(2) = f'(2)(x - 2)$

$y - 4 = 4(x - 2)$

$y = 4x - 4$

x	1.9	1.99	2	2.01	2.1
$f(x) = x^2$	3.6100	3.9601	4	4.0401	4.4100
$T(x) = 4x - 4$	3.6000	3.9600	4	4.0400	4.4000

3. $f(x) = x^5$

$f'(x) = 5x^4$

Tangent line at (2, 32): $y - f(2) = f'(2)(x - 2)$

$y - 32 = 80(x - 2)$

$y = 80x - 128$

x	1.9	1.99	2	2.01	2.1
$f(x) = x^5$	24.7610	31.2080	32	32.8080	40.8410
$T(x) = 80x - 128$	24.0000	31.2000	32	32.8000	40.0000

5. $f(x) = \sin x$

$f'(x) = \cos x$

Tangent line at (2, sin 2):

$y - f(2) = f'(2)(x - 2)$

$y - \sin 2 = (\cos 2)(x - 2)$

$y = (\cos 2)(x - 2) + \sin 2$

x	1.9	1.99	2	2.01	2.1
$f(x) = \sin x$	0.9463	0.9134	0.9093	0.9051	0.8632
$T(x) = (\cos 2)(x - 2) + \sin 2$	0.9509	0.9135	0.9093	0.9051	0.8677

7. $y = f(x) = \frac{1}{2}x^3, f'(x) = \frac{3}{2}x^2, x = 2, \Delta x = dx = 0.1$

$\Delta y = f(x + \Delta x) - f(x)$

$= f(2.1) - f(2)$

$= 0.6305$

$dy = f'(x)dx$

$= f'(2)(0.1)$

$= 6(0.1) = 0.6$

9. $y = f(x) = x^4 + 1, f'(x) = 4x^3, x = -1, \Delta x = dx = 0.01$

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x)\\ &= f(-0.99) - f(-1)\\ &= [(-0.99)^4 + 1] - [(-1)^4 + 1] \approx -0.0394\end{aligned}$$

$$\begin{aligned}dy &= f'(x)\,dx\\ &= f'(-1)(0.01)\\ &= (-4)(0.01) = -0.04\end{aligned}$$

11. $y = 3x^2 - 4$

$dy = 6x\,dx$

13. $y = \dfrac{x + 1}{2x - 1}$

$dy = \dfrac{-3}{(2x - 1)^2}\,dx$

15. $y = x\sqrt{1 - x^2}$

$dy = \left(x\dfrac{-x}{\sqrt{1 - x^2}} + \sqrt{1 - x^2}\right)dx = \dfrac{1 - 2x^2}{\sqrt{1 - x^2}}\,dx$

17. $y = 2x - \cot^2 x$

$dy = (2 + 2\cot x \csc^2 x)dx$

$= (2 + 2\cot x + 2\cot^3 x)dx$

19. $y = \dfrac{1}{3}\cos\left(\dfrac{6\pi x - 1}{2}\right)$

$dy = -\pi \sin\left(\dfrac{6\pi x - 1}{2}\right)dx$

21. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$\approx 1 + (1)(-0.1) = 0.9$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$\approx 1 + (1)(0.04) = 1.04$

23. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$\approx 1 + \left(-\frac{1}{2}\right)(-0.1) = 1.05$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$\approx 1 + \left(-\frac{1}{2}\right)(0.04) = 0.98$

25. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$\approx 8 + \left(-\frac{1}{2}\right)(-0.07) = 8.035$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$\approx 8 + \left(-\frac{1}{2}\right)(0.1) = 7.95$

27. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$\approx 8 + 0(-0.07) = 8$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$\approx 8 + 0(0.1) = 8$

29. $A = x^2$

$x = 12$

$\Delta x = dx = \pm\frac{1}{64}$

$dA = 2x\,dx$

$\Delta A \approx dA = 2(12)\left(\pm\frac{1}{64}\right)$

$= \pm\frac{3}{8}$ square inches

31. $A = \pi r^2$

$r = 14$

$\Delta r = dr = \pm\frac{1}{4}$

$\Delta A \approx dA = 2\pi r\,dr = \pi(28)\left(\pm\frac{1}{4}\right)$

$= \pm 7\pi$ square inches

33. (a) $x = 15$ centimeter

$\Delta x = dx = \pm 0.05$ centimeters

$A = x^2$

$dA = 2x\,dx = 2(15)(\pm 0.05)$.

$= \pm 1.5$ square centimeters

Percentage error:

$$\frac{dA}{A} = \frac{\pm 1.5}{(15)^2} = 0.00666\ldots = \frac{2}{3}\%$$

(b) $$\frac{dA}{A} = \frac{2x\,dx}{x^2} = \frac{2\,dx}{x} \leq 0.025$$

$$\frac{dx}{x} \leq \frac{0.025}{2} = 0.0125 = 1.25\%$$

35. $r = 6$ inches

$\Delta r = dr = \pm 0.02$ inches

(a) $V = \frac{4}{3}\pi r^3$

$dV = 4\pi r^2\,dr = 4\pi(6)^2(\pm 0.02) = \pm 2.88\pi$ cubic inches

(b) $S = 4\pi r^2$

$dS = 8\pi r\,dr = 8\pi(6)(\pm 0.02) = \pm 0.96\pi$ square inches

(c) Relative error: $$\frac{dV}{V} = \frac{4\pi r^2\,dr}{(4/3)\pi r^3} = \frac{3dr}{r}$$

$$= \frac{3}{6}(0.02) = 0.01 = 1\%$$

Relative error: $$\frac{dS}{S} = \frac{8\pi r\,dr}{4\pi r^2} = \frac{2dr}{r}$$

$$= \frac{2(0.02)}{6} = 0.000666\ldots = \frac{2}{3}\%$$

37. $V = \pi r^2 h = 40\pi r^2$, $r = 5$ cm, $h = 40$ cm, $dr = 0.2$ cm

$\Delta V \approx dV = 80\pi r\,dr = 80\pi(5)(0.2) = 80\pi$ cm^3

39. (a) $T = 2\pi\sqrt{L/g}$

$$dT = \frac{\pi}{g\sqrt{L/g}}\,dL$$

Relative error:

$$\frac{dT}{T} = \frac{(\pi\,dL)/\left(g\sqrt{L/g}\right)}{2\pi\sqrt{L/g}}$$

$$= \frac{dL}{2L}$$

$$= \frac{1}{2}(\text{relative error in } L)$$

$$= \frac{1}{2}(0.005) = 0.0025$$

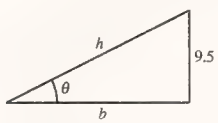
Percentage error: $\frac{dT}{T}(100) = 0.25\% = \frac{1}{4}\%$

(b) $(0.0025)(3600)(24) = 216$ seconds

$= 3.6$ minutes

41. $\theta = 26°45' = 26.75°$

$d\theta = \pm 15' = \pm 0.25°$



(a) $h = 9.5 \csc\theta$

$dh = -9.5 \csc\theta \cot\theta\,d\theta$

$$\frac{dh}{h} = -\cot\theta\,d\theta$$

$$\left|\frac{dh}{h}\right| = (\cot 26.75°)(0.25°)$$

Converting to radians, $(\cot 0.4669)(0.0044)$
$\approx 0.0087 = 0.87\%$ (in radians).

(b) $$\left|\frac{dh}{h}\right| = \cot\theta\,d\theta \leq 0.02$$

$$\frac{d\theta}{\theta} \leq \frac{0.02}{\theta(\cot\theta)} = \frac{0.02\tan\theta}{\theta}$$

$$\frac{d\theta}{\theta} \leq \frac{0.02\tan 26.75°}{26.75°} \approx \frac{0.02\tan 0.4669}{0.4669}$$

$\approx 0.0216 = 2.16\%$ (in radians)

43. $r = \frac{v_0^2}{32}(\sin 2\theta)$

$v_0 = 2200$ ft/sec

θ changes from 10° to 11°

$dr = \frac{(2200)^2}{16}(\cos 2\theta)\,d\theta$

$\theta = 10\left(\frac{\pi}{180}\right)$

$d\theta = (11 - 10)\frac{\pi}{180}$

$\Delta r \approx dr$

$= \frac{(2200)^2}{16}\cos\left(\frac{20\pi}{180}\right)\left(\frac{\pi}{180}\right) \approx 4961$ feet

≈ 4961 feet

45. Let $f(x) = \sqrt{x}, x = 100, dx = -0.6$.

$f(x + \Delta x) \approx f(x) + f'(x)\,dx$

$= \sqrt{x} + \frac{1}{2\sqrt{x}}\,dx$

$f(x + \Delta x) = \sqrt{99.4}$

$\approx \sqrt{100} + \frac{1}{2\sqrt{100}}(-0.6) = 9.97$

Using a calculator: $\sqrt{99.4} \approx 9.96995$

47. Let $f(x) = \sqrt[4]{x}, x = 625, dx = -1$.

$f(x + \Delta x) \approx f(x) + f'(x)\,dx = \sqrt[4]{x} + \frac{1}{4^4\sqrt{x^3}}\,dx$

$f(x + \Delta x) = \sqrt[4]{624} \approx \sqrt[4]{625} + \frac{1}{4\left(\sqrt[4]{625}\right)^3}(-1)$

$= 5 - \frac{1}{500} = 4.998$

Using a calculator, $\sqrt[4]{624} \approx 4.9980$.

49. Let $f(x) = \sqrt{x}, x = 4, dx = 0.02, f'(x) = 1/\left(2\sqrt{x}\right)$.

Then

$f(4.02) \approx f(4) + f'(4)\,dx$

$\sqrt{4.02} \approx \sqrt{4} + \frac{1}{2\sqrt{4}}(0.02) = 2 + \frac{1}{4}(0.02)$.

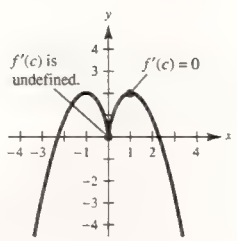
51. In general, when $\Delta x \to 0$, dy approaches Δy.

53. True

55. True

Review Exercises for Chapter 3

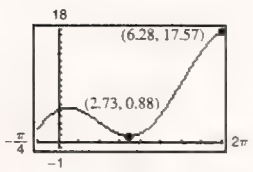
1. A number c in the domain of f is a critical number if $f'(c) = 0$ or f' is undefined at c.



3. $g(x) = 2x + 5\cos x, [0, 2\pi]$

$g'(x) = 2 - 5\sin x$

$= 0$ when $\sin x = \frac{2}{5}$.



Critical numbers: $x \approx 0.41, x \approx 2.73$

Left endpoint: (0, 5)

Critical number: (0.41, 5.41)

Critical number: (2.73, 0.88) Minimum

Right endpoint: $(2\pi, 17.57)$ Maximum

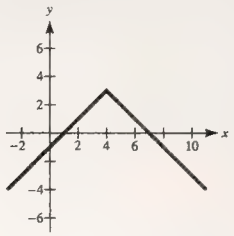
5. Yes. $f(-3) = f(2) = 0$. f is continuous on $[-3, 2]$, differentiable on $(-3, 2)$.

$f'(x) = (x + 3)(3x - 1) = 0$ for $x = \frac{1}{3}$.

$c = \frac{1}{3}$ satisfies $f'(c) - 0$.

7. $f(x) = 3 - |x - 4|$

(a)



$f(1) = f(7) = 0$

(b) f is not differentiable at $x = 4$.

9. $f(x) = x^{2/3}, 1 \le x \le 8$

$$f'(x) = \frac{2}{3}x^{-1/3}$$

$$\frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{8 - 1} = \frac{3}{7}$$

$$f'(c) = \frac{2}{3}c^{-1/3} = \frac{3}{7}$$

$$c = \left(\frac{14}{9}\right)^3 = \frac{2744}{729} \approx 3.764$$

11. $f(x) = x - \cos x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$$f'(x) = 1 + \sin x$$

$$\frac{f(b) - f(a)}{b - a} = \frac{(\pi/2) - (-\pi/2)}{(\pi/2) - (-\pi/2)} = 1$$

$$f'(c) = 1 + \sin c = 1$$

$$c = 0$$

13. $f(x) = Ax^2 + Bx + C$

$$f'(x) = 2Ax + B$$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{A(x_2^2 - x_1^2) + B(x_2 - x_1)}{x_2 - x_1}$$

$$= A(x_1 + x_2) + B$$

$$f'(c) = 2Ac + B = A(x_1 + x_2) + B$$

$$2Ac = A(x_1 + x_2)$$

$$c = \frac{x_1 + x_2}{2} = \text{Midpoint of } [x_1, x_2]$$

15. $f(x) = (x - 1)^2(x - 3)$

$$f'(x) = (x - 1)^2(1) + (x - 3)(2)(x - 1)$$

$$= (x - 1)(3x - 7)$$

Critical numbers: $x = 1$ and $x = \frac{7}{3}$

Interval:	$-\infty < x < 1$	$1 < x < \frac{7}{3}$	$\frac{7}{3} < x < \infty$
Sign of $f'(x)$:	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion:	Increasing	Decreasing	Increasing

17. $h(x) = \sqrt{x}(x - 3) = x^{3/2} - 3x^{1/2}$

Domain: $(0, \infty)$

$$h'(x) = \frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}$$

$$= \frac{3}{2}x^{-1/2}(x - 1) = \frac{3(x - 1)}{2\sqrt{x}}$$

Critical number: $x = 1$

Interval:	$0 < x < 1$	$1 < x < \infty$
Sign of $h'(x)$:	$h'(x) < 0$	$h'(x) > 0$
Conclusion:	Decreasing	Increasing

19. $h(t) = \frac{1}{4}t^4 - 8t$

$h'(t) = t^3 - 8 = 0$ when $t = 2$.

Relative minimum: $(2, -12)$

Test Interval:	$-\infty < t < 2$	$2 < t < \infty$
Sign of $h'(t)$:	$h'(t) < 0$	$h'(t) > 0$
Conclusion:	Decreasing	Increasing

21. $y = \frac{1}{3}\cos(12t) - \frac{1}{4}\sin(12t)$

$v = y' = -4\sin(12t) - 3\cos(12t)$

(a) When $t = \frac{\pi}{8}$, $y = \frac{1}{4}$ inch and $v = y' = 4$ inches/second.

(b) $y' = -4\sin(12t) - 3\cos(12t) = 0$ when $\frac{\sin(12t)}{\cos(12t)} = -\frac{3}{4} \Rightarrow \tan(12t) = -\frac{3}{4}$.

Therefore, $\sin(12t) = -\frac{3}{5}$ and $\cos(12t) = \frac{4}{5}$. The maximum displacement is

$$y = \left(\frac{1}{3}\right)\left(\frac{4}{5}\right) - \frac{1}{4}\left(-\frac{3}{5}\right) = \frac{5}{12} \text{ inch.}$$

(c) Period: $\frac{2\pi}{12} = \frac{\pi}{6}$

Frequency: $\frac{1}{\pi/6} = \frac{6}{\pi}$

23. $f(x) = x + \cos x,\ 0 \le x \le 2\pi$

$f'(x) = 1 - \sin x$

$f''(x) = -\cos x = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points of inflection: $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

Test Interval:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f''(x)$:	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

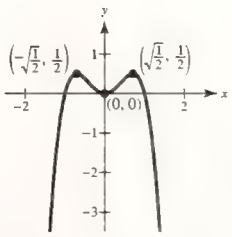
25. $g(x) = 2x^2(1 - x^2)$

$g'(x) = -4x(2x^2 - 1)$ Critical numbers: $x = 0, \pm\frac{1}{\sqrt{2}}$

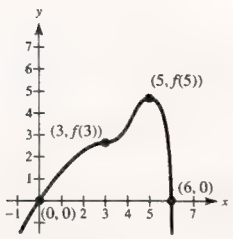
$g''(x) = 4 - 24x^2$

$g''(0) = 4 > 0$ Relative minimum at $(0, 0)$

$g''\left(\pm\frac{1}{\sqrt{2}}\right) = -8 < 0$ Relative maximums at $\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$



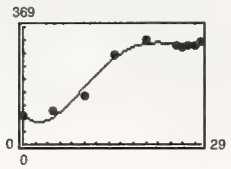
27.



29. The first derivative is positive and the second derivative is negative. The graph is increasing and is concave down.

31. (a) $D = 0.0034t^4 - 0.2352t^3 + 4.9423t^2 - 20.8641t + 94.4025$

(b)



(c) Maximum at (21.9, 319.5) ($\approx$1992)

Minimum at (2.6, 69.6) ($\approx$1972)

(d) Outlays increasing at greatest rate at the point of inflection (9.8, 173.7) ($\approx$1979)

33. $\lim_{x\to\infty} \dfrac{2x^2}{3x^2+5} = \lim_{x\to\infty} \dfrac{2}{3+5/x^2} = \dfrac{2}{3}$

35. $\lim_{x\to\infty} \dfrac{5\cos x}{x} = 0$, since $|5\cos x| \le 5$.

37. $h(x) = \dfrac{2x+3}{x-4}$

Discontinuity: $x = 4$

$$\lim_{x\to\infty} \frac{2x+3}{x-4} = \lim_{x\to\infty} \frac{2+(3/x)}{1-(4/x)} = 2$$

Vertical asymptote: $x = 4$

Horizontal asymptote: $y = 2$

39. $f(x) = \dfrac{3}{x} - 2$

Discontinuity: $x = 0$

$$\lim_{x\to\infty} \left(\frac{3}{x} - 2\right) = -2$$

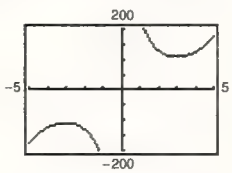
Vertical asymptote: $x = 0$

Horizontal asymptote: $y = -2$

41. $f(x) = x^3 + \dfrac{243}{x}$

Relative minimum: (3, 108)

Relative maximum: $(-3, -108)$

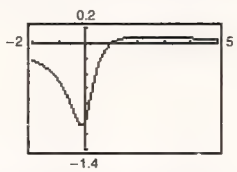


Vertical asymptote: $x = 0$

43. $f(x) = \dfrac{x-1}{1+3x^2}$

Relative minimum: $(-0.155, -1.077)$

Relative maximum: (2.155, 0.077)



Horizontal asymptote: $y = 0$

45. $f(x) = 4x - x^2 = x(4-x)$

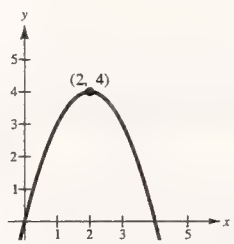
Domain: $(-\infty, \infty)$; Range: $(-\infty, 4)$

$f'(x) = 4 - 2x = 0$ when $x = 2$.

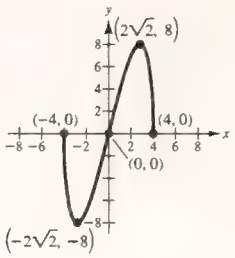
$f''(x) = -2$

Therefore, (2, 4) is a relative maximum.

Intercepts: (0, 0), (4, 0)



47. $f(x) = x\sqrt{16 - x^2}$, Domain: $[-4, 4]$, Range: $[-8, 8]$



Domain: $[-4, 4]$; Range: $[-8, 8]$

$f'(x) = \dfrac{16 - 2x^2}{\sqrt{16 - x^2}} = 0$ when $x = \pm 2\sqrt{2}$ and undefined when $x = \pm 4$.

$f''(x) = \dfrac{2x(x^2 - 24)}{(16 - x^2)^{3/2}}$

$f'(-2\sqrt{2}) > 0$

Therefore, $(-2\sqrt{2}, -8)$ is a relative minimum.

$f''(2\sqrt{2}) < 0$

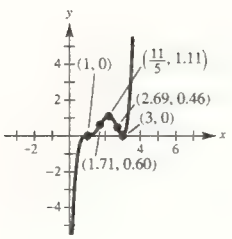
Therefore, $(2\sqrt{2}, 8)$ is a relative maximum.

Point of inflection: $(0, 0)$

Intercepts: $(-4, 0)$, $(0, 0)$, $(4, 0)$

Symmetry with respect to origin

49. $f(x) = (x - 1)^3(x - 3)^2$



Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

$f'(x) = (x - 1)^2(x - 3)(5x - 11) = 0$ when $x = 1, \dfrac{11}{5}, 3$.

$f''(x) = 4(x - 1)(5x^2 - 22x + 23) = 0$ when $x = 1, \dfrac{11 \pm \sqrt{6}}{5}$.

$f''(3) > 0$

Therefore, $(3, 0)$ is a relative minimum.

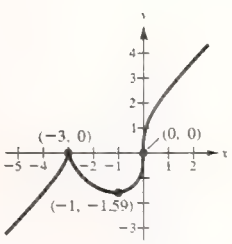
$f''\left(\dfrac{11}{5}\right) < 0$

Therefore, $\left(\dfrac{11}{5}, \dfrac{3456}{3125}\right)$ is a relative maximum.

Points of inflection: $(1, 0)$, $\left(\dfrac{11 - \sqrt{6}}{5}, 0.60\right)$, $\left(\dfrac{11 + \sqrt{6}}{5}, 0.46\right)$

Intercepts: $(0, -9)$, $(1, 0)$, $(3, 0)$

51. $f(x) = x^{1/3}(x + 3)^{2/3}$



Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

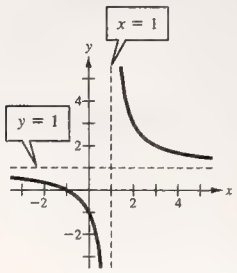
$f'(x) = \dfrac{x + 1}{(x + 3)^{1/3}x^{2/3}} = 0$ when $x = -1$ and undefined when $x = -3, 0$.

$f''(x) = \dfrac{-2}{x^{5/3}(x + 3)^{4/3}}$ is undefined when $x = 0, -3$.

By the First Derivative Test $(-3, 0)$ is a relative maximum and $(-1, -\sqrt[3]{4})$ is a relative minimum. $(0, 0)$ is a point of inflection.

Intercepts: $(-3, 0)$, $(0, 0)$

53. $f(x) = \dfrac{x+1}{x-1}$



Domain: $(-\infty, 1), (1, \infty)$; Range: $(-\infty, 1), (1, \infty)$

$f'(x) = \dfrac{-2}{(x-1)^2} < 0$ if $x \neq 1$.

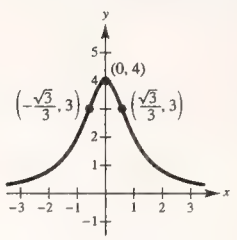
$f''(x) = \dfrac{4}{(x-1)^3}$

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 1$

Intercepts: $(-1, 0), (0, -1)$

55. $f(x) = \dfrac{4}{1+x^2}$



Domain: $(-\infty, \infty)$; Range: $(0, 4]$

$f'(x) = \dfrac{-8x}{(1+x^2)^2} = 0$ when $x = 0$.

$f''(x) = \dfrac{-8(1-3x^2)}{(1+x^2)^3} = 0$ when $x = \pm\dfrac{\sqrt{3}}{3}$.

$f''(0) < 0$

Therefore, $(0, 4)$ is a relative maximum.

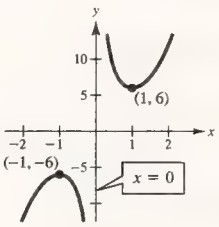
Points of inflection: $\left(\pm\sqrt{3}/3, 3\right)$

Intercept: $(0, 4)$

Symmetric to the y-axis

Horizontal asymptote: $y = 0$

57. $f(x) = x^3 + x + \dfrac{4}{x}$



Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, -6], [6, \infty)$

$f'(x) = 3x^2 + 1 - \dfrac{4}{x^2} = \dfrac{3x^4 + x^2 - 4}{x^2} = 0$ when $x = \pm 1$.

$f''(x) = 6x + \dfrac{8}{x^3} = \dfrac{6x^4 + 8}{x^3} \neq 0$

$f''(-1) < 0$

Therefore, $(-1, -6)$ is a relative maximum.

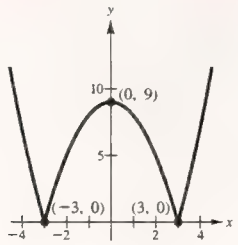
$f''(1) > 0$

Therefore, $(1, 6)$ is a relative minimum.

Vertical asymptote: $x = 0$

Symmetric with respect to origin

59. $f(x) = |x^2 - 9|$



Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$f'(x) = \dfrac{2x(x^2 - 9)}{|x^2 - 9|} = 0$ when $x = 0$ and is undefined when $x = \pm 3$.

$f''(x) = \dfrac{2(x^2 - 9)}{|x^2 - 9|}$ is undefined at $x = \pm 3$.

$f''(0) < 0$

Therefore, $(0, 9)$ is a relative maximum.

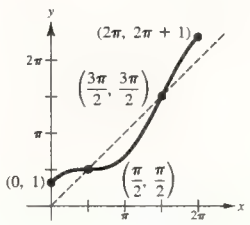
Relative minima: $(\pm 3, 0)$

Points of inflection: $(\pm 3, 0)$

Intercepts: $(\pm 3, 0)$, $(0, 9)$

Symmetric to the y-axis

61. $f(x) = x + \cos x$



Domain: $[0, 2\pi]$; Range: $[1, 1 + 2\pi]$

$f'(x) = 1 - \sin x \geq 0$, f is increasing.

$f''(x) = -\cos x = 0$ when $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Points of inflection: $\left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, \dfrac{3\pi}{2}\right)$

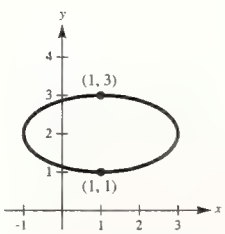
Intercept: $(0, 1)$

63. $x^2 + 4y^2 - 2x - 16y + 13 = 0$

(a) $(x^2 - 2x + 1) + 4(y^2 - 4y + 4) = -13 + 1 + 16$

$$(x - 1)^2 + 4(y - 2)^2 = 4$$

$$\frac{(x - 1)^2}{4} + \frac{(y - 2)^2}{1} = 1$$



The graph is an ellipse:

Maximum: $(1, 3)$

Minimum: $(1, 1)$

(b) $x^2 + 4y^2 - 2x - 16y + 13 = 0$

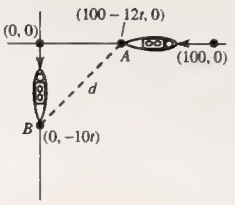
$$2x + 8y\frac{dy}{dx} - 2 - 16\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(8y - 16) = 2 - 2x$$

$$\frac{dy}{dx} = \frac{2 - 2x}{8y - 16} = \frac{1 - x}{4y - 8}$$

The critical numbers are $x = 1$ and $y = 2$. These correspond to the points $(1, 1)$, $(1, 3)$, $(2, -1)$, and $(2, 3)$. Hence, the maximum is $(1, 3)$ and the minimum is $(1, 1)$.

65. Let $t = 0$ at noon.



$$L = d^2 = (100 - 12t)^2 + (-10t)^2 = 10{,}000 - 2400t + 244t^2$$

$$\frac{dL}{dt} = -2400 + 488t = 0 \text{ when } t = \frac{300}{61} \approx 4.92 \text{ hr.}$$

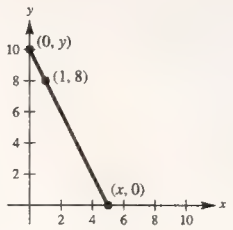
Ship A at (40.98, 0); Ship B at (0, −49.18)

$$d^2 = 10{,}000 - 2400t + 244t^2$$

$$\approx 4098.36 \text{ when } t \approx 4.92 \approx 4{:}55 \text{ P.M.}.$$

$$d \approx 64 \text{ km}$$

67. We have points $(0, y)$, $(x, 0)$, and $(1, 8)$. Thus,



$$m = \frac{y - 8}{0 - 1} = \frac{0 - 8}{x - 1} \text{ or } y = \frac{8x}{x - 1}.$$

Let $f(x) = L^2 = x^2 + \left(\frac{8x}{x-1}\right)^2$.

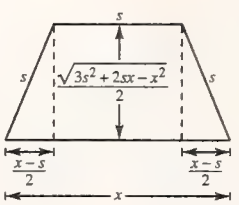
$$f'(x) = 2x + 128\left(\frac{x}{x-1}\right)\left[\frac{(x-1) - x}{(x-1)^2}\right] = 0$$

$$x - \frac{64x}{(x-1)^3} = 0$$

$$x[(x-1)^3 - 64] = 0 \text{ when } x = 0, 5 \text{ (minimum).}$$

Vertices of triangle: (0, 0), (5, 0), (0, 10)

69. A = (Average of bases)(Height)



$$= \left(\frac{x+s}{2}\right)\frac{\sqrt{3s^2 + 2sx - x^2}}{2} \text{ (see figure)}$$

$$\frac{dA}{dx} = \frac{1}{4}\left[\frac{(s-x)(s+x)}{\sqrt{3s^2 + 2sx - x^2}} + \sqrt{3s^2 + 2sx - x^2}\right]$$

$$= \frac{2(2s - x)(s + x)}{4\sqrt{3s^2 + 2sx - x^2}} = 0 \text{ when } x = 2s.$$

A is a maximum when $x = 2s$.

71. You can form a right triangle with vertices $(0, 0)$, $(x, 0)$ and $(0, y)$. Assume that the hypotenuse of length L passes through $(4, 6)$.

$$m = \frac{y - 6}{0 - 4} = \frac{6 - 0}{4 - x} \text{ or } y = \frac{6x}{x - 4}$$

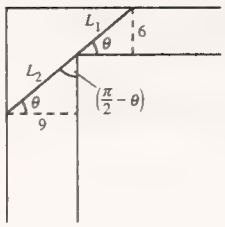
Let $f(x) = L^2 = x^2 + y^2 = x^2 + \left(\frac{6x}{x-4}\right)^2$.

$$f'(x) = 2x + 72\left(\frac{x}{x-4}\right)\left[\frac{-4}{(x-4)^2}\right] = 0$$

$$x[(x-4)^3 - 144] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{144}.$$

$$L \approx 14.05 \text{ feet}$$

73. $\csc\theta = \dfrac{L_1}{6}$ or $L_1 = 6\csc\theta$ (see figure)



$$\csc\left(\frac{\pi}{2}-\theta\right) = \frac{L_2}{9} \text{ or } L_2 = 9\csc\left(\frac{\pi}{2}-\theta\right)$$

$$L = L_1 + L_2 = 6\csc\theta + 9\csc\left(\frac{\pi}{2}-\theta\right) = 6\csc\theta + 9\sec\theta$$

$$\frac{dL}{d\theta} = -6\csc\theta\cot\theta + 9\sec\theta\tan\theta = 0$$

$$\tan^3\theta = \frac{2}{3} \Rightarrow \tan\theta = \frac{\sqrt[3]{2}}{\sqrt[3]{3}}$$

$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\left(\frac{2}{3}\right)^{2/3}} = \frac{\sqrt{3^{2/3}+2^{2/3}}}{3^{1/3}}$$

$$\csc\theta = \frac{\sec\theta}{\tan\theta} = \frac{\sqrt{3^{2/3}+2^{2/3}}}{2^{1/3}}$$

$$L = 6\frac{(3^{2/3}+2^{2/3})^{1/2}}{2^{1/3}} + 9\frac{(3^{2/3}+2^{2/3})^{1/2}}{3^{1/3}} = 3(3^{2/3}+2^{2/3})^{3/2} \text{ ft} \approx 21.07 \text{ ft (Compare to Exercise 72 using } a = 9 \text{ and } b = 6.)$$

75. Total cost = (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{600}+5\right)\left(\frac{110}{v}\right) = \frac{11v}{60} + \frac{550}{v}$$

$$\frac{dT}{dv} = \frac{11}{60} - \frac{550}{v^2} = \frac{11v^2 - 33{,}000}{60v^2}$$

$= 0$ when $v = \sqrt{3000} = 10\sqrt{30} \approx 54.8$ mph.

$$\frac{d^2T}{dv^2} = \frac{1100}{v^3} > 0 \text{ when } v = 10\sqrt{30} \text{ so this value yields a minimum.}$$

77. $f(x) = x^3 - 3x - 1$

From the graph you can see that $f(x)$ has three real zeros.

$f'(x) = 3x^2 - 3$

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	−1.5000	0.1250	3.7500	0.0333	−1.5333
2	−1.5333	−0.0049	4.0530	−0.0012	−1.5321

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	−0.5000	0.3750	−2.2500	−0.1667	−0.3333
2	−0.3333	−0.0371	−2.6667	0.0139	−0.3472
3	−0.3472	−0.0003	−2.6384	0.0001	−0.3473

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	−1.9000	0.1590	7.8300	0.0203	1.8797
2	1.8797	0.0024	7.5998	0.0003	1.8794

The three real zeros of $f(x)$ are $x \approx -1.532$, $x \approx -0.347$, and $x \approx 1.879$.

79. Find the zeros of $f(x) = x^4 - x - 3$.

$f'(x) = 4x^3 - 1$

From the graph you can see that $f(x)$ has two real zeros.

f changes sign in $[-2, -1]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.2000	0.2736	-7.9120	-0.0346	-1.1654
2	-1.1654	0.0100	-7.3312	-0.0014	-1.1640

On the interval $[-2, -1]$: $x \approx -1.164$.

f changes sign in $[1, 2]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.5000	0.5625	12.5000	0.0450	1.4550
2	1.4550	0.0268	11.3211	0.0024	1.4526
3	1.4526	-0.0003	11.2602	0.0000	1.4526

On the interval $[1, 2]$; $x \approx 1.453$.

81. $y = x(1 - \cos x) = x - x\cos x$

$$\frac{dy}{dx} = 1 + x\sin x - \cos x$$

$$dy = (1 + x\sin x - \cos x)\,dx$$

83.
$$S = 4\pi r^2.\ dr = \Delta r = \pm 0.025$$

$$dS = 8\pi r\,dr = 8\pi(9)(\pm 0.025)$$

$$= \pm 1.8\pi \text{ square cm}$$

$$\frac{dS}{S}(100) = \frac{8\pi r\,dr}{4\pi r^2}(100) = \frac{2\,dr}{r}(100)$$

$$= \frac{2(\pm 0.025)}{9}(100) \approx \pm 0.56\%$$

$$V = \frac{4}{3}\pi r^3$$

$$dV = 4\pi r^2\,dr = 4\pi(9)^2(\pm 0.025)$$

$$= \pm 8.1\pi \text{ cubic cm}$$

$$\frac{dV}{V}(100) = \frac{4\pi r^2\,dr}{(4/3)\pi r^3}(100) = \frac{3\,dr}{r}(100)$$

$$= \frac{3(\pm 0.025)}{9}(100) \approx \pm 0.83\%$$

Problem Solving for Chapter 3

1. Assume $y_1 < d < y_2$. Let $g(x) = f(x) - d(x - a)$. g is continuous on $[a, b]$ and therefore has a minimum $(c, g(c))$ on $[a, b]$. The point c cannot be an endpoint of $[a, b]$ because

$g'(a) = f'(a) - d = y_1 - d < 0$

$g'(b) = f'(b) - d = y_2 - d > 0$

Hence, $a < c < b$ and $g'(c) = 0 \Rightarrow f'(c) = d$.

3. (a) For $a = -3, -2, -1, 0$, p has a relative maximum at $(0, 0)$.

For $a = 1, 2, 3$, p has a relative maximum at $(0, 0)$ and 2 relative minima.

(b) $p'(x) = 4ax^3 - 12x = 4x(ax^2 - 3) = 0 \Rightarrow x = 0, \pm\sqrt{\frac{3}{a}}$

$p''(x) = 12ax^2 - 12 = 12(ax^2 - 1)$

For $x = 0, p''(0) = -12 < 0 \Rightarrow p$ has a relative maximum at $(0, 0)$.

(c) If $a > 0, x = \pm\sqrt{\frac{3}{a}}$ are the remaining critical numbers.

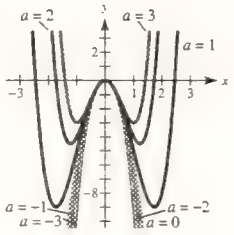
$p''\left(\pm\sqrt{\frac{3}{a}}\right) = 12\left(\frac{3}{a}\right) - 12 = 24 > 0 \Rightarrow p$ has relative minima for $a > 0$.

(d) $(0, 0)$ lies on $y = -3x^2$.

Let $x = \pm\sqrt{\frac{3}{a}}$. Then

$$p(x) = a\left(\frac{3}{a}\right)^2 - 6\left(\frac{3}{a}\right) = \frac{9}{a} - \frac{18}{a} = -\frac{9}{a}.$$

Thus, $y = -\frac{9}{a} = -3\left(\pm\sqrt{\frac{3}{a}}\right)^2 = -3x^2$ is satisfied by all the relative extrema of p.



5. $p(x) = x^4 + ax^2 + 1$

(a) $p'(x) = 4x^4 + 2ax = 2x(2x^2 + a)$

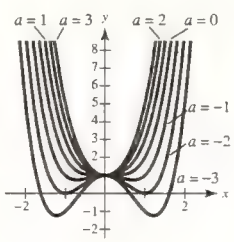
$p''(x) = 16x^2 + 2a$

For $a \geq 0$, there is one relative minimum at $(0, 0)$.

(b) For $a < 0$, there is a relative maximum at $(0, 1)$.

(c) For $a < 0$, there are two relative minima at $x = \pm\sqrt{-\frac{a}{2}}$.

(d) There are either 1 or 3 critical points. The above analysis shows that there cannot be exactly two relative extrema.



7. $f(x) = \frac{c}{x} + x^2$

$$f'(x) = -\frac{c}{x^2} + 2x = 0 \Rightarrow \frac{c}{x^2} = 2x \Rightarrow x^3 = \frac{c}{2} \Rightarrow x = \sqrt[3]{\frac{c}{x}}$$

$$f''(x) = \frac{2c}{x^3} + 2$$

If $c = 0, f(x) = x^2$ has a relative minimum, but no relative maximum.

If $c > 0, x = \sqrt[3]{\frac{c}{2}}$ is a relative minimum, because $f''\left(\sqrt[3]{\frac{c}{2}}\right) > 0$.

If $c < 0, x = \sqrt[3]{\frac{c}{2}}$ is a relative minimum too.

Answer: all c.

9. Set $\dfrac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2} = k.$

Define $F(x) = f(x) - f(a) - f'(a)(x - a) - k(x - a)^2$.

$F(a) = 0, F(b) = f(b) - f(a) - f'(a)(b - a) - k(b - a)^2 = 0$

F is continuous on $[a, b]$ and differentiable on (a, b).

There exists $c_1, a < c_1 < b$, satisfying $F'(c_1) = 0$.

$F'(x) = f'(x) - f'(a) - 2k(x - a)$ satisfies the hypothesis of Rolle's Theorem on $[a, c_1]$:

$F'(a) = 0, F'(c_1) = 0.$

There exists $c_2, a < c_2 < c_1$ satisfying $F''(c_2) = 0$.

Finally, $F''(x) = f''(x) - 2k$ and $F''(c_2) = 0$ implies that

$$k = \frac{f''(c_2)}{2}.$$

Thus, $k = \dfrac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2} = \dfrac{f''(c_2)}{2} \Rightarrow f(b) = f(a) + f'(a)(b - a) + \dfrac{1}{2}f''(c_2)(b - a)^2.$

11. $E(\phi) = \dfrac{\tan\phi(1 - 0.1\tan\phi)}{0.1 + \tan\phi} = \dfrac{10\tan\phi - \tan^2\phi}{1 + 10\tan\phi}$

$$E'(\phi) = \frac{(1 + 10\tan\phi)(10\sec^2\phi - 2\tan\phi\sec^2\phi) - (10\tan\phi - \tan^2\phi)10\sec^2\phi}{(1 + 10\tan\phi)} = 0$$

$$\Rightarrow (1 + 10\tan\phi)(10\sec^2\phi - 2\tan\phi\sec^2\phi) = (10\tan\phi - \tan^2\phi)10\sec^2\phi$$

$$\Rightarrow 10\sec^2\phi - 2\tan\phi\sec^2\phi + 100\tan\phi\sec^2\phi - 20\tan^2\phi\sec^2\phi$$
$$= 100\tan\phi\sec^2\phi - 10\tan^2\phi\sec^2\phi$$

$$\Rightarrow 10 - 2\tan\phi = 10\tan^2\phi$$

$$\Rightarrow 10\tan^2\phi + 2\tan\phi - 10 = 0$$

$$\tan\phi = \frac{-2 \pm \sqrt{4 + 400}}{20} \approx 0.90499, -1.10499$$

Using the positive value, $\phi \approx 0.7356$, or $42.14°$.

13. $v = -2400\pi\sin\theta$

$v' = -2400\pi\cos\theta = 0$

$\theta = \dfrac{\pi}{2} + 2n\pi, \dfrac{3\pi}{2} + 2n\pi$, n an integer

15. The line has equation $\frac{x}{3} + \frac{y}{4} = 1$ or $y = -\frac{4}{3}x + 4$.

Rectangle:

$$\text{Area} = A = xy = x\left(-\frac{4}{3}x + 4\right) = -\frac{4}{3}x^2 + 4x.$$

$$A'(x) = -\frac{8}{3}x + 4 = 0 \Rightarrow \frac{8}{3}x = 4 \Rightarrow x = \frac{3}{2}$$

Dimensions: $\frac{3}{2} \times 2$ Calculus was helpful.

Circle: The distance from the center (r, r) to the line $\frac{x}{3} + \frac{y}{4} - 1 = 0$ must be r:

$$r = \frac{\left|\frac{r}{3} + \frac{r}{4} - 1\right|}{\sqrt{\frac{1}{9} + \frac{1}{16}}} = \frac{12}{5}\left|\frac{7r - 12}{12}\right| = \frac{|7r - 12|}{5}$$

$5r = |7r - 12| \Rightarrow r = 1$ or $r = 6$.

Clearly, $r = 1$.

Semicircle: The center lies on the line $\frac{x}{3} + \frac{y}{4} = 1$ and satisfies $x = y = r$.

Thus $\frac{r}{3} + \frac{r}{4} = 1 \Rightarrow \frac{7}{12}r = 1 \Rightarrow r = \frac{12}{7}$. No calculus necessary.

17. $y = (1 + x^2)^{-1}$

$$y' = \frac{-2x}{(1 + x^2)^2}$$

$$y'' = \frac{2(3x^2 - 1)}{(x^2 + 1)^3} = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$$

y'': $+++$ $----$ $----$ $+++$ at $-\frac{\sqrt{3}}{3}$, 0, $\frac{\sqrt{3}}{3}$

The tangent line has greatest slope at $\left(-\frac{\sqrt{3}}{3}, \frac{3}{4}\right)$ and least slope at $\left(\frac{\sqrt{3}}{3}, \frac{3}{4}\right)$.

19. (a)

x	0.1	0.2	0.3	0.4	0.5	1.0
$\sin x$	0.09983	0.19867	0.29552	0.38942	0.47943	0.84147

$\sin x \leq x$

(b) Let $f(x) = \sin x$. Then $f'(x) = \cos x$ and on $[0, x]$ you have by the Mean Value Theorem,

$$f'(c) = \frac{f(x) - f(0)}{x - 0}, \quad 0 < c < x$$

$$\cos(c) = \frac{\sin x}{x}$$

Hence, $\left|\frac{\sin x}{x}\right| = |\cos(c)| \leq 1$

$$\Rightarrow |\sin x| \leq |x|$$

$$\Rightarrow \sin x \leq x$$

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Integration

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CHAPTER 4
Integration

Section 4.1 Antiderivatives and Indefinite Integration

Solutions to Odd-Numbered Exercises

1. $\dfrac{d}{dx}\left(\dfrac{3}{x^3}+C\right)=\dfrac{d}{dx}(3x^{-3}+C)=-9x^{-4}=\dfrac{-9}{x^4}$

3. $\dfrac{d}{dx}\left(\dfrac{1}{3}x^3-4x+C\right)=x^2-4=(x-2)(x+2)$

5. $\dfrac{dy}{dt}=3t^2$

$y=t^3+C$

Check: $\dfrac{d}{dt}[t^3+C]=3t^2$

7. $\dfrac{dy}{dx}=x^{3/2}$

$y=\dfrac{2}{5}x^{5/2}+C$

Check: $\dfrac{d}{dx}\left[\dfrac{2}{5}x^{5/2}+C\right]=x^{3/2}$

	Given	*Rewrite*	*Integrate*	*Simplify*
9.	$\int\sqrt[3]{x}\,dx$	$\int x^{1/3}\,dx$	$\dfrac{x^{4/3}}{4/3}+C$	$\dfrac{3}{4}x^{4/3}+C$
11.	$\int\dfrac{1}{x\sqrt{x}}\,dx$	$\int x^{-3/2}\,dx$	$\dfrac{x^{-1/2}}{-1/2}+C$	$-\dfrac{2}{\sqrt{x}}+C$
13.	$\int\dfrac{1}{2x^3}\,dx$	$\dfrac{1}{2}\int x^{-3}\,dx$	$\dfrac{1}{2}\left(\dfrac{x^{-2}}{-2}\right)+C$	$-\dfrac{1}{4x^2}+C$

15. $\int(x+3)dx=\dfrac{x^2}{2}+3x+C$

Check: $\dfrac{d}{dx}\left[\dfrac{x^2}{2}+3x+C\right]=x+3$

17. $\int(2x-3x^2)dx=x^2-x^3+C$

Check: $\dfrac{d}{dx}[x^2-x^3+C]=2x-3x^2$

19. $\int(x^3+2)\,dx=\dfrac{1}{4}x^4+2x+C$

Check: $\dfrac{d}{dx}\left(\dfrac{1}{4}x^4+2x+C\right)=x^3+2$

21. $\int(x^{3/2}+2x+1)\,dx=\dfrac{2}{5}x^{5/2}+x^2+x+C$

Check: $\dfrac{d}{dx}\left(\dfrac{2}{5}x^{5/2}+x^2+x+C\right)=x^{3/2}+2x+1$

23. $\int\sqrt[3]{x^2}\,dx=\int x^{2/3}\,dx=\dfrac{x^{5/3}}{5/3}+C=\dfrac{3}{5}x^{5/3}+C$

Check: $\dfrac{d}{dx}\left(\dfrac{3}{5}x^{5/3}+C\right)=x^{2/3}=\sqrt[3]{x^2}$

25. $\int\dfrac{1}{x^3}\,dx=\int x^{-3}\,dx=\dfrac{x^{-2}}{-2}+C=-\dfrac{1}{2x^2}+C$

Check: $\dfrac{d}{dx}\left(-\dfrac{1}{2x^2}+C\right)=\dfrac{1}{x^3}$

27. $\displaystyle\int \frac{x^2 + x + 1}{\sqrt{x}}\,dx = \int (x^{3/2} + x^{1/2} + x^{-1/2})\,dx = \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C = \frac{2}{15}x^{1/2}(3x^2 + 5x + 15) + C$

Check: $\displaystyle\frac{d}{dx}\left(\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C\right) = x^{3/2} + x^{1/2} + x^{-1/2} = \frac{x^2 + x + 1}{\sqrt{x}}$

29. $\displaystyle\int (x + 1)(3x - 2)\,dx = \int (3x^2 + x - 2)\,dx$

$= x^3 + \frac{1}{2}x^2 - 2x + C$

Check: $\displaystyle\frac{d}{dx}\left(x^3 + \frac{1}{2}x^2 - 2x + C\right) = 3x^2 + x - 2$

$= (x + 1)(3x - 2)$

31. $\displaystyle\int y^2\sqrt{y}\,dy = \int y^{5/2}\,dy = \frac{2}{7}y^{7/2} + C$

Check: $\displaystyle\frac{d}{dy}\left(\frac{2}{7}y^{7/2} + C\right) = y^{5/2} = y^2\sqrt{y}$

33. $\displaystyle\int dx = \int 1\,dx = x + C$

Check: $\displaystyle\frac{d}{dx}(x + C) = 1$

35. $\displaystyle\int (2\sin x + 3\cos x)\,dx = -2\cos x + 3\sin x + C$

Check: $\displaystyle\frac{d}{dx}(-2\cos x + 3\sin x + C) = 2\sin x + 3\cos x$

37. $\displaystyle\int (1 - \csc t\cot t)\,dt = t + \csc t + C$

Check: $\displaystyle\frac{d}{dt}(t + \csc t + C) = 1 - \csc t\cot t$

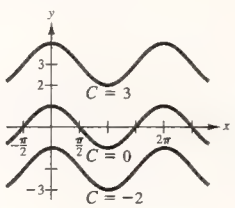
39. $\displaystyle\int (\sec^2\theta - \sin\theta)\,d\theta = \tan\theta + \cos\theta + C$

Check: $\displaystyle\frac{d}{d\theta}(\tan\theta + \cos\theta + C) = \sec^2\theta - \sin\theta$

41. $\displaystyle\int (\tan^2 y + 1)\,dy = \int \sec^2 y\,dy = \tan y + C$

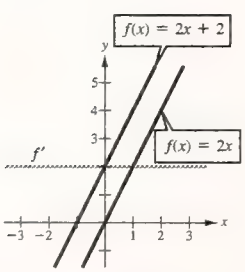
Check: $\displaystyle\frac{d}{dy}(\tan y + C) = \sec^2 y = \tan^2 y + 1$

43. $f(x) = \cos x$



45. $f'(x) = 2$

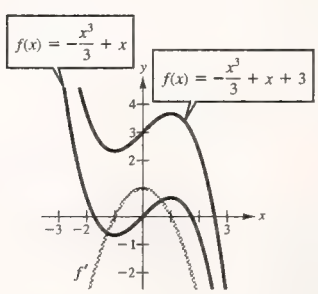
$f(x) = 2x + C$



Answers will vary.

47. $f'(x) = 1 - x^2$

$\displaystyle f(x) = x - \frac{x^3}{3} + C$



Answers will vary.

49. $\displaystyle\frac{dy}{dx} = 2x - 1,\ (1, 1)$

$\displaystyle y = \int (2x - 1)\,dx = x^2 - x + C$

$1 = (1)^2 - (1) + C \Rightarrow C = 1$

$y = x^2 - x + 1$

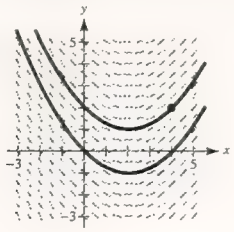
51. $\frac{dy}{dx} = \cos x,\ (0, 4)$

$$y = \int \cos x\, dx = \sin x + C$$

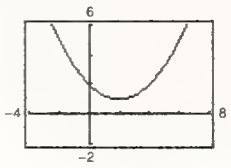
$$4 = \sin 0 + C \Rightarrow C = 4$$

$$y = \sin x + 4$$

53. (a) Answers will vary.



(b) $\frac{dy}{dx} = \frac{1}{2}x - 1,\ (4, 2)$



$$y = \frac{x^2}{4} - x + C$$

$$2 = \frac{4^2}{4} - 4 + C$$

$$2 = C$$

$$y = \frac{x^2}{4} - x + 2$$

55. $f'(x) = 4x, f(0) = 6$

$$f(x) = \int 4x\, dx = 2x^2 + C$$

$$f(0) = 6 = 2(0)^2 + C \Rightarrow C = 6$$

$$f(x) = 2x^2 + 6$$

57. $h'(t) = 8t^3 + 5, h(1) = -4$

$$h(t) = \int (8t^3 + 5)dt = 2t^4 + 5t + C$$

$$h(1) = -4 = 2 + 5 + C \Rightarrow C = -11$$

$$h(t) = 2t^4 + 5t - 11$$

59. $f''(x) = 2$

$$f'(2) = 5$$

$$f(2) = 10$$

$$f'(x) = \int 2\, dx = 2x + C_1$$

$$f'(2) = 4 + C_1 = 5 \Rightarrow C_1 = 1$$

$$f'(x) = 2x + 1$$

$$f(x) = \int (2x + 1)\, dx = x^2 + x + C_2$$

$$f(2) = 6 + C_2 = 10 \Rightarrow C_2 = 4$$

$$f(x) = x^2 + x + 4$$

61. $f''(x) = x^{-3/2}$

$$f'(4) = 2$$

$$f(0) = 0$$

$$f'(x) = \int x^{-3/2}\, dx = -2x^{-1/2} + C_1 = -\frac{2}{\sqrt{x}} + C_1$$

$$f'(4) = -\frac{2}{2} + C_1 = 2 \Rightarrow C_1 = 3$$

$$f'(x) = -\frac{2}{\sqrt{x}} + 3$$

$$f(x) = \int (-2x^{-1/2} + 3)\, dx = -4x^{1/2} + 3x + C_2$$

$$f(0) = 0 + 0 + C_2 = 0 \Rightarrow C_2 = 0$$

$$f(x) = -4x^{1/2} + 3x = -4\sqrt{x} + 3x$$

63. (a) $h(t) = \int (1.5t + 5)\, dt = 0.75t^2 + 5t + C$

$$h(0) = 0 + 0 + C = 12 \Rightarrow C = 12$$

$$h(t) = 0.75t^2 + 5t + 12$$

(b) $h(6) = 0.75(6)^2 + 5(6) + 12 = 69$ cm

65. $f(0) = -4$. Graph of f' is given.

(a) $f'(4) \approx -1.0$

(b) No. The slopes of the tangent lines are greater than 2 on $[0, 2]$. Therefore, f must increase more than 4 units on $[0, 4]$.

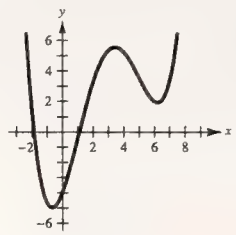
(c) No, $f(5) < f(4)$ because f is decreasing on $[4, 5]$.

(d) f is an maximum at $x = 3.5$ because $f'(3.5) \approx 0$ and the first derivative test.

(e) f is concave upward when f' is increasing on $(-\infty, 1)$ and $(5, \infty)$. f is concave downward on $(1, 5)$. Points of inflection at $x = 1, 5$.

(f) f'' is a minimum at $x = 3$.

(g)



67. $a(t) = -32$ ft/sec^2

$$v(t) = \int -32\,dt = -32t + C_1$$

$$v(0) = 60 = C_1$$

$$s(t) = \int (-32t + 60)dt = -16t^2 + 60t + C_2$$

$$s(0) = 6 = C_2$$

$s(t) = -16t^2 + 60t + 6$ Position function

The ball reaches its maximim height when

$$v(t) = -32t + 60 = 0$$

$$32t = 60$$

$$t = \frac{15}{8} \text{ seconds}$$

$$s\left(\frac{15}{18}\right) = -16\left(\frac{15}{8}\right)^2 + 60\left(\frac{15}{8}\right) + 6 \approx 62.26 \text{ feet}$$

69. From Exercise 68, we have:

$$s(t) = -16t^2 + v_0 t$$

$s'(t) = -32t + v_0 = 0$ when $t = \dfrac{v_0}{32}$ = time to reach maximum height.

$$s\left(\frac{v_0}{32}\right) = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) = 550$$

$$-\frac{v_0^2}{64} + \frac{v_0^2}{32} = 550$$

$$v_0^2 = 35{,}200$$

$$v_0 \approx 187.617 \text{ ft/sec}$$

71. $a(t) = -9.8$

$$v(t) = \int -9.8\,dt = -9.8t + C_1$$

$$v(0) = v_0 = C_1 \Rightarrow v(t) = -9.8t + v_0$$

$$f(t) = \int (-9.8t + v_0)\,dt = -4.9t^2 + v_0 t + C_2$$

$$f(0) = s_0 = C_2 \Rightarrow f(t) = -4.9t^2 + v_0 t + s_0$$

73. From Exercise 71, $f(t) = -4.9t^2 + 10t + 2$.

$v(t) = -9.8t + 10 = 0$ (Maximum height when $v = 0$.)

$$9.8t = 10$$

$$t = \frac{10}{9.8}$$

$$f\left(\frac{10}{9.8}\right) \approx 7.1 \text{ m}$$

75. $a = -1.6$

$v(t) = \int -1.6\,dt = -1.6t + v_0 = -1.6t$, since the stone was dropped, $v_0 = 0$.

$$s(t) = \int (-1.6t)\,dt = -0.8t^2 + s_0$$

$$s(20) = 0 \Rightarrow -0.8(20)^2 + s_0 = 0$$

$$s_0 = 320$$

Thus, the height of the cliff is 320 meters.

$$v(t) = -1.6t$$

$$v(20) = -32 \text{ m/sec}$$

77. $x(t) = t^3 - 6t^2 + 9t - 2 \qquad 0 \le t \le 5$

(a) $v(t) = x'(t) = 3t^2 - 12t + 9$

$= 3(t^2 - 4t + 3) = 3(t - 1)(t - 3)$

$a(t) = v'(t) = 6t - 12 = 6(t - 2)$

(b) $v(t) > 0$ when $0 < t < 1$ or $3 < t < 5$.

(c) $a(t) = 6(t - 2) = 0$ when $t = 2$.

$v(2) = 3(1)(-1) = -3$

79. $v(t) = \dfrac{1}{\sqrt{t}} = t^{-1/2} \qquad t > 0$

$x(t) = \displaystyle\int v(t)\,dt = 2t^{1/2} + C$

$x(1) = 4 = 2(1) + C \Rightarrow C = 2$

$x(t) = 2t^{1/2} + 2$ position function

$a(t) = v'(t) = -\dfrac{1}{2}t^{-3/2} = \dfrac{-1}{2t^{3/2}}$ acceleration

81. (a) $v(0) = 25 \text{ km/hr} = 25 \cdot \dfrac{1000}{3600} = \dfrac{250}{36} \text{ m/sec}$

$v(13) = 80 \text{ km/hr} = 80 \cdot \dfrac{1000}{3600} = \dfrac{800}{36} \text{ m/sec}$

$a(t) = a$ (constant acceleration)

$v(t) = at + C$

$v(0) = \dfrac{250}{36} \Rightarrow v(t) = at + \dfrac{250}{36}$

$v(13) = \dfrac{800}{36} = 13a + \dfrac{250}{36}$

$\dfrac{550}{36} = 13a$

$a = \dfrac{550}{468} = \dfrac{275}{234} \approx 1.175 \text{ m/sec}^2$

(b) $s(t) = a\dfrac{t^2}{2} + \dfrac{250}{36}t \qquad (s(0) = 0)$

$s(13) = \dfrac{275}{234}\dfrac{(13)^2}{2} + \dfrac{250}{36}(13) \approx 189.58 \text{ m}$

83. Truck: $v(t) = 30$

$s(t) = 30t$ (Let $s(0) = 0$.)

Automobile: $a(t) = 6$

$v(t) = 6t$ (Let $v(0) = 0$.)

$s(t) = 3t^2$ (Let $s(0) = 0$.)

At the point where the automobile overtakes the truck:

$30t = 3t^2$

$0 = 3t^2 - 30t$

$0 = 3t(t - 10)$ when $t = 10$ sec.

(a) $s(10) = 3(10)^2 = 300$ ft

(b) $v(10) = 6(10) = 60$ ft/sec ≈ 41 mph

85. $\dfrac{(1 \text{ mi/hr})(5280 \text{ ft/mi})}{(3600 \text{ sec/hr})} = \dfrac{22}{15} \text{ ft/sec}$

(a)

t	0	5	10	15	20	25	30
V_1(ft/sec)	0	3.67	10.27	23.47	42.53	66	95.33
V_2(ft/sec)	0	30.8	55.73	74.8	88	93.87	95.33

(b) $V_1(t) = 0.1068t^2 - 0.0416t + 0.3679$

$V_2(t) = -0.1208t^2 + 6.7991t - 0.0707$

(c) $S_1(t) = \displaystyle\int V_1(t)\,dt = \frac{0.1068}{3}t^3 - \frac{0.0416}{2}t^2 + 0.3679t$

$S_2(t) = \displaystyle\int V_2(t)\,dt = -\frac{0.1208t^3}{3} + \frac{6.7991t^2}{2} - 0.0707t$

[In both cases, the constant of integration is 0 because $S_1(0) = S_2(0) = 0$]

$S_1(30) \approx 953.5$ feet

$S_2(30) \approx 1970.3$ feet

The second car was going faster than the first until the end.

87. $a(t) = k$

$v(t) = kt$

$s(t) = \frac{k}{2}t^2$ since $v(0) = s(0) = 0$.

At the time of lift-off, $kt = 160$ and $(k/2)t^2 = 0.7$. Since $(k/2)t^2 = 0.7$,

$$t = \sqrt{\frac{1.4}{k}}$$

$$v\left(\sqrt{\frac{1.4}{k}}\right) = k\sqrt{\frac{1.4}{k}} = 160$$

$$1.4k = 160^2 \Rightarrow k = \frac{160^2}{1.4}$$

$$\approx 18{,}285.714 \text{ mi/hr}^2$$

$$\approx 7.45 \text{ ft/sec}^2.$$

89. True

91. True

93. False. For example, $\int x \cdot x\,dx \neq \int x\,dx \cdot \int x\,dx$ because $\frac{x^3}{3} + C \neq \left(\frac{x^2}{2} + C_1\right)\left(\frac{x^2}{2} + C_2\right)$

95. $f'(x) = \begin{cases} 1, & 0 \le x < 2 \\ 3x, & 2 \le x \le 5 \end{cases}$

$$f(x) = \begin{cases} x + C_1, & 0 \le x < 2 \\ \frac{3x^2}{2} + C_2, & 2 \le x \le 5 \end{cases}$$

$f(1) = 3 \Rightarrow 1 + C_1 = 3 \Rightarrow C_1 = 2$

f is continuous: Values must agree at $x = 2$:

$4 = 6 + C_2 \Rightarrow C_2 = -2$

$$f(x) = \begin{cases} x + 2, & 0 \le x < 2 \\ \frac{3x^2}{2} - 2, & 2 \le x \le 5 \end{cases}$$

The left and right hand derivatives at $x = 2$ do not agree. Hence f is not differentiable at $x = 2$.

Section 4.2 Area

1. $\sum_{i=1}^{5}(2i + 1) = 2\sum_{i=1}^{5} i + \sum_{i=1}^{5} 1 = 2(1 + 2 + 3 + 4 + 5) + 5 = 35$

3. $\sum_{k=0}^{4} \frac{1}{k^2 + 1} = 1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} = \frac{158}{85}$

5. $\sum_{k=1}^{4} c = c + c + c + c = 4c$

7. $\sum_{i=1}^{9} \frac{1}{3i}$

9. $\sum_{j=1}^{8}\left[5\left(\frac{j}{8}\right) + 3\right]$

11. $\frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3 - \left(\frac{2i}{n}\right)\right]$

13. $\frac{3}{n}\sum_{i=1}^{n}\left[2\left(1 + \frac{3i}{n}\right)^2\right]$

15. $\sum_{i=1}^{20} 2i = 2\sum_{i=1}^{20} i = 2\left[\frac{20(21)}{2}\right] = 420$

17. $\sum_{i=1}^{20}(i - 1)^2 = \sum_{i=1}^{19} i^2$

$$= \left[\frac{19(20)(39)}{6}\right] = 2470$$

19. $\displaystyle\sum_{i=1}^{15} i(i-1)^2 = \sum_{i=1}^{15} i^3 - 2\sum_{i=1}^{15} i^2 + \sum_{i=1}^{15} i$

$$= \frac{15^2(16)^2}{4} - 2\frac{15(16)(31)}{6} + \frac{15(16)}{2}$$

$$= 14{,}400 - 2{,}480 + 120$$

$$= 12{,}040$$

21. sum seq($x\boxed{\wedge}2 + 3, x, 1, 20, 1$) = 2930 *(TI-82)*

$$\sum_{i=1}^{20}(i^2+3) = \frac{20(20+1)(2(20)+1)}{6} + 3(20)$$

$$= \frac{(20)(21)(41)}{6} + 60 = 2930$$

23. $S = \left[3 + 4 + \frac{9}{2} + 5\right](1) = \frac{33}{2} = 16.5$

$s = \left[1 + 3 + 4 + \frac{9}{2}\right](1) = \frac{25}{2} = 12.5$

25. $S = [3 + 3 + 5](1) = 11$

$s = [2 + 2 + 3](1) = 7$

27. $S(4) = \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) + \sqrt{1}\left(\frac{1}{4}\right) = \frac{1+\sqrt{2}+\sqrt{3}+2}{8} \approx 0.768$

$s(4) = 0\left(\frac{1}{4}\right) + \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) = \frac{1+\sqrt{2}+\sqrt{3}}{8} \approx 0.518$

29. $S(5) = 1\left(\frac{1}{5}\right) + \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \approx 0.746$

$s(5) = \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5}\right) = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \approx 0.646$

31. $\displaystyle\lim_{n\to\infty}\left[\left(\frac{81}{n^4}\right)\frac{n^2(n+1)^2}{4}\right] = \frac{81}{4}\lim_{n\to\infty}\left[\frac{n^4+2n^3+n^2}{n^4}\right] = \frac{81}{4}(1) = \frac{81}{4}$

33. $\displaystyle\lim_{n\to\infty}\left[\left(\frac{18}{n^2}\right)\frac{n(n+1)}{2}\right] = \frac{18}{2}\lim_{n\to\infty}\left[\frac{n^2+n}{n^2}\right] = \frac{18}{2}(1) = 9$

35. $\displaystyle\sum_{i=1}^{n}\frac{2i+1}{n^2} = \frac{1}{n^2}\sum_{i=1}^{n}(2i+1) = \frac{1}{n^2}\left[2\frac{n(n+1)}{2} + n\right] = \frac{n+2}{n} = S(n)$

$S(10) = \frac{12}{10} = 1.2$

$S(100) = 1.02$

$S(1000) = 1.002$

$S(10{,}000) = 1.0002$

37. $\displaystyle\sum_{k=1}^{n}\frac{6k(k-1)}{n^3} = \frac{6}{n^3}\sum_{k=1}^{n}(k^2-k) = \frac{6}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right]$

$$= \frac{6}{n^2}\left[\frac{2n^2+3n+1-3n-3}{6}\right] = \frac{1}{n^2}[2n^2-2] = S(n)$$

$S(10) = 1.98$

$S(100) = 1.9998$

$S(1000) = 1.999998$

$S(10{,}000) = 1.99999998$

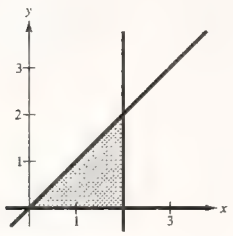
39. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{16i}{n^2}\right) = \lim_{n\to\infty}\frac{16}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty}\frac{16}{n^2}\left(\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\left[8\left(\frac{n^2+n}{n^2}\right)\right] = 8\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 8$

41. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n^3}(i-1)^2 = \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n-1}i^2 = \lim_{n\to\infty}\frac{1}{n^3}\left[\frac{(n-1)(n)(2n-1)}{6}\right]$

$$= \lim_{n\to\infty}\frac{1}{6}\left[\frac{2n^3-3n^2+n}{n^3}\right] = \lim_{n\to\infty}\left[\frac{1}{6}\left(\frac{2-(3/n)+(1/n^2)}{1}\right)\right] = \frac{1}{3}$$

43. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)\left(\frac{2}{n}\right) = 2\lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^{n}1+\frac{1}{n}\sum_{i=1}^{n}i\right] = 2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{1}{n}\left(\frac{n(n+1)}{2}\right)\right] = 2\lim_{n\to\infty}\left[1+\frac{n^2+n}{2n^2}\right] = 2\left(1+\frac{1}{2}\right) = 3$

45. (a)



(b) $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

Endpoints:

$$0 < 1\left(\frac{2}{n}\right) < 2\left(\frac{2}{n}\right) < \cdots < (n-1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right) = 2$$

(c) Since $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$= \sum_{i=1}^{n} f\left(\frac{2i-2}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\frac{2}{n} = \sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(e)

x	5	10	50	100
$s(n)$	1.6	1.8	1.96	1.98
$S(n)$	2.4	2.2	2.04	2.02

(f) $\lim_{n\to\infty}\sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n}(i-1)$

$$= \lim_{n\to\infty}\frac{4}{n^2}\left[\frac{n(n+1)}{2}-n\right]$$

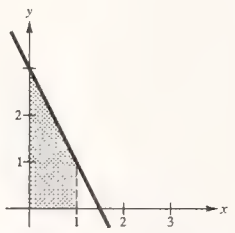
$$= \lim_{n\to\infty}\left[\frac{2(n+1)}{n}-\frac{4}{n}\right] = 2$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n}i$$

$$= \lim_{n\to\infty}\left(\frac{4}{n^2}\right)\frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty}\frac{2(n+1)}{n} = 2$$

47. $y = -2x + 3$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \frac{1-0}{n} = \frac{1}{n}\right)$

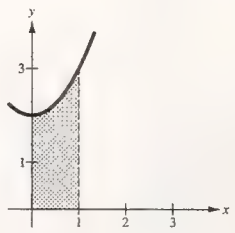


$$s(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[-2\left(\frac{i}{n}\right)+3\right]\left(\frac{1}{n}\right)$$

$$= 3 - \frac{2}{n^2}\sum_{i=1}^{n} i = 3 - \frac{2(n+1)n}{2n^2} = 2 - \frac{1}{n}$$

Area $= \lim_{n\to\infty} s(n) = 2$

49. $y = x^2 + 2$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \frac{1}{n}\right)$

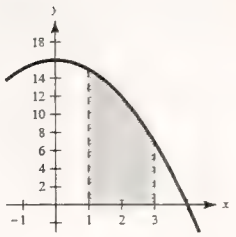


$$S(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^2+2\right]\left(\frac{1}{n}\right)$$

$$= \left[\frac{1}{n^3}\sum_{i=1}^{n} i^2\right] + 2 = \frac{n(n+1)(2n+1)}{6n^3} + 2 = \frac{1}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right) + 2$$

Area $= \lim_{n\to\infty} S(n) = \frac{7}{3}$

51. $y = 16 - x^2$ on $[1, 3]$. $\left(\textbf{Note: } \Delta x = \frac{2}{n}\right)$



$$s(n) = \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[16 - \left(1 + \frac{2i}{n}\right)^2\right]\left(\frac{2}{n}\right)$$

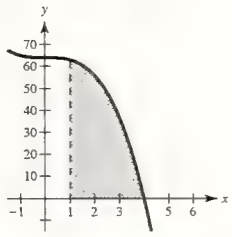
$$= \frac{2}{n}\sum_{i=1}^{n}\left[15 - \frac{4i^2}{n^2} - \frac{4i}{n}\right]$$

$$= \frac{2}{n}\left[15n - \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{4}{n}\frac{n(n+1)}{2}\right]$$

$$= 30 - \frac{8}{6n^2}(n+1)(2n+1) - \frac{4}{n}(n+1)$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 30 - \frac{8}{3} - 4 = \frac{70}{3} = 23\frac{1}{3}$$

53. $y = 64 - x^3$ on $[1, 4]$. $\left(\textbf{Note: } \Delta x = \frac{4-1}{n} = \frac{3}{n}\right)$



$$s(n) = \sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left[64 - \left(1 + \frac{3i}{n}\right)^3\right]\left(\frac{3}{n}\right)$$

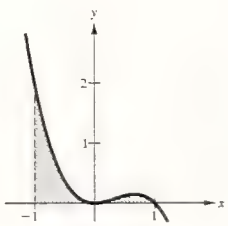
$$= \frac{3}{n}\sum_{i=1}^{n}\left[63 - \frac{27i^3}{n^3} - \frac{27i^2}{n^2} - \frac{9i}{n}\right]$$

$$= \frac{3}{n}\left[63n - \frac{27}{n^3}\frac{n^2(n+1)^2}{4} - \frac{27}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{9}{n}\frac{n(n+1)}{2}\right]$$

$$= 189 - \frac{81}{4n^2}(n+1)^2 - \frac{81}{6n^2}(n+1)(2n+1) - \frac{27}{2}\frac{n+1}{n}$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 189 - \frac{81}{4} - 27 - \frac{27}{2} = \frac{513}{4} = 128.25$$

55. $y = x^2 - x^3$ on $[-1, 1]$. $\left(\textbf{Note: } \Delta x = \frac{1-(-1)}{n} = \frac{2}{n}\right)$



Again, $T(n)$ is neither an upper nor a lower sum.

$$T(n) = \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[\left(-1 + \frac{2i}{n}\right)^2 - \left(-1 + \frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left[\left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right)\right]\left(\frac{2}{n}\right)$$

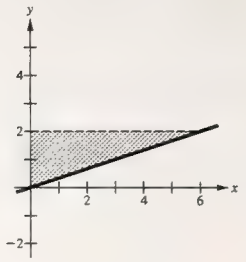
$$= \sum_{i=1}^{n}\left[2 - \frac{10i}{n} + \frac{16i^2}{n^2} - \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right) = \frac{4}{n}\sum_{i=1}^{n} 1 - \frac{20}{n^2}\sum_{i=1}^{n} i + \frac{32}{n^3}\sum_{i=1}^{n} i^2 - \frac{16}{n^4}\sum_{i=1}^{n} i^3$$

$$= \frac{4}{n}(n) - \frac{20}{n^2}\cdot\frac{n(n+1)}{2} + \frac{32}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{16}{n^4}\cdot\frac{n^2(n+1)^2}{4}$$

$$= 4 - 10\left(1 + \frac{1}{n}\right) + \frac{16}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) - 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$$

$$\text{Area} = \lim_{n\to\infty} T(n) = 4 - 10 + \frac{32}{3} - 4 = \frac{2}{3}$$

57. $f(y) = 3y,\ 0 \le y \le 2 \quad \left(\text{Note: } \Delta y = \dfrac{2-0}{n} = \dfrac{2}{n}\right)$

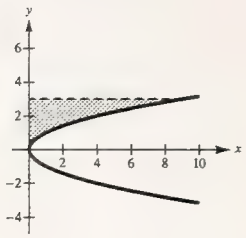


$$S(n) = \sum_{i=1}^{n} f(m_i)\,\Delta y = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n} 3\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \frac{12}{n^2}\sum_{i=1}^{n} i = \left(\frac{12}{n^2}\right)\cdot\frac{n(n+1)}{2} = \frac{6(n+1)}{n} = 6 + \frac{6}{n}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(6 + \frac{6}{n}\right) = 6$$

59. $f(y) = y^2,\ 0 \le y \le 3 \quad \left(\text{Note: } \Delta y = \dfrac{3-0}{n} = \dfrac{3}{n}\right)$

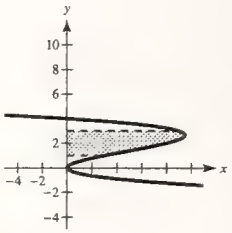


$$S(n) = \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^{n} \left(\frac{3i}{n}\right)^2\left(\frac{3}{n}\right) = \frac{27}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{9}{n^2}\left(\frac{2n^2+3n+1}{2}\right) = 9 + \frac{27}{2n} + \frac{9}{2n^2}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(9 + \frac{27}{2n} + \frac{9}{2n^2}\right) = 9$$

61. $g(y) = 4y^2 - y^3,\ 1 \le y \le 3. \quad \left(\text{Note: } \Delta y = \dfrac{3-1}{n} = \dfrac{2}{n}\right)$



$$S(n) = \sum_{i=1}^{n} g\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left[4\left(1 + \frac{2i}{n}\right)^2 - \left(1 + \frac{2i}{n}\right)^3\right]\frac{2}{n}$$

$$= \frac{2}{n}\sum_{i=1}^{n} 4\left[1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right] - \left[1 + \frac{6i}{n} + \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right]$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left[3 + \frac{10i}{n} + \frac{4i^2}{n^2} - \frac{8i^3}{n^3}\right] = \frac{2}{n}\left[3n + \frac{10}{n}\frac{n(n+1)}{2} + \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{8}{n^2}\frac{n^2(n+1)^2}{4}\right]$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 6 + 10 + \frac{8}{3} - 4 = \frac{44}{3}$$

63. $f(x) = x^2 + 3,\ 0 \le x \le 2,\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = \frac{1}{2},\ c_1 = \frac{1}{4},\ c_2 = \frac{3}{4},\ c_3 = \frac{5}{4},\ c_4 = \frac{7}{4}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}[c_i^2 + 3]\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\left[\left(\frac{1}{16} + 3\right) + \left(\frac{9}{16} + 3\right) + \left(\frac{25}{16} + 3\right) + \left(\frac{49}{16} + 3\right)\right]$$

$$= \frac{69}{8}$$

65. $f(x) = \tan x,\ 0 \le x \le \dfrac{\pi}{4},\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = \frac{\pi}{16},\ c_1 = \frac{\pi}{32},\ c_2 = \frac{3\pi}{32},\ c_3 = \frac{5\pi}{32},\ c_4 = \frac{7\pi}{32}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}(\tan c_i)\left(\frac{\pi}{16}\right)$$

$$= \frac{\pi}{16}\left(\tan\frac{\pi}{32} + \tan\frac{3\pi}{32} + \tan\frac{5\pi}{32} + \tan\frac{7\pi}{32}\right) \approx 0.345$$

67. $f(x) = \sqrt{x}$ on $[0, 4]$.

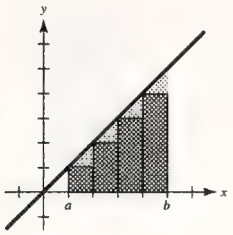
n	4	8	12	16	20
Approximate area	5.3838	5.3523	5.3439	5.3403	5.3384

(Exact value is 16/3)

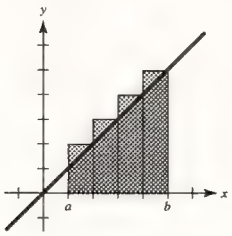
69. $f(x) = \tan\left(\frac{\pi x}{8}\right)$ on $[1, 3]$.

n	4	8	12	16	20
Approximate area	2.2223	2.2387	2.2418	2.2430	2.2435

71. We can use the line $y = x$ bounded by $x = a$ and $x = b$. The sum of the areas of these inscribed rectangles is the lower sum.



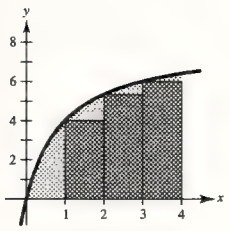
The sum of the areas of these circumscribed rectangles is the upper sum.



We can see that the rectangles do not contain all of the area in the first graph and the rectangles in the second graph cover more than the area of the region.

The exact value of the area lies between these two sums.

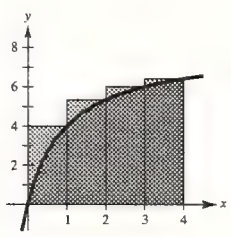
73. (a)



Lower sum:

$$s(4) = 0 + 4 + 5\tfrac{1}{3} + 6 = 15\tfrac{1}{3} = \tfrac{46}{3} \approx 15.333$$

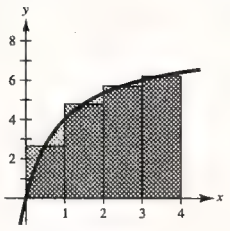
(b)



Upper sum:

$$S(4) = 4 + 5\tfrac{1}{3} + 6 + 6\tfrac{2}{5} = 21\tfrac{11}{15} = \tfrac{326}{15} \approx 21.733$$

(c)



Midpoint Rule:

$$M(4) = 2\tfrac{2}{3} + 4\tfrac{4}{5} + 5\tfrac{5}{7} + 6\tfrac{2}{9} = \tfrac{6112}{315} \approx 19.403$$

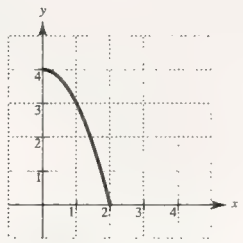
(d) In each case, $\Delta x = 4/n$. The lower sum uses left endpoints, $(i - 1)(4/n)$. The upper sum uses right endpoints, $(i)(4/n)$. The Midpoint Rule uses midpoints, $\left(i - \frac{1}{2}\right)(4/n)$.

(e)

n	4	8	20	100	200
$s(n)$	15.333	17.368	18.459	18.995	19.06
$S(n)$	21.733	20.568	19.739	19.251	19.188
$M(n)$	19.403	19.201	19.137	19.125	19.125

(f) $s(n)$ increases because the lower sum approaches the exact value as n increases. $S(n)$ decreases because the upper sum approaches the exact value as n increases. Because of the shape of the graph, the lower sum is always smaller than the exact value, whereas the upper sum is always larger.

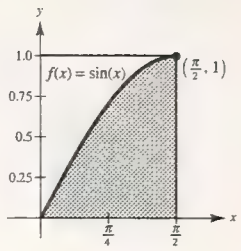
75.



b. $A \approx 6$ square units

77. True. (Theorem 4.2 (2))

79. $f(x) = \sin x, \left[0, \frac{\pi}{2}\right]$



Let A_1 = area bounded by $f(x) = \sin x$, the x-axis, $x = 0$ and $x = \pi/2$. Let A_2 = area of the rectangle bounded by $y = 1$, $y = 0$, $x = 0$, and $x = \pi/2$. Thus, $A_2 = (\pi/2)(1) \approx 1.570796$.
In this program, the computer is generating N_2 pairs of random points in the rectangle whose area is represented by A_2. It is keeping track of how many of these points, N_1, lie in the region whose area is represented by A_1. Since the points are randomly generated, we assume that

$$\frac{A_1}{A_2} \approx \frac{N_1}{N_2} \Rightarrow A_1 \approx \frac{N_1}{N_2} A_2.$$

The larger N_2 is the better the approximation to A_1.

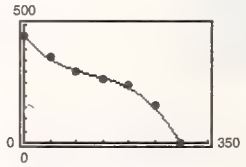
81. Suppose there are n rows in the figure. The stars on the left total $1 + 2 + \cdots + n$, as do the stars on the right. There are $n(n + 1)$ stars in total, hence

$$2[1 + 2 + \cdots + n] = n(n + 1)$$

$$1 + 2 + \cdots + n = \tfrac{1}{2}(n)(n + 1).$$

83. (a) $y = (-4.09 \times 10^{-5})x^3 + 0.016x^2 - 2.67x + 452.9$

(b)

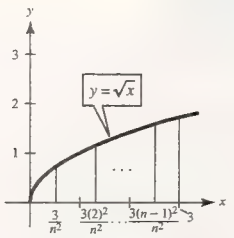


(c) Using the integration capability of a graphing utility, you obtain

$A \approx 76{,}897.5$ ft^2.

Section 4.3 Riemann Sums and Definite Integrals

1. $f(x) = \sqrt{x}, y = 0, x = 0, x = 3, c_i = \dfrac{3i^2}{n^2}$



$$\Delta x_i = \frac{3i^2}{n^2} - \frac{3(i-1)^2}{n^2} = \frac{3}{n^2}(2i - 1)$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{3i^2}{n^2}} \frac{3}{n^2}(2i - 1)$$

$$= \lim_{n\to\infty} \frac{3\sqrt{3}}{n^3} \sum_{i=1}^{n} (2i^2 - i)$$

$$= \lim_{n\to\infty} \frac{3\sqrt{3}}{n^3} \left[2\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} 3\sqrt{3}\left[\frac{(n+1)(2n+1)}{3n^2} - \frac{n+1}{2n^2}\right]$$

$$= 3\sqrt{3}\left[\frac{2}{3} - 0\right] = 2\sqrt{3} \approx 3.464$$

3. $y = 6$ on $[4, 10]$. $\left(\textbf{Note: } \Delta x = \dfrac{10-4}{n} = \dfrac{6}{n},\ \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(4 + \frac{6i}{n}\right)\left(\frac{6}{n}\right) = \sum_{i=1}^{n} 6\left(\frac{6}{n}\right) = \sum_{i=1}^{n} \frac{36}{n} = 36$$

$$\int_4^{10} 6\,dx = \lim_{n\to\infty} 36 = 36$$

5. $y = x^3$ on $[-1, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-(-1)}{n} = \dfrac{2}{n},\ \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n} \left(-1 + \frac{2i}{n}\right)^3\left(\frac{2}{n}\right) = \sum_{i=1}^{n} \left[-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right)$$

$$= -2 + \frac{12}{n^2}\sum_{i=1}^{n} i - \frac{24}{n^3}\sum_{i=1}^{n} i^2 + \frac{16}{n^4}\sum_{i=1}^{n} i^3$$

$$= -2 + 6\left(1 + \frac{1}{n}\right) - 4\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{2}{n}$$

$$\int_{-1}^{1} x^3\,dx = \lim_{n\to\infty} \frac{2}{n} = 0$$

7. $y = x^2 + 1$ on $[1, 2]$. $\left(\textbf{Note: } \Delta x = \dfrac{2-1}{n} = \dfrac{1}{n},\ \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n} \left[\left(1 + \frac{i}{n}\right)^2 + 1\right]\left(\frac{1}{n}\right) = \sum_{i=1}^{n} \left[1 + \frac{2i}{n} + \frac{i^2}{n^2} + 1\right]\left(\frac{1}{n}\right)$$

$$= 2 + \frac{2}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2 = 2 + \left(1 + \frac{1}{n}\right) + \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}$$

$$\int_1^2 (x^2 + 1)\,dx = \lim_{n\to\infty} \left(\frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}\right) = \frac{10}{3}$$

9. $\lim\limits_{\|\Delta\|\to 0} \sum\limits_{i=1}^{n} (3c_i + 10)\,\Delta x_i = \int_{-1}^{5} (3x + 10)\,dx$

on the interval $[-1, 5]$.

11. $\lim\limits_{\|\Delta\|\to 0} \sum\limits_{i=1}^{n} \sqrt{c_i^2 + 4}\,\Delta x_i = \int_0^3 \sqrt{x^2 + 4}\,dx$

on the interval $[0, 3]$.

13. $\displaystyle\int_0^5 3\,dx$

15. $\displaystyle\int_{-4}^{4} (4 - |x|)\,dx$

17. $\displaystyle\int_{-2}^{2} (4 - x^2)\,dx$

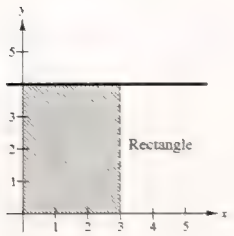
19. $\displaystyle\int_0^{\pi} \sin x\,dx$

21. $\displaystyle\int_0^2 y^3\,dy$

23. Rectangle

$$A = bh = 3(4)$$

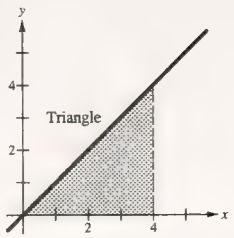
$$A = \int_0^3 4\,dx = 12$$



25. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(4)(4)$$

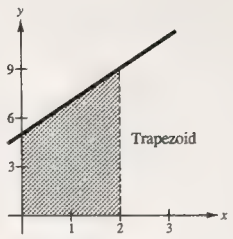
$$A = \int_0^4 x\,dx = 8$$



27. Trapezoid

$$A = \frac{b_1 + b_2}{2}h = \left(\frac{5+9}{2}\right)2$$

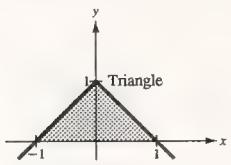
$$A = \int_0^2 (2x + 5)\,dx = 14$$



29. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(2)(1)$$

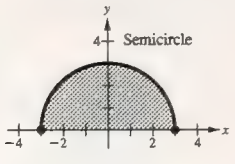
$$A = \int_{-1}^{1} (1 - |x|)\,dx = 1$$



31. Semicircle

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(3)^2$$

$$A = \int_{-3}^{3} \sqrt{9 - x^2}\,dx = \frac{9\pi}{2}$$



In Exercises 33–39, $\int_2^4 x^3\,dx = 60$, $\int_2^4 x\,dx = 6$, $\int_2^4 dx = 2$

33. $\displaystyle\int_4^2 x\,dx = -\int_2^4 x\,dx = -6$

35. $\displaystyle\int_2^4 4x\,dx = 4\int_2^4 x\,dx = 4(6) = 24$

37. $\displaystyle\int_2^4 (x - 8)\,dx = \int_2^4 x\,dx - 8\int_2^4 dx = 6 - 8(2) = -10$

39. $\displaystyle\int_2^4 \left(\frac{1}{2}x^3 - 3x + 2\right)dx = \frac{1}{2}\int_2^4 x^3\,dx - 3\int_2^4 x\,dx + 2\int_2^4 dx$

$$= \frac{1}{2}(60) - 3(6) + 2(2) = 16$$

41. (a) $\displaystyle\int_0^7 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx = 10 + 3 = 13$

(b) $\displaystyle\int_5^0 f(x)\,dx = -\int_0^5 f(x)\,dx = -10$

(c) $\displaystyle\int_5^5 f(x)\,dx = 0$

(d) $\displaystyle\int_0^5 3f(x)\,dx = 3\int_0^5 f(x)\,dx = 3(10) = 30$

43. (a) $\displaystyle\int_2^6 [f(x) + g(x)]\,dx = \int_2^6 f(x)\,dx + \int_2^6 g(x)\,dx$

$$= 10 + (-2) = 8$$

(b) $\displaystyle\int_2^6 [g(x) - f(x)]\,dx = \int_2^6 g(x)\,dx - \int_2^6 f(x)\,dx$

$$= -2 - 10 = -12$$

(c) $\displaystyle\int_2^6 2g(x)\,dx = 2\int_2^6 g(x)\,dx = 2(-2) = -4$

(d) $\displaystyle\int_2^6 3f(x)\,dx = 3\int_2^6 f(x)\,dx = 3(10) = 30$

45. (a) Quarter circle below x-axis: $-\frac{1}{4}\pi r^2 = -\frac{1}{4}\pi(2)^2 = -\pi$

(b) Triangle: $\frac{1}{2}bh = \frac{1}{2}(4)(2) = 4$

(c) Triangle + Semicircle below x-axis: $-\frac{1}{2}(2)(1) - \frac{1}{2}\pi(2)^2 = -(1 + 2\pi)$

(d) Sum of parts (b) and (c): $4 - (1 + 2\pi) = 3 - 2\pi$

(e) Sum of absolute values of (b) and (c): $4 + (1 + 2\pi) = 5 + 2\pi$

(f) Answer to (d) plus $2(10) = 20$: $(3 - 2\pi) + 20 = 23 - 2\pi$

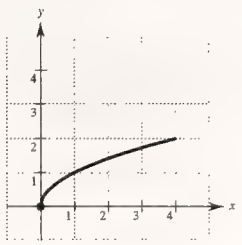
47. The left endpoint approximation will be greater than the actual area: $>$

49. Because the curve is concave upward, the midpoint approximation will be less than the actual area: $<$

51. $f(x) = \dfrac{1}{x-4}$

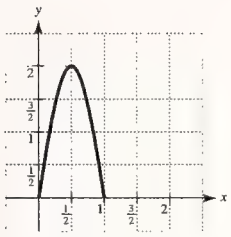
is not integrable on the interval $[3, 5]$ and f has a discontinuity at $x = 4$.

53.



a. $A \approx 5$ square units

55.



d. $\displaystyle\int_0^1 2 \sin \pi x \, dx \approx \frac{1}{2}(1)(2) \approx 1$

57. $\displaystyle\int_0^3 x\sqrt{3-x}\, dx$

n	4	8	12	16	20
$L(n)$	3.6830	3.9956	4.0707	4.1016	4.1177
$M(n)$	4.3082	4.2076	4.1838	4.1740	4.1690
$R(n)$	3.6830	3.9956	4.0707	4.1016	4.1177

59. $\displaystyle\int_0^{\pi/2} \sin^2 x \, dx$

n	4	8	12	16	20
$L(n)$	0.5890	0.6872	0.7199	0.7363	0.7461
$M(n)$	0.7854	0.7854	0.7854	0.7854	0.7854
$R(n)$	0.9817	0.8836	0.8508	0.8345	0.8247

61. True

63. True

65. False

$\displaystyle\int_0^2 (-x)\, dx = -2$

67. $f(x) = x^2 + 3x$, $[0, 8]$

$x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 7, x_4 = 8$

$\Delta x_1 = 1, \Delta x_2 = 2, \Delta x_3 = 4, \Delta x_4 = 1$

$c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 8$

$$\sum_{i=1}^{4} f(c_i)\,\Delta x = f(1)\,\Delta x_1 + f(2)\,\Delta x_2 + f(5)\,\Delta x_3 + f(8)\,\Delta x_4$$

$$= (4)(1) + (10)(2) + (40)(4) + (88)(1) = 272$$

69. $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$

is not integrable on the interval $[0, 1]$. As $\|\Delta\| \to 0, f(c_i) = 1$ or $f(c_i) = 0$ in each subinterval since there are an infinite number of both rational and irrational numbers in any interval, no matter how small.

71. Let $f(x) = x^2$, $0 \le x \le 1$, and $\Delta x_i = 1/n$. The appropriate Riemann Sum is

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2.$$

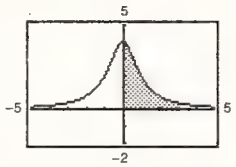
$$\lim_{n\to\infty} \frac{1}{n^3}[1^2 + 2^2 + 3^2 + \cdots + n^2] = \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n(2n+1)(n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{2n^2 + 3n + 1}{6n^2} = \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}$$

Section 4.4 The Fundamental Theorem of Calculus

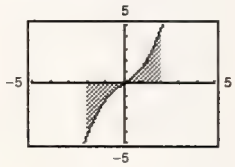
1. $f(x) = \dfrac{4}{x^2 + 1}$

$\displaystyle\int_0^{\pi} \frac{4}{x^2 + 1}\,dx$ is positive.



3. $f(x) = x\sqrt{x^2 + 1}$

$\displaystyle\int_{-2}^{2} x\sqrt{x^2 + 1}\,dx = 0$



5. $\displaystyle\int_0^1 2x\,dx = \left[x^2\right]_0^1 = 1 - 0 = 1$

7. $\displaystyle\int_{-1}^{0} (x - 2)\,dx = \left[\frac{x^2}{2} - 2x\right]_{-1}^{0} = 0 - \left(\frac{1}{2} + 2\right) = -\frac{5}{2}$

9. $\displaystyle\int_{-1}^{1} (t^2 - 2)\,dt = \left[\frac{t^3}{3} - 2t\right]_{-1}^{1} = \left(\frac{1}{3} - 2\right) - \left(-\frac{1}{3} + 2\right) = -\frac{10}{3}$

11. $\displaystyle\int_0^1 (2t - 1)^2\,dt = \int_0^1 (4t^2 - 4t + 1)\,dt = \left[\frac{4}{3}t^3 - 2t^2 + t\right]_0^1 = \frac{4}{3} - 2 + 1 = \frac{1}{3}$

13. $\displaystyle\int_1^2 \left(\frac{3}{x^2} - 1\right) dx = \left[-\frac{3}{x} - x\right]_1^2 = \left(-\frac{3}{2} - 2\right) - (-3 - 1) = \frac{1}{2}$

15. $\displaystyle\int_1^4 \frac{u - 2}{\sqrt{u}}\,du = \int_1^4 (u^{1/2} - 2u^{-1/2})\,du = \left[\frac{2}{3}u^{3/2} - 4u^{1/2}\right]_1^4 = \left[\frac{2}{3}\left(\sqrt{4}\right)^3 - 4\sqrt{4}\right] - \left[\frac{2}{3} - 4\right] = \frac{2}{3}$

17. $\displaystyle\int_{-1}^{1} \left(\sqrt[3]{t} - 2\right) dt = \left[\frac{3}{4}t^{4/3} - 2t\right]_{-1}^{1} = \left(\frac{3}{4} - 2\right) - \left(\frac{3}{4} + 2\right) = -4$

19. $\displaystyle\int_0^1 \frac{x - \sqrt{x}}{3}\,dx = \frac{1}{3}\int_0^1 (x - x^{1/2})\,dx = \frac{1}{3}\left[\frac{x^2}{2} - \frac{2}{3}x^{3/2}\right]_0^1 = \frac{1}{3}\left(\frac{1}{2} - \frac{2}{3}\right) = -\frac{1}{18}$

21. $\displaystyle\int_{-1}^{0} (t^{1/3} - t^{2/3})\,dt = \left[\frac{3}{4}t^{4/3} - \frac{3}{5}t^{5/3}\right]_{-1}^{0} = 0 - \left(\frac{3}{4} + \frac{3}{5}\right) = -\frac{27}{20}$

23. $\displaystyle\int_0^3 |2x - 3|\,dx = \int_0^{3/2} (3 - 2x)\,dx + \int_{3/2}^{3} (2x - 3)\,dx$ $\left(\text{split up the integral at the zero } x = \frac{3}{2}\right)$

$$= \left[3x - x^2\right]_0^{3/2} + \left[x^2 - 3x\right]_{3/2}^{3} = \left(\frac{9}{2} - \frac{9}{4}\right) - 0 + (9 - 9) - \left(\frac{9}{4} - \frac{9}{2}\right) = 2\left(\frac{9}{2} - \frac{9}{4}\right) = \frac{9}{2}$$

25. $\displaystyle\int_0^3 |x^2 - 4|\,dx = \int_0^2 (4 - x^2)\,dx + \int_2^3 (x^2 - 4)\,dx$

$$= \left[4x - \frac{x^3}{3}\right]_0^2 + \left[\frac{x^3}{3} - 4x\right]_2^3$$

$$= \left(8 - \frac{8}{3}\right) + (9 - 12) - \left(\frac{8}{3} - 8\right)$$

$$= \frac{23}{3}$$

27. $\displaystyle\int_0^{\pi} (1 + \sin x)\,dx = \Big[x - \cos x\Big]_0^{\pi} = (\pi + 1) - (0 - 1) = 2 + \pi$

29. $\displaystyle\int_{-\pi/6}^{\pi/6} \sec^2 x\,dx = \Big[\tan x\Big]_{-\pi/6}^{\pi/6} = \frac{\sqrt{3}}{3} - \left(-\frac{\sqrt{3}}{3}\right) = \frac{2\sqrt{3}}{3}$

31. $\displaystyle\int_{-\pi/3}^{\pi/3} 4 \sec\theta \tan\theta\,d\theta = \Big[4 \sec\theta\Big]_{-\pi/3}^{\pi/3} = 4(2) - 4(2) = 0$

33. $\displaystyle\int_0^3 10{,}000(t - 6)\,dt = 10{,}000\left[\frac{t^2}{2} - 6t\right]_0^3 = -\$135{,}000$

35. $\displaystyle A = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$

37. $\displaystyle A = \int_0^3 (3 - x)\sqrt{x}\,dx = \int_0^3 (3x^{1/2} - x^{3/2})\,dx = \left[2x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^3 = \left[\frac{x\sqrt{x}}{5}(10 - 2x)\right]_0^3 = \frac{12\sqrt{3}}{5}$

39. $\displaystyle A = \int_0^{\pi/2} \cos x\,dx = \Big[\sin x\Big]_0^{\pi/2} = 1$

41. Since $y \geq 0$ on $[0, 2]$,

$$A = \int_0^2 (3x^2 + 1)\,dx = \Big[x^3 + x\Big]_0^2 = 8 + 2 = 10.$$

43. Since $y \geq 0$ on $[0, 2]$,

$$A = \int_0^2 (x^3 + x)\,dx = \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^2 = 4 + 2 = 6.$$

45. $\displaystyle\int_0^2 \left(x - 2\sqrt{x}\right)dx = \left[\frac{x^2}{2} - \frac{4x^{3/2}}{3}\right]_0^2 = 2 - \frac{8\sqrt{2}}{3}$

$$f(c)(2 - 0) = \frac{6 - 8\sqrt{2}}{3}$$

$$c - 2\sqrt{c} = \frac{3 - 4\sqrt{2}}{3}$$

$$c - 2\sqrt{c} + 1 = \frac{3 - 4\sqrt{2}}{3} + 1$$

$$\left(\sqrt{c} - 1\right)^2 = \frac{6 - 4\sqrt{2}}{3}$$

$$\sqrt{c} - 1 = \pm\sqrt{\frac{6 - 4\sqrt{2}}{3}}$$

$$c = \left[1 \pm \sqrt{\frac{6 - 4\sqrt{2}}{3}}\right]^2$$

$$c \approx 0.4380 \text{ or } c \approx 1.7908$$

47. $\displaystyle\int_{-\pi/4}^{\pi/4} 2\sec^2 x\,dx = \Big[2\tan x\Big]_{-\pi/4}^{\pi/4} = 2(1) - 2(-1) = 4$

$$f(c)\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 4$$

$$2\sec^2 c = \frac{8}{\pi}$$

$$\sec^2 c = \frac{4}{\pi}$$

$$\sec c = \pm\frac{2}{\sqrt{\pi}}$$

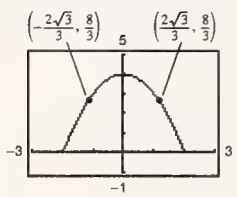
$$c = \pm\operatorname{arcsec}\left(\frac{2}{\sqrt{\pi}}\right)$$

$$= \pm\arccos\frac{\sqrt{\pi}}{2} \approx \pm 0.4817$$

49. $\displaystyle\frac{1}{2-(-2)}\int_{-2}^{2}(4 - x^2)\,dx = \frac{1}{4}\left[4x - \frac{1}{3}x^3\right]_{-2}^{2} = \frac{1}{4}\left[\left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)\right] = \frac{8}{3}$

Average value $= \dfrac{8}{3}$

$4 - x^2 = \dfrac{8}{3}$ when $x^2 = 4 - \dfrac{8}{3}$ or $x = \pm\dfrac{2\sqrt{3}}{3} \approx \pm 1.155$.

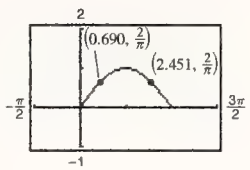


51. $\displaystyle\frac{1}{\pi - 0}\int_0^{\pi}\sin x\,dx = \left[-\frac{1}{\pi}\cos x\right]_0^{\pi} = \frac{2}{\pi}$

Average value $= \dfrac{2}{\pi}$

$\sin x = \dfrac{2}{\pi}$

$x \approx 0.690, 2.451$



53. If f is continuous on $[a, b]$ and $F'(x) = f(x)$ on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

55. $\displaystyle\int_0^2 f(x)\,dx = -(\text{area of region } A) = -1.5$

57. $\displaystyle\int_0^6 |f(x)|\,dx = -\int_0^2 f(x)\,dx + \int_2^6 f(x)\,dx = 1.5 + 5.0 = 6.5$

59. $\displaystyle\int_0^6 [2 + f(x)]\,dx = \int_0^6 2\,dx + \int_0^6 f(x)\,dx$

$= 12 + 3.5 = 15.5$

61. (a) $F(x) = k\sec^2 x$

$F(0) = k = 500$

$F(x) = 500\sec^2 x$

(b) $\displaystyle\frac{1}{\pi/3 - 0}\int_0^{\pi/3} 500\sec^2 x\,dx = \frac{1500}{\pi}\Big[\tan x\Big]_0^{\pi/3}$

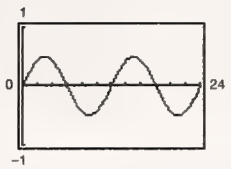
$= \dfrac{1500}{\pi}(\sqrt{3} - 0)$

≈ 826.99 newtons

≈ 827 newtons

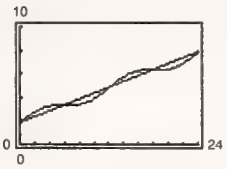
63. $\displaystyle\frac{1}{5 - 0}\int_0^5 (0.1729t + 0.1552t^2 - 0.0374t^3)\,dt \approx \frac{1}{5}\Big[0.08645t^2 + 0.05073t^3 - 0.00935t^4\Big]_0^5 \approx 0.5318$ liter

65. (a)



The area above the x-axis equals the area below the x-axis. Thus, the average value is zero.

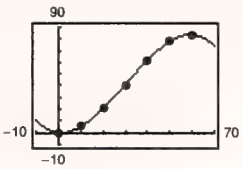
(b)



The average value of S appears to be g.

67. (a) $v = -8.61 \times 10^{-4}t^3 + 0.0782t^2 - 0.208t + 0.0952$

(b)



(c) $\displaystyle\int_0^{60} v(t)\,dt = \left[\frac{-8.61 \times 10^{-4}t^4}{4} + \frac{0.0782t^3}{3} - \frac{0.208t^2}{2} + 0.0952t\right]_0^{60} \approx 2476 \text{ meters}$

69. $\displaystyle F(x) = \int_0^x (t-5)\,dt = \left[\frac{t^2}{2} - 5t\right]_0^x = \frac{x^2}{2} - 5x$

$\displaystyle F(2) = \frac{4}{2} - 5(2) = -8$

$\displaystyle F(5) = \frac{25}{2} - 5(5) = -\frac{25}{2}$

$\displaystyle F(8) = \frac{64}{2} - 5(8) = -8$

71. $\displaystyle F(x) = \int_1^x \frac{10}{v^2}\,dv = \int_1^x 10v^{-2}\,dv = \left[\frac{-10}{v}\right]_1^x$

$\displaystyle = -\frac{10}{x} + 10 = 10\left(1 - \frac{1}{x}\right)$

$\displaystyle F(2) = 10\left(\frac{1}{2}\right) = 5$

$\displaystyle F(5) = 10\left(\frac{4}{5}\right) = 8$

$\displaystyle F(8) = 10\left(\frac{7}{8}\right) = \frac{35}{4}$

73. $\displaystyle F(x) = \int_1^x \cos\theta\,d\theta = \sin\theta\Big]_1^x = \sin x - \sin 1$

$F(2) = \sin 2 - \sin 1 = 0.0678$

$F(5) = \sin 5 - \sin 1 \approx -1.8004$

$F(8) = \sin 8 - \sin 1 \approx 0.1479$

75. (a) $\displaystyle\int_0^x (t+2)\,dt = \left[\frac{t^2}{2} + 2t\right]_0^x = \frac{1}{2}x^2 + 2x$

(b) $\displaystyle\frac{d}{dx}\left[\frac{1}{2}x^2 + 2x\right] = x + 2$

77. (a) $\displaystyle\int_8^x \sqrt[3]{t}\,dt = \left[\frac{3}{4}t^{4/3}\right]_8^x = \frac{3}{4}(x^{4/3} - 16) = \frac{3}{4}x^{4/3} - 12$

(b) $\displaystyle\frac{d}{dx}\left[\frac{3}{4}x^{4/3} - 12\right] = x^{1/3} = \sqrt[3]{x}$

79. (a) $\displaystyle\int_{x/4}^x \sec^2 t\,dt = \Big[\tan t\Big]_{x/4}^x = \tan x - 1$

(b) $\displaystyle\frac{d}{dx}[\tan x - 1] = \sec^2 x$

81. $\displaystyle F(x) = \int_{-2}^x (t^2 - 2t)\,dt$

$F'(x) = x^2 - 2x$

83. $\displaystyle F(x) = \int_{-1}^x \sqrt{t^4 + 1}\,dt$

$F'(x) = \sqrt{x^4 + 1}$

85. $\displaystyle F(x) = \int_0^x t\cos t\,dt$

$F'(x) = x\cos x$

87. $F(x) = \int_x^{x+2} (4t + 1)\,dt$

$= \left[2t^2 + t\right]_x^{x+2}$

$= [2(x + 2)^2 + (x + 2)] - [2x^2 + x]$

$= 8x + 10$

$F'(x) = 8$

Alternate solution:

$F(x) = \int_x^{x+2} (4t + 1)\,dt$

$= \int_x^0 (4t + 1)\,dt + \int_0^{x+2} (4t + 1)\,dt$

$= -\int_0^x (4t + 1)\,dt + \int_0^{x+2} (4t + 1)\,dt$

$F'(x) = -(4x + 1) + 4(x + 2) + 1 = 8$

89. $F(x) = \int_0^{\sin x} \sqrt{t}\,dt = \left[\frac{2}{3}t^{3/2}\right]_0^{\sin x} = \frac{2}{3}(\sin x)^{3/2}$

$F'(x) = (\sin x)^{1/2}\cos x = \cos x\sqrt{\sin x}$

Alternate solution

$F(x) = \int_0^{\sin x} \sqrt{t}\,dt$

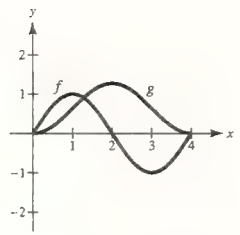
$F'(x) = \sqrt{\sin x}\,\frac{d}{dx}(\sin x) = \sqrt{\sin x}(\cos x)$

91. $F(x) = \int_0^{x^3} \sin t^2\,dt$

$F'(x) = \sin(x^3)^2 \cdot 3x^2 = 3x^2 \sin x^6$

93. $g(x) = \int_0^x f(t)\,dt$

$g(0) = 0, g(1) \approx \frac{1}{2}, g(2) \approx 1, g(3) \approx \frac{1}{2}, g(4) = 0$



g has a relative maximum at $x = 2$.

95. (a) $C(x) = 5000\left(25 + 3\int_0^x t^{1/4}\,dt\right)$

$= 5000\left(25 + 3\left[\frac{4}{5}t^{5/4}\right]_0^x\right)$

$= 5000\left(25 + \frac{12}{5}x^{5/4}\right) = 1000(125 + 12x^{5/4})$

(b) $C(1) = 1000(125 + 12(1)) = \$137{,}000$

$C(5) = 1000(125 + 12(5)^{5/4}) \approx \$214{,}721$

$C(10) = 1000(125 + 12(10)^{5/4}) \approx \$338{,}394$

97. True

99. False; $\int_{-1}^1 x^{-2}dx = \int_{-1}^0 x^{-2}dx + \int_0^1 x^{-2}dx$

Each of these integrals is infinite. $f(x) = x^{-2}$ has a nonremovable discontinuity at $x = 0$.

101. $f(x) = \int_0^{1/x} \frac{1}{t^2 + 1}\,dt + \int_0^x \frac{1}{t^2 + 1}\,dt$

By the Second Fundamental Theorem of Calculus, we have

$f'(x) = \frac{1}{(1/x)^2 + 1}\left(-\frac{1}{x^2}\right) + \frac{1}{x^2 + 1}$

$= -\frac{1}{1 + x^2} + \frac{1}{x^2 + 1} = 0.$

Since $f'(x) = 0, f(x)$ must be constant.

103. $x(t) = t^3 - 6t^2 + 9t - 2$

$x'(t) = 3t^2 - 12t + 9$

$= 3(t^2 - 4t + 3)$

$= 3(t - 3)(t - 1)$

$$\text{Total distance} = \int_0^5 |x'(t)|\,dt$$

$$= \int_0^5 3|(t - 3)(t - 1)|\,dt$$

$$= 3\int_0^1 (t^2 - 4t + 3)\,dt - 3\int_1^3 (t^2 - 4t + 3)\,dt + 3\int_3^5 (t^2 - 4t + 3)\,dt$$

$$= 4 + 4 + 20$$

$$= 28 \text{ units}$$

105. $$\text{Total distance} = \int_1^4 |x'(t)|\,dt$$

$$= \int_1^4 |v(t)|\,dt$$

$$= \int_1^4 \frac{1}{\sqrt{t}}\,dt$$

$$= 2t^{1/2}\Big]_1^4$$

$$= 2(2 - 1) = 2 \text{ units}$$

Section 4.5 Integration by Substitution

$\int f(g(x))g'(x)\,dx$	$u = g(x)$	$du = g'(x)\,dx$
1. $\int (5x^2 + 1)^2(10x)\,dx$	$5x^2 + 1$	$10x\,dx$
3. $\int \frac{x}{\sqrt{x^2 + 1}}\,dx$	$x^2 + 1$	$2x\,dx$
5. $\int \tan^2 x \sec^2 x\,dx$	$\tan x$	$\sec^2 x\,dx$

7. $$\int (1 + 2x)^4\,2\,dx = \frac{(1 + 2x)^5}{5} + C$$

Check: $\dfrac{d}{dx}\left[\dfrac{(1 + 2x)^5}{5} + C\right] = 2(1 + 2x)^4$

9. $$\int (9 - x^2)^{1/2}(-2x)\,dx = \frac{(9 - x^2)^{3/2}}{3/2} + C = \frac{2}{3}(9 - x^2)^{3/2} + C$$

Check: $\dfrac{d}{dx}\left[\dfrac{2}{3}(9 - x^2)^{3/2} + C\right] = \dfrac{2}{3} \cdot \dfrac{3}{2}(9 - x^2)^{1/2}(-2x) = \sqrt{9 - x^2}(-2x)$

11. $\displaystyle\int x^3(x^4+3)^2\,dx = \frac{1}{4}\int (x^4+3)^2(4x^3)\,dx = \frac{1}{4}\frac{(x^4+3)^3}{3} + C = \frac{(x^4+3)^3}{12} + C$

Check: $\displaystyle\frac{d}{dx}\left[\frac{(x^4+3)^3}{12} + C\right] = \frac{3(x^4+3)^2}{12}(4x^3) = (x^4+3)^2(x^3)$

13. $\displaystyle\int x^2(x^3-1)^4\,dx = \frac{1}{3}\int (x^3-1)^4(3x^2)\,dx = \frac{1}{3}\left[\frac{(x^3-1)^5}{5}\right] + C = \frac{(x^3-1)^5}{15} + C$

Check: $\displaystyle\frac{d}{dx}\left[\frac{(x^3-1)^5}{15} + C\right] = \frac{5(x^3-1)^4(3x^2)}{15} = x^2(x^3-1)^4$

15. $\displaystyle\int t\sqrt{t^2+2}\,dt = \frac{1}{2}\int (t^2+2)^{1/2}(2t)\,dt = \frac{1}{2}\frac{(t^2+2)^{3/2}}{3/2} + C = \frac{(t^2+2)^{3/2}}{3} + C$

Check: $\displaystyle\frac{d}{dt}\left[\frac{(t^2+2)^{3/2}}{3} + C\right] = \frac{3/2(t^2+2)^{1/2}(2t)}{3} = (t^2+2)^{1/2}t$

17. $\displaystyle\int 5x(1-x^2)^{1/3}\,dx = -\frac{5}{2}\int (1-x^2)^{1/3}(-2x)\,dx = -\frac{5}{2}\cdot\frac{(1-x^2)^{4/3}}{4/3} + C = -\frac{15}{8}(1-x^2)^{4/3} + C$

Check: $\displaystyle\frac{d}{dx}\left[-\frac{15}{8}(1-x^2)^{4/3} + C\right] = -\frac{15}{8}\cdot\frac{4}{3}(1-x^2)^{1/3}(-2x) = 5x(1-x^2)^{1/3} = 5x\sqrt[3]{1-x^2}$

19. $\displaystyle\int \frac{x}{(1-x^2)^3}\,dx = -\frac{1}{2}\int (1-x^2)^{-3}(-2x)\,dx = -\frac{1}{2}\frac{(1-x^2)^{-2}}{-2} + C = \frac{1}{4(1-x^2)^2} + C$

Check: $\displaystyle\frac{d}{dx}\left[\frac{1}{4(1-x^2)^2} + C\right] = \frac{1}{4}(-2)(1-x^2)^{-3}(-2x) = \frac{x}{(1-x^2)^3}$

21. $\displaystyle\int \frac{x^2}{(1+x^3)^2}\,dx = \frac{1}{3}\int (1+x^3)^{-2}(3x^2)\,dx = \frac{1}{3}\left[\frac{(1+x^3)^{-1}}{-1}\right] + C = -\frac{1}{3(1+x^3)} + C$

Check: $\displaystyle\frac{d}{dx}\left[-\frac{1}{3(1+x^3)} + C\right] = -\frac{1}{3}(-1)(1+x^3)^{-2}(3x^2) = \frac{x^2}{(1+x^3)^2}$

23. $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\,dx = -\frac{1}{2}\frac{(1-x^2)^{1/2}}{1/2} + C = -\sqrt{1-x^2} + C$

Check: $\displaystyle\frac{d}{dx}[-(1-x^2)^{1/2} + C] = -\frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{1-x^2}}$

25. $\displaystyle\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt = -\int \left(1+\frac{1}{t}\right)^3\left(-\frac{1}{t^2}\right)dt = -\frac{[1+(1/t)]^4}{4} + C$

Check: $\displaystyle\frac{d}{dt}\left[-\frac{[1+(1/t)]^4}{4} + C\right] = -\frac{1}{4}(4)\left(1+\frac{1}{t}\right)^3\left(-\frac{1}{t^2}\right) = \frac{1}{t^2}\left(1+\frac{1}{t}\right)^3$

27. $\displaystyle\int \frac{1}{\sqrt{2x}}\,dx = \frac{1}{2}\int (2x)^{-1/2}\,2\,dx = \frac{1}{2}\left[\frac{(2x)^{1/2}}{1/2}\right] + C = \sqrt{2x} + C$

Check: $\displaystyle\frac{d}{dx}\left[\sqrt{2x} + C\right] = \frac{1}{2}(2x)^{-1/2}(2) = \frac{1}{\sqrt{2x}}$

29. $\displaystyle\int \frac{x^2+3x+7}{\sqrt{x}}\,dx = \int (x^{3/2}+3x^{1/2}+7x^{-1/2})\,dx = \frac{2}{5}x^{5/2}+2x^{3/2}+14x^{1/2}+C = \frac{2}{5}\sqrt{x}(x^2+5x+35)+C$

Check: $\displaystyle\frac{d}{dx}\left[\frac{2}{5}x^{5/2}+2x^{3/2}+14x^{1/2}+C\right] = \frac{x^2+3x+7}{\sqrt{x}}$

31. $\displaystyle\int t^2\left(t-\frac{2}{t}\right)dt = \int (t^3-2t)\,dt = \frac{1}{4}t^4-t^2+C$

Check: $\displaystyle\frac{d}{dt}\left[\frac{1}{4}t^4-t^2+C\right] = t^3-2t = t^2\left(t-\frac{2}{t}\right)$

33. $\displaystyle\int (9-y)\sqrt{y}\,dy = \int (9y^{1/2}-y^{3/2})\,dy = 9\left(\frac{2}{3}y^{3/2}\right)-\frac{2}{5}y^{5/2}+C = \frac{2}{5}y^{3/2}(15-y)+C$

Check: $\displaystyle\frac{d}{dy}\left[\frac{2}{5}y^{3/2}(15-y)+C\right] = \frac{d}{dy}\left[6y^{3/2}-\frac{2}{5}y^{5/2}+C\right] = 9y^{1/2}-y^{3/2} = (9-y)\sqrt{y}$

35. $\displaystyle y = \int\left[4x+\frac{4x}{\sqrt{16-x^2}}\right]dx$

$\displaystyle = 4\int x\,dx - 2\int (16-x^2)^{-1/2}(-2x)\,dx$

$\displaystyle = 4\left(\frac{x^2}{2}\right)-2\left[\frac{(16-x^2)^{1/2}}{1/2}\right]+C$

$= 2x^2-4\sqrt{16-x^2}+C$

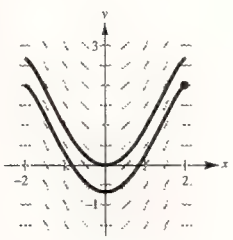
37. $\displaystyle y = \int \frac{x+1}{(x^2+2x-3)^2}\,dx$

$\displaystyle = \frac{1}{2}\int (x^2+2x-3)^{-2}(2x+2)\,dx$

$\displaystyle = \frac{1}{2}\left[\frac{(x^2+2x-3)^{-1}}{-1}\right]+C$

$\displaystyle = -\frac{1}{2(x^2+2x-3)}+C$

39. (a)



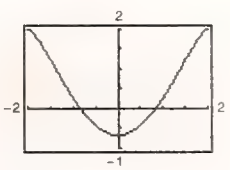
(b) $\displaystyle\frac{dy}{dx} = x\sqrt{4-x^2}$, $(2, 2)$

$\displaystyle y = \int x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int (4-x^2)^{1/2}(-2x\,dx)$

$\displaystyle = -\frac{1}{2}\cdot\frac{2}{3}(4-x^2)^{3/2}+C = -\frac{1}{3}(4-x^2)^{3/2}+C$

$(2, 2)$: $\displaystyle 2 = -\frac{1}{3}(4-2^2)^{3/2}+C \Rightarrow C = 2$

$\displaystyle y = -\frac{1}{3}(4-x^2)^{3/2}+2$



41. $\displaystyle\int \pi \sin \pi x\,dx = -\cos \pi x + C$

43. $\displaystyle\int \sin 2x\,dx = \frac{1}{2}\int (\sin 2x)(2x)\,dx = -\frac{1}{2}\cos 2x + C$

45. $\displaystyle\int \frac{1}{\theta^2}\cos\frac{1}{\theta}\,d\theta = -\int \cos\frac{1}{\theta}\left(-\frac{1}{\theta^2}\right)d\theta = -\sin\frac{1}{\theta}+C$

47. $\displaystyle\int \sin 2x \cos 2x\, dx = \frac{1}{2}\int (\sin 2x)(2\cos 2x)\, dx = \frac{1}{2}\frac{(\sin 2x)^2}{2} + C = \frac{1}{4}\sin^2 2x + C$ OR

$\displaystyle\int \sin 2x \cos 2x\, dx = -\frac{1}{2}\int (\cos 2x)(-2\sin 2x)\, dx = -\frac{1}{2}\frac{(\cos 2x)^2}{2} + C_1 = -\frac{1}{4}\cos^2 2x + C_1$ OR

$\displaystyle\int \sin 2x \cos 2x\, dx = \frac{1}{2}\int 2\sin 2x \cos 2x\, dx = \frac{1}{2}\int \sin 4x\, dx - \frac{1}{8}\cos 4x + C_2$

49. $\displaystyle\int \tan^4 x \sec^2 x\, dx = \frac{\tan^5 x}{5} + C = \frac{1}{5}\tan^5 x + C$

51. $\displaystyle\int \frac{\csc^2 x}{\cot^3 x}\, dx = -\int (\cot x)^{-3}(-\csc^2 x)\, dx$

$$= -\frac{(\cot x)^{-2}}{-2} + C = \frac{1}{2\cot^2 x} + C = \frac{1}{2}\tan^2 x + C = \frac{1}{2}(\sec^2 x - 1) + C = \frac{1}{2}\sec^2 x + C_1$$

53. $\displaystyle\int \cot^2 x\, dx = \int (\csc^2 x - 1)\, dx = -\cot x - x + C$

55. $\displaystyle f(x) = \int \cos\frac{x}{2}\, dx = 2\sin\frac{x}{2} + C$

Since $f(0) = 3 = 2\sin 0 + C$, $C = 3$. Thus,

$$f(x) = 2\sin\frac{x}{2} + 3.$$

57. $u = x + 2$, $x = u - 2$, $dx = du$

$$\int x\sqrt{x+2}\, dx = \int (u-2)\sqrt{u}\, du$$

$$= \int (u^{3/2} - 2u^{1/2})\, du$$

$$= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C$$

$$= \frac{2u^{3/2}}{15}(3u - 10) + C$$

$$= \frac{2}{15}(x+2)^{3/2}[3(x+2) - 10] + C$$

$$= \frac{2}{15}(x+2)^{3/2}(3x - 4) + C$$

59. $u = 1 - x$, $x = 1 - u$, $dx = -du$

$$\int x^2\sqrt{1-x}\, dx = -\int (1-u)^2\sqrt{u}\, du$$

$$= -\int (u^{1/2} - 2u^{3/2} + u^{5/2})\, du$$

$$= -\left(\frac{2}{3}u^{3/2} - \frac{4}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right) + C$$

$$= -\frac{2u^{3/2}}{105}(35 - 42u + 15u^2) + C$$

$$= -\frac{2}{105}(1-x)^{3/2}[35 - 42(1-x) + 15(1-x)^2] + C$$

$$= -\frac{2}{105}(1-x)^{3/2}(15x^2 + 12x + 8) + C$$

61. $u = 2x - 1, x = \frac{1}{2}(u + 1), dx = \frac{1}{2}\,du$

$$\begin{aligned}
\int \frac{x^2 - 1}{\sqrt{2x - 1}}\,dx &= \int \frac{[(1/2)(u + 1)]^2 - 1}{\sqrt{u}}\,\frac{1}{2}\,du \\
&= \frac{1}{8}\int u^{-1/2}[(u^2 + 2u + 1) - 4]\,du \\
&= \frac{1}{8}\int (u^{3/2} + 2u^{1/2} - 3u^{-1/2})\,du \\
&= \frac{1}{8}\left(\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2} - 6u^{1/2}\right) + C \\
&= \frac{u^{1/2}}{60}(3u^2 + 10u - 45) + C \\
&= \frac{\sqrt{2x - 1}}{60}[3(2x - 1)^2 + 10(2x - 1) - 45] + C \\
&= \frac{1}{60}\sqrt{2x - 1}(12x^2 + 8x - 52) + C \\
&= \frac{1}{15}\sqrt{2x - 1}(3x^2 + 2x - 13) + C
\end{aligned}$$

63. $u = x + 1, x = u - 1, dx = du$

$$\begin{aligned}
\int \frac{-x}{(x + 1) - \sqrt{x + 1}}\,dx &= \int \frac{-(u - 1)}{u - \sqrt{u}}\,du \\
&= -\int \frac{(\sqrt{u} + 1)(\sqrt{u} - 1)}{\sqrt{u}(\sqrt{u} - 1)}\,du \\
&= -\int (1 + u^{-1/2})\,du \\
&= -(u + 2u^{1/2}) + C \\
&= -u - 2\sqrt{u} + C \\
&= -(x + 1) - 2\sqrt{x + 1} + C \\
&= -x - 2\sqrt{x + 1} - 1 + C \\
&= -\left(x + 2\sqrt{x + 1}\right) + C_1
\end{aligned}$$

where $C_1 = -1 + C$.

65. Let $u = x^2 + 1$, $du = 2x\,dx$.

$$\int_{-1}^{1} x(x^2 + 1)^3\,dx = \frac{1}{2}\int_{-1}^{1} (x^2 + 1)^3(2x)\,dx = \left[\frac{1}{8}(x^2 + 1)^4\right]_{-1}^{1} = 0$$

67. Let $u = x^3 + 1$, $du = 3x^2\,dx$

$$\begin{aligned}
\int_1^2 2x^2\sqrt{x^3 + 1}\,dx &= 2 \cdot \frac{1}{3}\int_1^2 (x^3 + 1)^{1/2}(3x^2)\,dx \\
&= \left[\frac{2}{3}\frac{(x^3 + 1)^{3/2}}{3/2}\right]_1^2 \\
&= \frac{4}{9}\left[(x^3 + 1)^{3/2}\right]_1^2 \\
&= \frac{4}{9}\left[27 - 2\sqrt{2}\right] = 12 - \frac{8}{9}\sqrt{2}
\end{aligned}$$

69. Let $u = 2x + 1$, $du = 2\,dx$.

$$\int_0^4 \frac{1}{\sqrt{2x+1}}\,dx = \frac{1}{2}\int_0^4 (2x+1)^{-1/2}(2)\,dx = \Big[\sqrt{2x+1}\Big]_0^4 = \sqrt{9} - \sqrt{1} = 2$$

71. Let $u = 1 + \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$.

$$\int_1^9 \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx = 2\int_1^9 (1+\sqrt{x})^{-2}\left(\frac{1}{2\sqrt{x}}\right)dx = \left[-\frac{2}{1+\sqrt{x}}\right]_1^9 = -\frac{1}{2} + 1 = \frac{1}{2}$$

73. $u = 2 - x$, $x = 2 - u$, $dx = -du$

When $x = 1$, $u = 1$. When $x = 2$, $u = 0$.

$$\int_1^2 (x-1)\sqrt{2-x}\,dx = \int_1^0 -[(2-u)-1]\sqrt{u}\,du = \int_1^0 (u^{3/2} - u^{1/2})\,du = \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^0 = -\left[\frac{2}{5} - \frac{2}{3}\right] = \frac{4}{15}$$

75. $\displaystyle\int_0^{\pi/2} \cos\left(\frac{2}{3}x\right)dx = \left[\frac{3}{2}\sin\left(\frac{2}{3}x\right)\right]_0^{\pi/2} = \frac{3}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{4}$

77. $u = x + 1$, $x = u - 1$, $dx = du$

When $x = 0$, $u = 1$. When $x = 7$, $u = 8$.

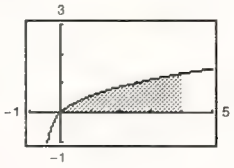
$$\text{Area} = \int_0^7 x\sqrt[3]{x+1}\,dx = \int_1^8 (u-1)\sqrt[3]{u}\,du$$

$$= \int_1^8 (u^{4/3} - u^{1/3})\,du = \left[\frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3}\right]_1^8 = \left(\frac{384}{7} - 12\right) - \left(\frac{3}{7} - \frac{3}{4}\right) = \frac{1209}{28}$$

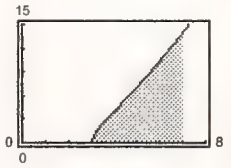
79. $\displaystyle A = \int_0^{\pi} (2\sin x + \sin 2x)\,dx = -\left[2\cos x + \frac{1}{2}\cos 2x\right]_0^{\pi} = 4$

81. $\displaystyle \text{Area} = \int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right)dx = 2\int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)dx = \left[2\tan\left(\frac{x}{2}\right)\right]_{\pi/2}^{2\pi/3} = 2\left(\sqrt{3} - 1\right)$

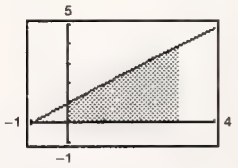
83. $\displaystyle\int_0^4 \frac{x}{\sqrt{2x+1}}\,dx \approx 3.333 = \frac{10}{3}$



85. $\displaystyle\int_3^7 x\sqrt{x-3}\,dx \approx 28.8 = \frac{144}{5}$



87. $\displaystyle\int_0^3 \left(\theta + \cos\frac{\theta}{6}\right)d\theta \approx 7.377$



89. $\displaystyle\int (2x-1)^2\,dx = \frac{1}{2}\int (2x-1)^2\,2\,dx = \frac{1}{6}(2x-1)^3 + C_1 = \frac{4}{3}x^3 - 2x^2 + x - \frac{1}{6} + C_1$

$$\int (2x-1)^2\,dx = \int (4x^2 - 4x + 1)\,dx = \frac{4}{3}x^3 - 2x^2 + x + C_2$$

They differ by a constant: $C_2 = C_1 - \dfrac{1}{6}$.

91. $f(x) = x^2(x^2 + 1)$ is even.

$$\int_{-2}^{2} x^2(x^2 + 1)\,dx = 2\int_{0}^{2} (x^4 + x^2)\,dx = 2\left[\frac{x^5}{5} + \frac{x^3}{3}\right]_0^2$$

$$= 2\left[\frac{32}{5} + \frac{8}{3}\right] = \frac{272}{15}$$

93. $f(x) = x(x^2 + 1)^3$ is odd.

$$\int_{-2}^{2} x(x^2 + 1)^3\,dx = 0$$

95. $\displaystyle\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$; the function x^2 is an even function.

(a) $\displaystyle\int_{-2}^{0} x^2\,dx = \int_0^2 x^2\,dx = \frac{8}{3}$

(b) $\displaystyle\int_{-2}^{2} x^2\,dx = 2\int_0^2 x^2\,dx = \frac{16}{3}$

(c) $\displaystyle\int_0^2 (-x^2)\,dx = -\int_0^2 x^2\,dx = -\frac{8}{3}$

(d) $\displaystyle\int_{-2}^{0} 3x^2\,dx = 3\int_0^2 x^2\,dx = 8$

97. $$\int_{-4}^{4} (x^3 + 6x^2 - 2x - 3)\,dx = \int_{-4}^{4} (x^3 - 2x)\,dx + \int_{-4}^{4} (6x^2 - 3)\,dx = 0 + 2\int_0^4 (6x^2 - 3)\,dx = 2\left[2x^3 - 3x\right]_0^4 = 232$$

99. Answers will vary. See "Guidelines for Making a Change of Variables" on page 292.

101. $f(x) = x(x^2 + 1)^2$ is odd. Hence, $\displaystyle\int_{-2}^{2} x(x^2 + 1)^2\,dx = 0$.

103. $$\frac{dV}{dt} = \frac{k}{(t + 1)^2}$$

$$V(t) = \int \frac{k}{(t + 1)^2}\,dt = -\frac{k}{t + 1} + C$$

$$V(0) = -k + C = 500{,}000$$

$$V(1) = -\frac{1}{2}k + C = 400{,}000$$

Solving this system yields $k = -200{,}000$ and $C = 300{,}000$. Thus,

$$V(t) = \frac{200{,}000}{t + 1} + 300{,}000.$$

When $t = 4$, $V(4) = \$340{,}000$.

105. $$\frac{1}{b - a}\int_a^b \left[74.50 + 43.75 \sin\frac{\pi t}{6}\right] dt = \frac{1}{b - a}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_a^b$$

(a) $\displaystyle\frac{1}{3}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_0^3 = \frac{1}{3}\left(223.5 + \frac{262.5}{\pi}\right) \approx 102.352$ thousand units

(b) $\displaystyle\frac{1}{3}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_3^6 = \frac{1}{3}\left(447 + \frac{262.5}{\pi} - 223.5\right) \approx 102.352$ thousand units

(c) $\displaystyle\frac{1}{12}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_0^{12} = \frac{1}{12}\left(894 - \frac{262.5}{\pi} + \frac{262.5}{\pi}\right) = 74.5$ thousand units

107. $\dfrac{1}{b-a}\displaystyle\int_a^b [2\sin(60\pi t) + \cos(120\pi t)]\,dt = \dfrac{1}{b-a}\left[-\dfrac{1}{30\pi}\cos(60\pi t) + \dfrac{1}{120\pi}\sin(120\pi t)\right]_a^b$

(a) $\dfrac{1}{(1/60)-0}\left[-\dfrac{1}{30\pi}\cos(60\pi t) + \dfrac{1}{120\pi}\sin(120\pi t)\right]_0^{1/60} = 60\left[\left(\dfrac{1}{30\pi}+0\right) - \left(-\dfrac{1}{30\pi}\right)\right] = \dfrac{4}{\pi} \approx 1.273 \text{ amps}$

(b) $\dfrac{1}{(1/240)-0}\left[-\dfrac{1}{30\pi}\cos(60\pi t) + \dfrac{1}{120\pi}\sin(120\pi t)\right]_0^{1/240} = 240\left[\left(-\dfrac{1}{30\sqrt{2}\pi}+\dfrac{1}{120\pi}\right) - \left(-\dfrac{1}{30\pi}\right)\right]$

$= \dfrac{2}{\pi}(5-2\sqrt{2}) \approx 1.382 \text{ amps}$

(c) $\dfrac{1}{(1/30)-0}\left[-\dfrac{1}{30\pi}\cos(60\pi t) + \dfrac{1}{120\pi}\sin(120\pi t)\right]_0^{1/30} = 30\left[\left(\dfrac{1}{30\pi}\right) - \left(-\dfrac{1}{30\pi}\right)\right] = 0 \text{ amps}$

109. False

$$\int (2x+1)^2\,dx = \frac{1}{2}\int (2x+1)^2\,2\,dx = \frac{1}{6}(2x+1)^3 + C$$

111. True

$$\int_{-10}^{10} (ax^3+bx^2+cx+d)\,dx = \int_{-10}^{10} (ax^3+cx)\,dx + \int_{-10}^{10} (bx^2+d)\,dx = 0 + 2\int_0^{10} (bx^2+d)\,dx$$

Odd Even

113. True

$$4\int \sin x\cos x\,dx = 2\int \sin 2x\,dx = -\cos 2x + C$$

115. Let $u = x + h$, then $du = dx$. When $x = a$, $u = a + h$. When $x = b$, $u = b + h$. Thus,

$$\int_a^b f(x+h)\,dx = \int_{a+h}^{b+h} f(u)\,du = \int_{a+h}^{b+h} f(x)\,dx.$$

Section 4.6 Numerical Integration

1. Exact: $\displaystyle\int_0^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^2 = \frac{8}{3} \approx 2.6667$

Trapezoidal: $\displaystyle\int_0^2 x^2\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)^2 + 2(1)^2 + 2\left(\frac{3}{2}\right)^2 + (2)^2\right] = \frac{11}{4} = 2.7500$

Simpson's: $\displaystyle\int_0^2 x^2\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^2 + 2(1)^2 + 4\left(\frac{3}{2}\right)^2 + (2)^2\right] = \frac{8}{3} \approx 2.6667$

3. Exact: $\displaystyle\int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = 4.000$

Trapezoidal: $\displaystyle\int_0^2 x^3\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)^3 + 2(1)^3 + 2\left(\frac{3}{2}\right)^3 + (2)^3\right] = \frac{17}{4} = 4.2500$

Simpson's: $\displaystyle\int_0^2 x^3\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^3 + 2(1)^3 + 4\left(\frac{3}{2}\right)^3 + (2)^3\right] = \frac{24}{6} = 4.0000$

5. Exact: $\displaystyle\int_0^2 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^2 = 4.0000$

Trapezoidal: $\displaystyle\int_0^2 x^3\,dx \approx \frac{1}{8}\left[0 + 2\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 2\left(\frac{3}{4}\right)^3 + 2(1)^3 + 2\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 2\left(\frac{7}{4}\right)^3 + 8\right] = 4.0625$

Simpson's: $\displaystyle\int_0^2 x^3\,dx \approx \frac{1}{12}\left[0 + 4\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 4\left(\frac{3}{4}\right)^3 + 2(1)^3 + 4\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 4\left(\frac{7}{4}\right)^3 + 8\right] = 4.0000$

7. Exact: $\displaystyle\int_4^9 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_4^9 = 18 - \frac{16}{3} = \frac{38}{3} \approx 12.6667$

Trapezoidal: $\displaystyle\int_4^9 \sqrt{x}\,dx \approx \frac{5}{16}\left[2 + 2\sqrt{\frac{37}{8}} + 2\sqrt{\frac{21}{4}} + 2\sqrt{\frac{47}{8}} + 2\sqrt{\frac{26}{4}} + 2\sqrt{\frac{57}{8}} + 2\sqrt{\frac{31}{4}} + 2\sqrt{\frac{67}{8}} + 3\right]$

≈ 12.6640

Simpson's: $\displaystyle\int_4^9 \sqrt{x}\,dx \approx \frac{5}{24}\left[2 + 4\sqrt{\frac{37}{8}} + \sqrt{21} + 4\sqrt{\frac{47}{8}} + \sqrt{26} + 4\sqrt{\frac{57}{8}} + \sqrt{31} + 4\sqrt{\frac{67}{8}} + 3\right] \approx 12.6667$

9. Exact: $\displaystyle\int_1^2 \frac{1}{(x+1)^2}\,dx = \left[-\frac{1}{x+1}\right]_1^2 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6} \approx 0.1667$

Trapezoidal: $\displaystyle\int_1^2 \frac{1}{(x+1)^2}\,dx \approx \frac{1}{8}\left[\frac{1}{4} + 2\left(\frac{1}{((5/4)+1)^2}\right) + 2\left(\frac{1}{((3/2)+1)^2}\right) + 2\left(\frac{1}{((7/4)+1)^2}\right) + \frac{1}{9}\right]$

$= \dfrac{1}{8}\left(\dfrac{1}{4} + \dfrac{32}{81} + \dfrac{8}{25} + \dfrac{32}{121} + \dfrac{1}{9}\right) \approx 0.1676$

Simpson's: $\displaystyle\int_1^2 \frac{1}{(x+1)^2}\,dx \approx \frac{1}{12}\left[\frac{1}{4} + 4\left(\frac{1}{((5/4)+1)^2}\right) + 2\left(\frac{1}{((3/2)+1)^2}\right) + 4\left(\frac{1}{((7/4)+1)^2}\right) + \frac{1}{9}\right]$

$= \dfrac{1}{12}\left(\dfrac{1}{4} + \dfrac{64}{81} + \dfrac{8}{25} + \dfrac{64}{121} + \dfrac{1}{9}\right) \approx 0.1667$

11. Trapezoidal: $\displaystyle\int_0^2 \sqrt{1+x^3}\,dx \approx \frac{1}{4}[1 + 2\sqrt{1+(1/8)} + 2\sqrt{2} + 2\sqrt{1+(27/8)} + 3] \approx 3.283$

Simpson's: $\displaystyle\int_0^2 \sqrt{1+x^3}\,dx \approx \frac{1}{6}[1 + 4\sqrt{1+(1/8)} + 2\sqrt{2} + 4\sqrt{1+(27/8)} + 3] \approx 3.240$

Graphing utility: 3.241

13. $\displaystyle\int_0^1 \sqrt{x}\sqrt{1-x}\,dx = \int_0^1 \sqrt{x(1-x)}\,dx$

Trapezoidal: $\displaystyle\int_0^1 \sqrt{x(1-x)}\,dx \approx \frac{1}{8}\left[0 + 2\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 2\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)}\right] \approx 0.342$

Simpson's: $\displaystyle\int_0^1 \sqrt{x(1-x)}\,dx \approx \frac{1}{12}\left[0 + 4\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 4\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)}\right] \approx 0.372$

Graphing utility: 0.393

15. Trapezoidal: $\displaystyle\int_0^{\sqrt{\pi/2}} \cos(x^2)\,dx \approx \frac{\sqrt{\pi/2}}{8}\left[\cos 0 + 2\cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2\cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 2\cos\left(\frac{3\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2\right]$

≈ 0.957

Simpson's: $\displaystyle\int_0^{\sqrt{\pi/2}} \cos(x^2)\,dx \approx \frac{\sqrt{\pi/2}}{12}\left[\cos 0 + 4\cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2\cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 4\cos\left(\frac{3\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2\right]$

≈ 0.978

Graphing utility: 0.977

17. Trapezoidal: $\displaystyle\int_1^{1.1} \sin x^2\,dx \approx \frac{1}{80}[\sin(1) + 2\sin(1.025)^2 + 2\sin(1.05)^2 + 2\sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$

Simpson's: $\displaystyle\int_1^{1.1} \sin x^2\,dx \approx \frac{1}{120}[\sin(1) + 4\sin(1.025)^2 + 2\sin(1.05)^2 + 4\sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$

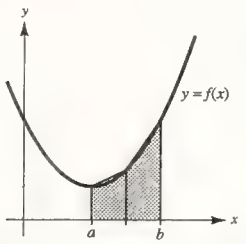
Graphing utility: 0.089

19. Trapezoidal: $\displaystyle\int_0^{\pi/4} x\tan x\,dx \approx \frac{\pi}{32}\left[0 + 2\left(\frac{\pi}{16}\right)\tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right)\tan\left(\frac{2\pi}{16}\right) + 2\left(\frac{3\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4}\right] \approx 0.194$

Simpson's: $\displaystyle\int_0^{\pi/4} x\tan x\,dx \approx \frac{\pi}{48}\left[0 + 4\left(\frac{\pi}{16}\right)\tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right)\tan\left(\frac{2\pi}{16}\right) + 4\left(\frac{3\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4}\right] \approx 0.186$

Graphing utility: 0.186

21. (a)



The Trapezoidal Rule overestimates the area if the graph of the integrand is concave up.

23. $f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = 6x$

$f'''(x) = 6$

$f^{(4)}(x) = 0$

(a) Trapezoidal: Error $\le \dfrac{(2-0)^3}{12(4^2)}(12) = 0.5$ since $f''(x)$ is maximum in $[0, 2]$ when $x = 2$.

(b) Simpson's: Error $\le \dfrac{(2-0)^5}{180(4^4)}(0) = 0$ since $f^{(4)}(x) = 0$.

25. $f''(x) = \dfrac{2}{x^3}$ in $[1, 3]$.

(a) $|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| = 2$.

Trapezoidal: Error $\le \dfrac{2^3}{12n^2}(2) < 0.00001$, $n^2 > 133{,}333.33$, $n > 365.15$; let $n = 366$.

$f^{(4)}(x) = \dfrac{24}{x^5}$ in $[1, 3]$

(b) $|f^{(4)}(x)|$ is maximum when $x = 1$ and $|f^{(4)}(1)| = 24$.

Simpson's: Error $\le \dfrac{2^5}{180n^4}(24) < 0.00001$, $n^4 > 426{,}666.67$, $n > 25.56$; let $n = 26$.

27. $f(x) = \sqrt{1 + x}$

(a) $f''(x) = -\dfrac{1}{4(1 + x)^{3/2}}$ in $[0, 2]$.

$|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = \dfrac{1}{4}$.

Trapezoidal: Error $\le \dfrac{8}{12n^2}\left(\dfrac{1}{4}\right) < 0.00001, n^2 > 16{,}666.67, n > 129.10$; let $n = 130$.

(b) $f^{(4)}(x) = \dfrac{-15}{16(1 + x)^{7/2}}$ in $[0, 2]$

$|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = \dfrac{15}{16}$.

Simpson's: Error $\le \dfrac{32}{180n^4}\left(\dfrac{15}{16}\right) < 0.00001, n^4 > 16{,}666.67, n > 11.36$; let $n = 12$.

29. $f(x) = \tan(x^2)$

(a) $f''(x) = 2\sec^2(x^2)[1 + 4x^2\tan(x^2)]$ in $[0, 1]$.

$|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 49.5305$.

Trapezoidal: Error $\le \dfrac{(1 - 0)^3}{12n^2}(49.5305) < 0.00001, n^2 > 412{,}754.17, n > 642.46$; let $n = 643$.

(b) $f^{(4)}(x) = 8\sec^2(x^2)[12x^2 + (3 + 32x^4)\tan(x^2) + 36x^2\tan^2(x^2) + 48x^4\tan^3(x^2)]$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x = 1$ and $|f^{(4)}(1)| \approx 9184.4734$.

Simpson's: Error $\le \dfrac{(1 - 0)^5}{180n^4}(9184.4734) < 0.00001, n^4 > 5{,}102{,}485.22, n > 47.53$; let $n = 48$.

31. Let $f(x) = Ax^3 + Bx^2 + Cx + D$. Then $f^{(4)}(x) = 0$.

Simpson's: Error $\le \dfrac{(b - a)^5}{180n^4}(0) = 0$

Therefore, Simpson's Rule is exact when approximating the integral of a cubic polynomial.

Example: $\displaystyle\int_0^1 x^3\,dx = \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^3 + 1\right] = \frac{1}{4}$

This is the exact value of the integral.

33. $f(x) = \sqrt{2 + 3x^2}$ on $[0, 4]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	12.7771	15.3965	18.4340	15.6055	15.4845
8	14.0868	15.4480	16.9152	15.5010	15.4662
10	14.3569	15.4544	16.6197	15.4883	15.4658
12	14.5386	15.4578	16.4242	15.4814	15.4657
16	14.7674	15.4613	16.1816	15.4745	15.4657
20	14.9056	15.4628	16.0370	15.4713	15.4657

35. $f(x) = \sin\sqrt{x}$ on $[0, 4]$.

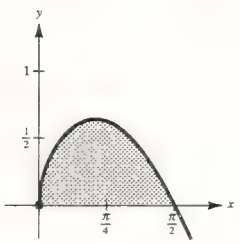
n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	2.8163	3.5456	3.7256	3.2709	3.3996
8	3.1809	3.5053	3.6356	3.4083	3.4541
10	3.2478	3.4990	3.6115	3.4296	3.4624
12	3.2909	3.4952	3.5940	3.4425	3.4674
16	3.3431	3.4910	3.5704	3.4568	3.4730
20	3.3734	3.4888	3.5552	3.4643	3.4759

37. $A = \displaystyle\int_0^{\pi/2} \sqrt{x}\cos x\,dx$

Simpson's Rule: $n = 14$

$$\int_0^{\pi/2} \sqrt{x}\cos x\,dx \approx \frac{\pi}{84}\left[\sqrt{0}\cos 0 + 4\sqrt{\frac{\pi}{28}}\cos\frac{\pi}{28} + 2\sqrt{\frac{\pi}{14}}\cos\frac{\pi}{14} + 4\sqrt{\frac{3\pi}{28}}\cos\frac{3\pi}{28} + \cdots + \sqrt{\frac{\pi}{2}}\cos\frac{\pi}{2}\right]$$

$$\approx 0.701$$



39. $W = \displaystyle\int_0^5 100x\sqrt{125 - x^3}\,dx$

Simpson's Rule: $n = 12$

$$\int_0^5 100x\sqrt{125 - x^3}\,dx \approx \frac{5}{3(12)}\left[0 + 400\left(\frac{5}{12}\right)\sqrt{125 - \left(\frac{5}{12}\right)^3} + 200\left(\frac{10}{12}\right)\sqrt{125 - \left(\frac{10}{12}\right)^3}\right.$$

$$\left. + 400\left(\frac{15}{12}\right)\sqrt{125 - \left(\frac{15}{12}\right)^3} + \cdots + 0\right] \approx 10{,}233.58 \text{ ft} \cdot \text{lb}$$

41. $\displaystyle\int_0^{1/2} \frac{6}{\sqrt{1 - x^2}}\,dx$ Simpson's Rule, $n = 6$

$$\pi \approx \frac{\left(\frac{1}{2} - 0\right)}{3(6)}[6 + 4(6.0209) + 2(6.0851) + 4(6.1968) + 2(6.3640) + 4(6.6002) + 6.9282]$$

$$\approx \frac{1}{36}[113.098] \approx 3.1416$$

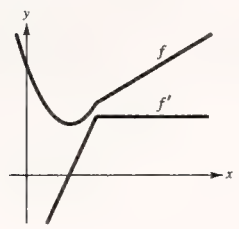
43. Area $\approx \dfrac{1000}{2(10)}[125 + 2(125) + 2(120) + 2(112) + 2(90) + 2(90) + 2(95) + 2(88) + 2(75) + 2(35)] = 89{,}250$ sq m

45. $\displaystyle\int_0^t \sin\sqrt{x}\,dx = 2,\ n = 10$

By trial and error, we obtain $t \approx 2.477$.

Review Exercises for Chapter 4

1.



3. $\int (2x^2 + x - 1)\,dx = \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + C$

5. $\int \frac{x^3 + 1}{x^2}\,dx = \int \left(x + \frac{1}{x^2}\right) dx = \frac{1}{2}x^2 - \frac{1}{x} + C$

7. $\int (4x - 3\sin x)\,dx = 2x^2 + 3\cos x + C$

9. $f'(x) = -2x,\ (-1, 1)$

$f(x) = \int -2x\,dx = -x^2 + C$

When $x = -1$:

$y = -1 + C = 1$

$C = 2$

$y = 2 - x^2$

11. $a(t) = a$

$v(t) = \int a\,dt = at + C_1$

$v(0) = 0 + C_1 = 0$ when $C_1 = 0$.

$v(t) = at$

$s(t) = \int at\,dt = \frac{a}{2}t^2 + C_2$

$s(0) = 0 + C_2 = 0$ when $C_2 = 0$.

$s(t) = \frac{a}{2}t^2$

$s(30) = \frac{a}{2}(30)^2 = 3600$ or

$a = \frac{2(3600)}{(30)^2} = 8$ ft/sec^2.

$v(30) = 8(30) = 240$ ft/sec

13. $a(t) = -32$

$v(t) = -32t + 96$

$s(t) = -16t^2 + 96t$

(a) $v(t) = -32t + 96 = 0$ when $t = 3$ sec.

(b) $s(3) = -144 + 288 = 144$ ft

(c) $v(t) = -32t + 96 = \frac{96}{2}$ when $t = \frac{3}{2}$ sec.

(d) $s\left(\frac{3}{2}\right) = -16\left(\frac{9}{4}\right) + 96\left(\frac{3}{2}\right) = 108$ ft

15. (a) $\sum_{i=1}^{10} (2i - 1)$

(b) $\sum_{i=1}^{n} i^3$

(c) $\sum_{i=1}^{10} (4i + 2)$

17. $y = \dfrac{10}{x^2 + 1}, \Delta x = \dfrac{1}{2}, n = 4$

$$S(n) = S(4) = \frac{1}{2}\left[\frac{10}{1} + \frac{10}{(1/2)^2 + 1} + \frac{10}{(1)^2 + 1} + \frac{10}{(3/2)^2 + 1}\right]$$

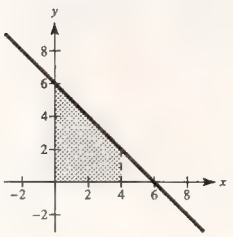
$$\approx 13.0385$$

$$s(n) = s(4) = \frac{1}{2}\left[\frac{10}{(1/2)^2 + 1} + \frac{10}{1 + 1} + \frac{10}{(3/2)^2 + 1} + \frac{10}{2^2 + 1}\right]$$

$$\approx 9.0385$$

$9.0385 <$ Area of Region < 13.0385

19. $y = 6 - x, \Delta x = \dfrac{4}{n}$, right endpoints



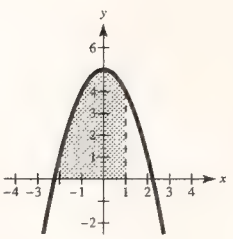
$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(ci)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(6 - \frac{4i}{n}\right)\frac{4}{n}$$

$$= \lim_{n\to\infty} \frac{4}{n}\left[6n - \frac{4}{n}\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left[24 - 8\frac{n+1}{n}\right] = 24 - 8 = 16$$

21. $y = 5 - x^2, \Delta x = \dfrac{3}{n}$



$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(ci)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[5 - \left(-2 + \frac{3i}{n}\right)^2\right]\left(\frac{3}{n}\right)$$

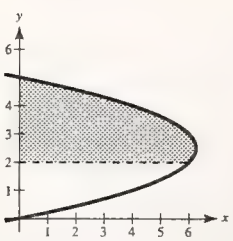
$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[1 + \frac{12i}{n} - \frac{9i^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[n + \frac{12}{n}\frac{n(n+1)}{2} - \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \left[3 + 18\frac{n+1}{n} - \frac{9}{2}\frac{(n+1)(2n+1)}{n^2}\right]$$

$$= 3 + 18 - 9 = 12$$

23. $x = 5y - y^2, 2 \le y \le 5, \Delta y = \dfrac{3}{n}$



$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \left[5\left(2 + \frac{3i}{n}\right) - \left(2 + \frac{3i}{n}\right)^2\right]\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[10 + \frac{15i}{n} - 4 - 12\frac{i}{n} - \frac{9i^2}{n^2}\right]$$

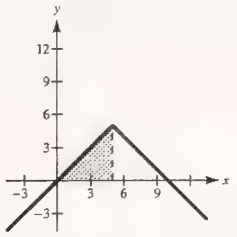
$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[6 + \frac{3i}{n} - \frac{9i^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[6n + \frac{3}{n}\frac{n(n+1)}{2} - \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \left[18 + \frac{9}{2} - 9\right] = \frac{27}{2}$$

25. $\lim_{\|\Delta\|\to\infty} \sum_{i=1}^{n} (2ci - 3)\,\Delta xi = \int_4^6 (2x - 3)\,dx$

27.



$$\int_0^5 \Big(5 - |x - 5|\,dx = \int_0^5 (5 - (5 - x))\,dx = \int_0^5 x\,dx = \frac{25}{2}$$

(triangle)

29. (a) $\displaystyle\int_2^6 [f(x) + g(x)]\,dx = \int_2^6 f(x)\,dx + \int_2^6 g(x)\,dx = 10 + 3 = 13$

(b) $\displaystyle\int_2^6 [f(x) - g(x)]\,dx = \int_2^6 f(x)\,dx - \int_2^6 g(x)\,dx = 10 - 3 = 7$

(c) $\displaystyle\int_2^6 [2f(x) - 3g(x)]\,dx = 2\int_2^6 f(x)\,dx - 3\int_2^6 g(x)\,dx = 2(10) - 3(3) = 11$

(d) $\displaystyle\int_2^6 5f(x)\,dx = 5\int_2^6 f(x)\,dx = 5(10) = 50$

31. $\displaystyle\int_1^8 \left(\sqrt[3]{x} + 1\right) dx = \left[\frac{3}{4}x^{4/3} + x\right]_1^8 = \left[\frac{3}{4}(16) + 8\right] - \left[\frac{3}{4} + 1\right] = \frac{73}{4}$ (c)

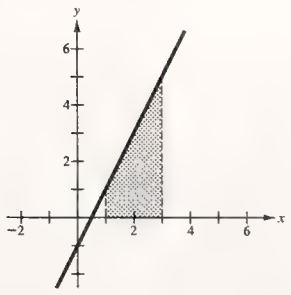
33. $\displaystyle\int_0^4 (2 + x)\,dx = \left[2x + \frac{x^2}{2}\right]_0^4 = 8 + \frac{16}{2} = 16$

35. $\displaystyle\int_{-1}^1 (4t^3 - 2t)\,dt = \left[t^4 - t^2\right]_{-1}^1 = 0$

37. $\displaystyle\int_4^9 x\sqrt{x}\,dx = \int_4^9 x^{3/2}\,dx = \left[\frac{2}{5}x^{5/2}\right]_4^9 = \frac{2}{5}\left[\left(\sqrt{9}\right)^5 - \left(\sqrt{4}\right)^5\right] = \frac{2}{5}(243 - 32) = \frac{422}{5}$

39. $\displaystyle\int_0^{3\pi/4} \sin\theta\,d\theta = \left[-\cos\theta\right]_0^{3\pi/4} = -\left(-\frac{\sqrt{2}}{2}\right) + 1 = 1 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2} + 2}{2}$

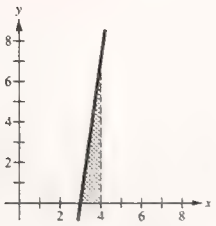
41. $\displaystyle\int_1^3 (2x - 1)\,dx = \left[x^2 - x\right]_1^3 = 6$



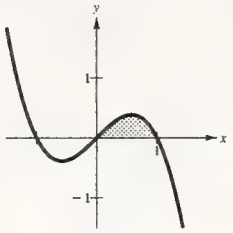
43. $\displaystyle\int_3^4 (x^2 - 9)\,dx = \left[\frac{x^3}{3} - 9x\right]_3^4$

$$= \left(\frac{64}{3} - 36\right) - (9 - 27)$$

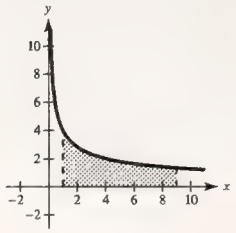
$$= \frac{64}{3} - \frac{54}{3} = \frac{10}{3}$$



45. $\displaystyle\int_0^1 (x - x^3)\,dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$



47. Area $\displaystyle= \int_1^9 \frac{4}{\sqrt{x}}\,dx = \left[\frac{4x^{1/2}}{(1/2)}\right]_1^9 = 8(3 - 1) = 16$

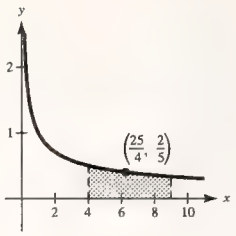


49. $\displaystyle\frac{1}{9-4}\int_4^9 \frac{1}{\sqrt{x}}\,dx = \left[\frac{1}{5}2\sqrt{x}\right]_4^9 = \frac{2}{5}(3 - 2) = \frac{2}{5}$ Average value

$$\frac{2}{5} = \frac{1}{\sqrt{x}}$$

$$\sqrt{x} = \frac{5}{2}$$

$$x = \frac{25}{4}$$



51. $F'(x) = x^2\sqrt{1 + x^3}$

53. $F'(x) = x^2 + 3x + 2$

55. $\displaystyle\int (x^2 + 1)^3\,dx = \int (x^6 + 3x^4 + 3x^2 + 1)\,dx = \frac{x^7}{7} + \frac{3}{5}x^5 + x^3 + x + C$

57. $u = x^3 + 3, du = 3x^2\,dx$

$$\int \frac{x^2}{\sqrt{x^3 + 3}}\,dx = \int (x^3 + 3)^{-1/2} x^2\,dx = \frac{1}{3}\int (x^3 + 3)^{-1/2} 3x^2\,dx = \frac{2}{3}(x^3 + 3)^{1/2} + C$$

59. $u = 1 - 3x^2, du = -6x\,dx$

$$\int x(1 - 3x^2)^4\,dx = -\frac{1}{6}\int (1 - 3x^2)^4(-6x\,dx) = -\frac{1}{30}(1 - 3x^2)^5 + C = \frac{1}{30}(3x^2 - 1)^5 + C$$

61. $\displaystyle\int \sin^3 x \cos x\,dx = \frac{1}{4}\sin^4 x + C$

63. $\displaystyle\int \frac{\sin\theta}{\sqrt{1 - \cos\theta}}\,dx = \int (1 - \cos\theta)^{-1/2} \sin\theta\,d\theta = 2(1 - \cos\theta)^{1/2} + C = 2\sqrt{1 - \cos\theta} + C$

65. $\displaystyle\int \tan^n x \sec^2 x\,dx = \frac{\tan^{n+1} x}{n + 1} + C, n \neq -1$

67. $\displaystyle\int (1 + \sec \pi x)^2 \sec \pi x \tan \pi x\,dx = \frac{1}{\pi}\int (1 + \sec \pi x)^2(\pi \sec \pi x \tan \pi x)\,dx = \frac{1}{3\pi}(1 + \sec \pi x)^3 + C$

69. $\displaystyle\int_{-1}^2 x(x^2 - 4)\,dx = \frac{1}{2}\int_{-1}^2 (x^2 - 4)(2x)\,dx = \frac{1}{2}\frac{(x^2 - 4)^2}{2}\Big]_{-1}^2 = \frac{1}{4}[0 - 9] = -\frac{9}{4}$

71. $\displaystyle\int_0^3 \frac{1}{\sqrt{1+x}}\,dx = \int_0^3 (1+x)^{-1/2}\,dx = \Big[2(1+x)^{1/2}\Big]_0^3 = 4 - 2 = 2$

73. $u = 1 - y, y = 1 - u, dy = -du$

When $y = 0$, $u = 1$. When $y = 1$, $u = 0$.

$$2\pi\int_0^1 (y+1)\sqrt{1-y}\,dy = 2\pi\int_1^0 -[(1-u)+1]\sqrt{u}\,du$$

$$= 2\pi\int_1^0 (u^{3/2} - 2u^{1/2})\,du = 2\pi\left[\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2}\right]_1^0 = \frac{28\pi}{15}$$

75. $\displaystyle\int_0^{\pi} \cos\left(\frac{x}{2}\right)dx = 2\int_0^{\pi} \cos\left(\frac{x}{2}\right)\frac{1}{2}\,dx = \left[2\sin\left(\frac{x}{2}\right)\right]_0^{\pi} = 2$

77. $u = 1 - x, x = 1 - u, dx = -du$

When $x = a$, $u = 1 - a$. When $x = b$, $u = 1 - b$.

$$P_{a,b} = \int_a^b \frac{15}{4}x\sqrt{1-x}\,dx = \frac{15}{4}\int_{1-a}^{1-b} -(1-u)\sqrt{u}\,du$$

$$= \frac{15}{4}\int_{1-a}^{1-b} (u^{3/2} - u^{1/2})\,du = \frac{15}{4}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_{1-a}^{1-b} = \frac{15}{4}\left[\frac{2u^{3/2}}{15}(3u-5)\right]_{1-a}^{1-b} = \left[-\frac{(1-x)^{3/2}}{2}(3x+2)\right]_a^b$$

(a) $P_{0.50,\,0.75} = \left[-\dfrac{(1-x)^{3/2}}{2}(3x+2)\right]_{0.50}^{0.75} = 0.353 = 35.3\%$

(b) $P_{0,b} = \left[-\dfrac{(1-x)^{3/2}}{2}(3x+2)\right]_0^b = -\dfrac{(1-b)^{3/2}}{2}(3b+2) + 1 = 0.5$

$(1-b)^{3/2}(3b+2) = 1$

$b \approx 0.586 = 58.6\%$

79. $p = 1.20 + 0.04t$

$C = \dfrac{15{,}000}{M}\displaystyle\int_t^{t+1} p\,ds$

(a) 2000 corresponds to $t = 10$.

$$C = \frac{15{,}000}{M}\int_{10}^{11} [1.20 + 0.04t]\,dt$$

$$= \frac{15{,}000}{M}\Big[1.20t + 0.02t^2\Big]_{10}^{11} = \frac{24{,}300}{M}$$

(b) 2005 corresponds to $t = 15$.

$$C = \frac{15{,}000}{M}\Big[1.20t + 0.02t^2\Big]_{15}^{16} = \frac{27{,}300}{M}$$

81. Trapezoidal Rule ($n = 4$): $\displaystyle\int_1^2 \frac{1}{1+x^3}\,dx \approx \frac{1}{8}\left[\frac{1}{1+1^3} + \frac{2}{1+(1.25)^3} + \frac{2}{1+(1.5)^3} + \frac{2}{1+(1.75)^3} + \frac{1}{1+2^3}\right] \approx 0.257$

Simpson's Rule ($n = 4$): $\displaystyle\int_1^2 \frac{1}{1+x^3}\,dx \approx \frac{1}{12}\left[\frac{1}{1+1^3} + \frac{4}{1+(1.25)^3} + \frac{2}{1+(1.5)^3} + \frac{4}{1+(1.75)^3} + \frac{1}{1+2^3}\right] \approx 0.254$

Graphing utility: 0.254

83. Trapezoidal Rule ($n = 4$): $\displaystyle\int_0^{\pi/2} \sqrt{x}\cos x\,dx \approx 0.637$

Simpson's Rule ($n = 4$): 0.685

Graphing Utility: 0.704

85. (a) $R < I < T < L$

(b) $S(4) = \dfrac{4-0}{3(4)}[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)]$

$$\approx \frac{1}{3}\left[4 + 4(2) + 2(1) + 4\left(\frac{1}{2}\right) + \frac{1}{4}\right] \approx 5.417$$

Problem Solving for Chapter 4

1. (a) $L(1) = \int_1^1 \frac{1}{t}\,dt = 0$

(b) $L'(x) = \frac{1}{x}$ by the Second Fundamental Theorem of Calculus.

$L'(1) = 1$

(c) $L(x) = 1 = \int_1^x \frac{1}{t}\,dt$ for $x \approx 2.718$

$\int_1^{2.718} \frac{1}{t}\,dt = 0.999896$ **(Note:** The exact value of x is e, the base of the natural logarithm function.)

(d) We first show that $\int_1^{x_1} \frac{1}{t}\,dt = \int_{1/x_1}^1 \frac{1}{t}\,dt.$

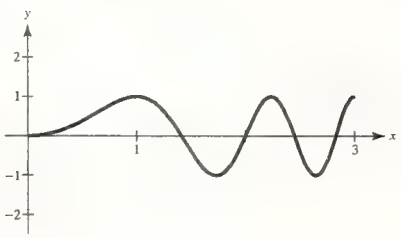
To see this, let $u = \frac{t}{x_1}$ and $du = \frac{1}{x_1}\,dt.$

Then $\int_1^{x_1} \frac{1}{t}\,dt = \int_{1/x_1}^1 \frac{1}{ux_1}(x_1\,du) = \int_{1/x_1}^1 \frac{1}{u}\,du = \int_{1/x_1}^1 \frac{1}{t}\,dt.$

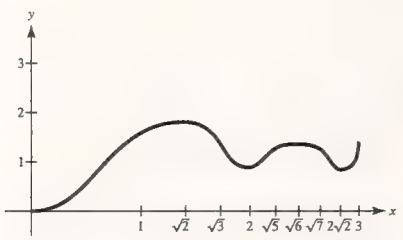
Now, $$\begin{aligned} L(x_1x_2) &= \int_1^{x_1x_2} \frac{1}{t}\,dt = \int_{1/x_1}^{x_2} \frac{1}{u}\,du \left(\text{using } u = \frac{t}{x_1}\right) \\ &= \int_{1/x_1}^1 \frac{1}{u}\,du + \int_1^{x_2} \frac{1}{u}\,du \\ &= \int_1^{x_1} \frac{1}{u}\,du + \int_1^{x_2} \frac{1}{u}\,du \\ &= L(x_1) + L(x_2). \end{aligned}$$

3. $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$

(a)



(b)



The zeros of $y = \sin\frac{\pi x^2}{2}$ correspond to the relative extrema of $S(x)$.

(c) $S'(x) = \sin\frac{\pi x^2}{2} = 0 \Rightarrow \frac{\pi x^2}{2} = n\pi \Rightarrow x^2 = 2n \Rightarrow x = \sqrt{2n}$, n integer.

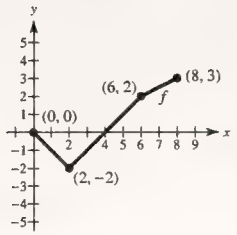
Relative maximum at $x = \sqrt{2} \approx 1.4142$ and $x = \sqrt{6} \approx 2.4495$

Relative minimum at $x = 2$ and $x = \sqrt{8} \approx 2.8284$

(d) $S''(x) = \cos\left(\frac{\pi x^2}{2}\right)(\pi x) = 0 \Rightarrow \frac{\pi x^2}{2} = \frac{\pi}{2} + n\pi \Rightarrow x^2 = 1 + 2n \Rightarrow x = \sqrt{1 + 2n}$, n integer

Points of inflection at $x = 1$, $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{7}$.

5. (a)



(b)

x	0	1	2	3	4	5	6	7	8
$F(x)$	0	$-\frac{1}{2}$	-2	$-\frac{7}{2}$	-4	$-\frac{7}{2}$	-2	$\frac{1}{4}$	3

(c) $f(x) = \begin{cases} -x, & 0 \le x < 2 \\ x - 4, & 2 \le x < 6 \\ \frac{1}{2}x - 1, & 6 \le x \le 8 \end{cases}$

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} (-x^2/2), & 0 \le x < 2 \\ (x^2/2) - 4x + 4, & 2 \le x < 6 \\ (1/4)x^2 - x - 5, & 6 \le x \le 8 \end{cases}$$

$F'(x) = f(x)$. F is decreasing on $(0, 4)$ and increasing on $(4, 8)$. Therefore, the minimum is -4 at $x = 4$, and the maximum is 3 at $x = 8$.

(d) $F''(x) = f'(x) = \begin{cases} -1, & 0 < x < 2 \\ 1, & 2 < x < 6 \\ \frac{1}{2}, & 6 < x < 8 \end{cases}$

$x = 2$ is a point of inflection, whereas $x = 6$ is not. (f is not continuous at $x = 6$.)

7. (a) $\displaystyle\int_{-1}^{1} \cos x\,dx \approx \cos\left(-\frac{1}{\sqrt{3}}\right) + \cos\left(\frac{1}{\sqrt{3}}\right) = 2\cos\left(\frac{1}{\sqrt{3}}\right) \approx 1.6758$

$$\int_{-1}^{1} \cos x\,dx = \sin x\Big]_{-1}^{1} = 2\sin(1) \approx 1.6829$$

Error. $|1.6829 - 1.6758| = 0.0071$

(b) $\displaystyle\int_{-1}^{1} \frac{1}{1+x^2}\,dx \approx \frac{1}{1+(1/3)} + \frac{1}{1+(1/3)} = \frac{3}{2}$

(Note: exact answer is $\pi/2 \approx 1.5708$)

(c) Let $p(x) = ax^3 + bx^2 + cx + d$.

$$\int_{-1}^{1} p(x)\,dx = \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_{-1}^{1} = \frac{2b}{3} + 2d$$

$$p\left(-\frac{1}{\sqrt{3}}\right) + p\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{b}{3} + d\right) + \left(\frac{b}{3} + d\right) = \frac{2b}{3} + 2d$$

9. Consider $F(x) = [f(x)]^2 \Rightarrow F'(x) = 2f(x)f'(x)$. Thus,

$$\int_a^b f(x)f'(x)\,dx = \int_a^b \frac{1}{2}F'(x)\,dx$$

$$= \left[\frac{1}{2}F(x)\right]_a^b$$

$$= \frac{1}{2}[F(b) - F(a)]$$

$$= \frac{1}{2}[f(b)^2 - f(a)^2]$$

11. Consider $\displaystyle\int_0^1 x^5\,dx = \frac{x^6}{6}\Big]_0^1 = \frac{1}{6}$.

The corresponding Riemann Sum using right endpoints is

$$S(n) = \frac{1}{n}\left[\left(\frac{1}{n}\right)^5 + \left(\frac{2}{n}\right)^5 + \cdots + \left(\frac{n}{n}\right)^5\right]$$

$$= \frac{1}{n^6}[1^5 + 2^5 + \cdots + n^5]$$

Thus, $\displaystyle\lim_{n\to\infty} S(n) = \lim_{n\to\infty} \frac{1^5 + 2^5 + \cdots + n^5}{n^6} = \frac{1}{6}$.

13. By Theorem 4.8, $0 < f(x) \le M \Rightarrow \displaystyle\int_a^b f(x)\,dx \le \int_a^b M\,dx = M(b-a)$.

Similarly, $m \le f(x) \Rightarrow m(b-a) = \displaystyle\int_a^b m\,dx \le \int_a^b f(x)\,dx$.

Thus, $m(b-a) \le \displaystyle\int_a^b f(x)\,dx \le M(b-a)$. On the interval $[0, 1]$, $1 \le \sqrt{1+x^4} \le \sqrt{2}$ and $b - a = 1$.

Thus, $1 \le \displaystyle\int_0^1 \sqrt{1+x^4}\,dx \le \sqrt{2}$. $\left(\textbf{Note: } \displaystyle\int_0^1 \sqrt{1+x^4}\,dx \approx 1.0894\right)$

15. Since $-|f(x)| \le f(x) \le |f(x)|$,

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx \Rightarrow \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

17. $\dfrac{1}{365}\displaystyle\int_0^{365} 100{,}000\left[1 + \sin\frac{2\pi(t-60)}{365}\right]dt = \frac{100{,}000}{365}\left[t - \frac{365}{2\pi}\cos\frac{2\pi(t-60)}{365}\right]_0^{365} = 100{,}000$ lbs.

CHAPTER 5
Logarithmic, Exponential, and Other Transcendental Functions

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CHAPTER 5
Logarithmic, Exponential, and Other Transcendental Functions

Section 5.1 The Natural Logarithmic Function: Differentiation

Solutions to Odd-Numbered Exercises

1. Simpson's Rule: $n = 10$

x	0.5	1.5	2	2.5	3	3.5	4
$\int_1^x \frac{1}{t}\,dt$	-0.6932	0.4055	0.6932	0.9163	1.0987	1.2529	1.3865

Note: $\int_1^{0.5} \frac{1}{t}\,dt = -\int_{0.5}^{1} \frac{1}{t}\,dt$

3. (a) $\ln 45 \approx 3.8067$

(b) $\int_1^{45} \frac{1}{t}\,dt \approx 3.8067$

5. (a) $\ln 0.8 \approx -0.2231$

(b) $\int_1^{0.8} \frac{1}{t}\,dt \approx -0.2231$

7. $f(x) = \ln x + 2$

Vertical shift 2 units upward

Matches (b)

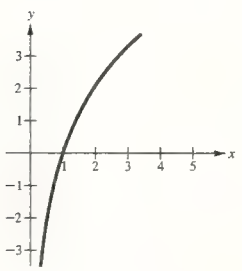
9. $f(x) = \ln(x - 1)$

Horizontal shift 1 unit to the right

Matches (a)

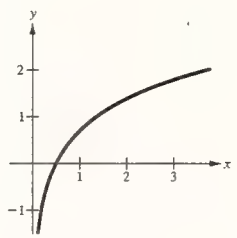
11. $f(x) = 3 \ln x$

Domain: $x > 0$



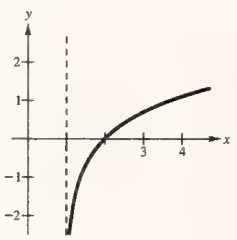
13. $f(x) = \ln 2x$

Domain: $x > 0$



15. $f(x) = \ln(x - 1)$

Domain: $x > 1$



17. (a) $\ln 6 = \ln 2 + \ln 3 \approx 1.7917$

(b) $\ln \frac{2}{3} = \ln 2 - \ln 3 \approx -0.4055$

(c) $\ln 81 = \ln 3^4 = 4 \ln 3 \approx 4.3944$

(d) $\ln \sqrt{3} = \ln 3^{1/2} = \frac{1}{2} \ln 3 \approx 0.5493$

19. $\ln \frac{2}{3} = \ln 2 - \ln 3$

21. $\ln \frac{xy}{z} = \ln x + \ln y - \ln z$

23. $\ln \sqrt[3]{a^2 + 1} = \ln(a^2 + 1)^{1/3} = \frac{1}{3} \ln(a^2 + 1)$

25. $\ln\left(\dfrac{x^2-1}{x^3}\right)^3 = 3[\ln(x^2-1) - \ln x^3]$

$= 3[\ln(x+1) + \ln(x-1) - 3\ln x]$

27. $\ln z(z-1)^2 = \ln z + \ln(z-1)^2$

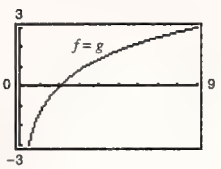
$= \ln z + 2\ln(z-1)$

29. $\ln(x-2) - \ln(x+2) = \ln\dfrac{x-2}{x+2}$

31. $\dfrac{1}{3}[2\ln(x+3) + \ln x - \ln(x^2-1)] = \dfrac{1}{3}\ln\dfrac{x(x+3)^2}{x^2-1} = \ln\sqrt[3]{\dfrac{x(x+3)^2}{x^2-1}}$

33. $2\ln 3 - \dfrac{1}{2}\ln(x^2+1) = \ln 9 - \ln\sqrt{x^2+1} = \ln\dfrac{9}{\sqrt{x^2+1}}$

35.



37. $\lim_{x\to 3^+}\ln(x-3) = -\infty$

39. $\lim_{x\to 2^-}\ln[x^2(3-x)] = \ln 4 \approx 1.3863$

41. $y = \ln x^3 = 3\ln x$

$y' = \dfrac{3}{x}$

At $(1, 0)$, $y' = 3$.

43. $y = \ln x^2 = 2\ln x$

$y' = \dfrac{2}{x}$

At $(1, 0)$, $y' = 2$.

45. $g(x) = \ln x^2 = 2\ln x$

$g'(x) = \dfrac{2}{x}$

47. $y = (\ln x)^4$

$\dfrac{dy}{dx} = 4(\ln x)^3\left(\dfrac{1}{x}\right) = \dfrac{4(\ln x)^3}{x}$

49. $y = \ln x\sqrt{x^2-1} = \ln x + \dfrac{1}{2}\ln(x^2-1)$

$\dfrac{dy}{dx} = \dfrac{1}{x} + \dfrac{1}{2}\left(\dfrac{2x}{x^2-1}\right) = \dfrac{2x^2-1}{x(x^2-1)}$

51. $f(x) = \ln\dfrac{x}{x^2+1} = \ln x - \ln(x^2+1)$

$f'(x) = \dfrac{1}{x} - \dfrac{2x}{x^2+1} = \dfrac{1-x^2}{x(x^2+1)}$

53. $g(t) = \dfrac{\ln t}{t^2}$

$g'(t) = \dfrac{t^2(1/t) - 2t\ln t}{t^4} = \dfrac{1-2\ln t}{t^3}$

55. $y = \ln(\ln x^2)$

$\dfrac{dy}{dx} = \dfrac{1}{\ln x^2}\dfrac{d}{dx}(\ln x^2) = \dfrac{(2x/x^2)}{\ln x^2} = \dfrac{2}{x\ln x^2} = \dfrac{1}{x\ln x}$

57. $y = \ln\sqrt{\dfrac{x+1}{x-1}} = \dfrac{1}{2}[\ln(x+1) - \ln(x-1)]$

$\dfrac{dy}{dx} = \dfrac{1}{2}\left[\dfrac{1}{x+1} - \dfrac{1}{x-1}\right] = \dfrac{1}{1-x^2}$

59. $f(x) = \ln\dfrac{\sqrt{4+x^2}}{x} = \dfrac{1}{2}\ln(4+x^2) - \ln x$

$f'(x) = \dfrac{x}{4+x^2} - \dfrac{1}{x} = \dfrac{-4}{x(x^2+4)}$

61. $y = \dfrac{-\sqrt{x^2+1}}{x} + \ln\left(x + \sqrt{x^2+1}\right)$

$$\frac{dy}{dx} = \frac{-x\left(x/\sqrt{x^2+1}\right) + \sqrt{x^2+1}}{x^2} + \left(\frac{1}{x+\sqrt{x^2+1}}\right)\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x^2\sqrt{x^2+1}} + \left(\frac{1}{x+\sqrt{x^2+1}}\right)\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right) = \frac{1}{x^2\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} = \frac{1+x^2}{x^2\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}}{x^2}$$

63. $y = \ln|\sin x|$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

65. $y = \ln\left|\dfrac{\cos x}{\cos x - 1}\right|$

$$= \ln|\cos x| - \ln|\cos x - 1|$$

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} - \frac{-\sin x}{\cos x - 1} = -\tan x + \frac{\sin x}{\cos x - 1}$$

67. $y = \ln\left|\dfrac{-1+\sin x}{2+\sin x}\right|$

$$= \ln|-1+\sin x| - \ln|2+\sin x|$$

$$\frac{dy}{dx} = \frac{\cos x}{-1+\sin x} - \frac{\cos x}{2+\sin x}$$

$$= \frac{3\cos x}{(\sin x - 1)(\sin x + 2)}$$

69. $f(x) = \sin 2x \ln x^2 = 2\sin 2x \ln x$

$$f'(x) = (2\sin 2x)\left(\frac{1}{x}\right) + 4\cos 2x \ln x$$

$$= \frac{2}{x}(\sin 2x + 2x\cos 2x \ln x)$$

$$= \frac{2}{x}(\sin 2x + x\cos 2x \ln x^2)$$

71. (a) $y = 3x^2 - \ln x, \quad (1, 3)$

$$\frac{dy}{dx} = 6x - \frac{1}{x}$$

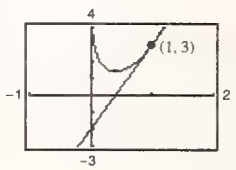
When $x = 1$, $\dfrac{dy}{dx} = 5$.

Tangent line: $y - 3 = 5(x - 1)$

$$y = 5x - 2$$

$$0 = 5x - y - 2$$

(b)



73. $x^2 - 3\ln y + y^2 = 10$

$$2x - \frac{3}{y}\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$2x = \frac{dy}{dx}\left(\frac{3}{y} - 2y\right)$$

$$\frac{dy}{dx} = \frac{2x}{(3/y) - 2y} = \frac{2xy}{3 - 2y^2}$$

75. $y = 2(\ln x) + 3$

$$y' = \frac{2}{x}$$

$$y'' = -\frac{2}{x^2}$$

$$xy'' + y' = x\left(-\frac{2}{x^2}\right) + \frac{2}{x} = 0$$

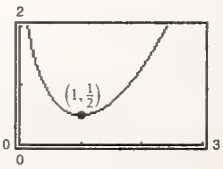
77. $y = \dfrac{x^2}{2} - \ln x$

Domain: $x > 0$

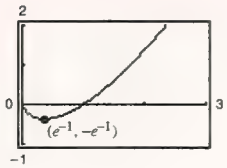
$y' = x - \dfrac{1}{x} = \dfrac{(x+1)(x-1)}{x} = 0$ when $x = 1$.

$y'' = 1 + \dfrac{1}{x^2} > 0$

Relative minimum: $\left(1, \frac{1}{2}\right)$



79. $y = x \ln x$



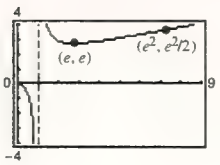
Domain: $x > 0$

$y' = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x = 0$ when $x = e^{-1}$.

$y'' = \frac{1}{x} > 0$

Relative minimum: $(e^{-1}, -e^{-1})$

81. $y = \frac{x}{\ln x}$



Domain: $0 < x < 1, x > 1$

$y' = \frac{(\ln x)(1) - (x)(1/x)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} = 0$ when $x = e$.

$y'' = \frac{(\ln x)^2(1/x) - (\ln x - 1)(2/x)\ln x}{(\ln x)^4} = \frac{2 - \ln x}{x(\ln x)^3} = 0$ when $x = e^2$.

Relative minimum: (e, e)

Point of inflection: $(e^2, e^2/2)$

83. $f(x) = \ln x, \quad f(1) = 0$

$f'(x) = \frac{1}{x}, \quad f'(1) = 1$

$f''(x) = -\frac{1}{x^2}, \quad f''(1) = -1$

$P_1(x) = f(1) + f'(1)(x - 1) = x - 1, \quad P_1(1) = 0$

$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2}f''(1)(x - 1)^2$

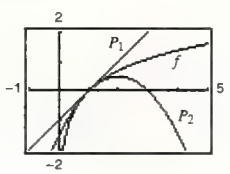
$= (x - 1) - \frac{1}{2}(x - 1)^2, \quad P_2(1) = 0$

$P_1'(x) = 1, \quad P_1'(1) = 1$

$P_2'(x) = 1 - (x - 1) = 2 - x, \quad P_2'(1) = 1$

$P_2''(x) = -1, \quad P_2''(1) = -1$

The values of f, P_1, P_2, and their first derivatives agree at $x = 1$. The values of the second derivatives of f and P_2 agree at $x = 1$.



85. Find x such that $\ln x = -x$.

$f(x) = (\ln x) + x = 0$

$f'(x) = \frac{1}{x} + 1$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n\left[\frac{1 - \ln x_n}{1 + x_n}\right]$

n	1	2	3
x_n	0.5	0.5644	0.5671
$f(x_n)$	−0.1931	−0.0076	−0.0001

Approximate root: $x = 0.567$

87. $y = x\sqrt{x^2 - 1}$

$\ln y = \ln x + \frac{1}{2}\ln(x^2 - 1)$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{x}{x^2 - 1}$

$\frac{dy}{dx} = y\left[\frac{2x^2 - 1}{x(x^2 - 1)}\right] = \frac{2x^2 - 1}{\sqrt{x^2 - 1}}$

89. $y = \dfrac{x^2\sqrt{3x-2}}{(x-1)^2}$

$$\ln y = 2\ln x + \frac{1}{2}\ln(3x-2) - 2\ln(x-1)$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x-1}$$

$$\frac{dy}{dx} = y\left[\frac{3x^2-15x+8}{2x(3x-2)(x-1)}\right]$$

$$= \frac{3x^3-15x^2+8x}{2(x-1)^3\sqrt{3x-2}}$$

91. $y = \dfrac{x(x-1)^{3/2}}{\sqrt{x+1}}$

$$\ln y = \ln x + \frac{3}{2}\ln(x-1) - \frac{1}{2}\ln(x+1)$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{3}{2}\left(\frac{1}{x-1}\right) - \frac{1}{2}\left(\frac{1}{x+1}\right)$$

$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{2}{x} + \frac{3}{x-1} - \frac{1}{x+1}\right]$$

$$= \frac{y}{2}\left[\frac{4x^2+4x-2}{x(x^2-1)}\right] = \frac{(2x^2+2x-1)\sqrt{x-1}}{(x+1)^{3/2}}$$

93. Answers will vary. See Theorems 5.1 and 5.2.

95. $\ln e^x = x$ because $f(x) = \ln x$ and $g(x) = e^x$ are inverse functions.

97. (a) $f(1) \neq f(3)$

(b) $f'(x) = 1 - \dfrac{2}{x} = 0$ for $x = 2$.

99. $$\beta = 10\log_{10}\left(\frac{I}{10^{-16}}\right) = \frac{10}{\ln 10}\ln\left(\frac{I}{10^{-16}}\right) = \frac{10}{\ln 10}[\ln I + 16\ln 10] = 160 + 10\log_{10} I$$

$$\beta(10^{-10}) = \frac{10}{\ln 10}[\ln 10^{-10} + 16\ln 10] = \frac{10}{\ln 10}[-10\ln 10 + 16\ln 10] = \frac{10}{\ln 10}[6\ln 10] = 60 \text{ decibels}$$

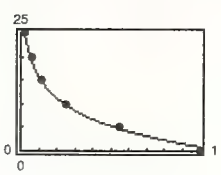
101. (a) You get an error message because $\ln h$ does not exist for $h = 0$.

(b) Reversing the data, you obtain

$$h = 0.8627 - 6.4474\ln p.$$

[Note: Fit a line to the data $(x, y) = (\ln p, h)$.]

(c)



(d) If $p = 0.75$, $h \approx 2.72$ km.

(e) If $h = 13$ km, $p \approx 0.15$ atmosphere.

(f) $h = 0.8627 - 6.4474\ln p$

$$1 = -6.4474\frac{1}{p}\frac{dp}{dh} \quad \text{(implicit differentiation)}$$

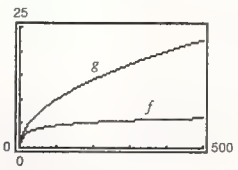
$$\frac{dp}{dh} = \frac{p}{-6.4474}$$

For $h = 5$, $p = 0.5264$ and $dp/dh = -0.0816$ atmos/km.

For $h = 20$, $p = 0.0514$ and $dp/dh = -0.0080$ atmos/km.

As the altitude increases, the rate of change of pressure decreases.

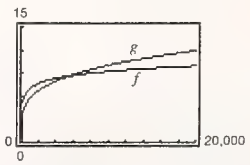
103. (a) $f(x) = \ln x$, $g(x) = \sqrt{x}$



$$f'(x) = \frac{1}{x},\ g'(x) = \frac{1}{2\sqrt{x}}$$

For $x > 4$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for "large" values of x.

(b) $f(x) = \ln x$, $g(x) = \sqrt[4]{x}$



$$f'(x) = \frac{1}{x},\ g'(x) = \frac{1}{4\sqrt[4]{x^3}}$$

For $x > 256$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for "large" values of x. $f(x) = \ln x$ increases very slowly for "large" values of x.

105. False

$\ln x + \ln 25 = \ln(25x) \neq \ln(x + 25)$

Section 5.2 The Natural Logarithmic Function: Integration

1. $\int \frac{5}{x}dx = 5\int \frac{1}{x}dx = 5 \ln |x| + C$

3. $u = x + 1, du = dx$

$$\int \frac{1}{x + 1}dx = \ln|x + 1| + C$$

5. $u = 3 - 2x, du = -2\,dx$

$$\int \frac{1}{3 - 2x}dx = -\frac{1}{2}\int \frac{1}{3 - 2x}(-2)\,dx$$

$$= -\frac{1}{2}\ln|3 - 2x| + C$$

7. $u = x^2 + 1, du = 2x\,dx$

$$\int \frac{x}{x^2 + 1}dx = \frac{1}{2}\int \frac{1}{x^2 + 1}(2x)\,dx$$

$$= \frac{1}{2}\ln(x^2 + 1) + C$$

$$= \ln\sqrt{x^2 + 1} + C$$

9. $\int \frac{x^2 - 4}{x}dx = \int \left(x - \frac{4}{x}\right)dx$

$$= \frac{x^2}{2} - 4 \ln|x| + C$$

11. $u = x^3 + 3x^2 + 9x, du = 3(x^2 + 2x + 3)\,dx$

$$\int \frac{x^2 + 2x + 3}{x^3 + 3x^2 + 9x}dx = \frac{1}{3}\int \frac{3(x^2 + 2x + 3)}{x^3 + 3x^2 + 9x}dx$$

$$= \frac{1}{3}\ln|x^3 + 3x^2 + 9x| + C$$

13. $\int \frac{x^2 - 3x + 2}{x + 1}dx = \int \left(x - 4 + \frac{6}{x + 1}\right)dx$

$$= \frac{x^2}{2} - 4x + 6 \ln|x + 1| + C$$

15. $\int \frac{x^3 - 3x^2 + 5}{x - 3}dx = \int \left(x^2 + \frac{5}{x - 3}\right)dx$

$$= \frac{x^3}{3} + 5 \ln|x - 3| + C$$

17. $\int \frac{x^4 + x - 4}{x^2 + 2}dx = \int \left(x^2 - 2 + \frac{x}{x^2 + 2}\right)dx$

$$= \frac{x^3}{3} - 2x + \frac{1}{2}\ln(x^2 + 2) + C$$

19. $u = \ln x, du = \frac{1}{x}dx$

$$\int \frac{(\ln x)^2}{x}dx = \frac{1}{3}(\ln x)^3 + C$$

21. $u = x + 1, du = dx$

$$\int \frac{1}{\sqrt{x + 1}}dx = \int (x + 1)^{-1/2}\,dx$$

$$= 2(x + 1)^{1/2} + C$$

$$= 2\sqrt{x + 1} + C$$

23. $\int \frac{2x}{(x - 1)^2}dx = \int \frac{2x - 2 + 2}{(x - 1)^2}dx$

$$= \int \frac{2(x - 1)}{(x - 1)^2}dx + 2\int \frac{1}{(x - 1)^2}dx$$

$$= 2\int \frac{1}{x - 1}dx + 2\int \frac{1}{(x - 1)^2}dx$$

$$= 2 \ln|x - 1| - \frac{2}{(x - 1)} + C$$

25. $u = 1 + \sqrt{2x},\ du = \dfrac{1}{\sqrt{2x}}\,dx \implies (u-1)\,du = dx$

$$\int \frac{1}{1+\sqrt{2x}}\,dx = \int \frac{(u-1)}{u}\,du = \int \left(1 - \frac{1}{u}\right) du$$
$$= u - \ln|u| + C_1$$
$$= \left(1 + \sqrt{2x}\right) - \ln\left|1 + \sqrt{2x}\right| + C_1$$
$$= \sqrt{2x} - \ln\left(1 + \sqrt{2x}\right) + C$$

where $C = C_1 + 1$.

27. $u = \sqrt{x} - 3,\ du = \dfrac{1}{2\sqrt{x}}\,dx \implies 2(u+3)\,du = dx$

$$\int \frac{\sqrt{x}}{\sqrt{x}-3}\,dx = 2\int \frac{(u+3)^2}{u}\,du = 2\int \frac{u^2+6u+9}{u}\,du = 2\int \left(u + 6 + \frac{9}{u}\right) du$$
$$= 2\left[\frac{u^2}{2} + 6u + 9\ln|u|\right] + C_1 = u^2 + 12u + 18\ln|u| + C_1$$
$$= \left(\sqrt{x}-3\right)^2 + 12\left(\sqrt{x}-3\right) + 18\ln\left|\sqrt{x}-3\right| + C_1$$
$$= x + 6\sqrt{x} + 18\ln\left|\sqrt{x}-3\right| + C \text{ where } C = C_1 - 27.$$

29. $\displaystyle\int \frac{\cos\theta}{\sin\theta}\,d\theta = \ln|\sin\theta| + C$

$(u = \sin\theta,\ du = \cos\theta\,d\theta)$

31. $\displaystyle\int \csc 2x\,dx = \frac{1}{2}\int (\csc 2x)(2)\,dx$

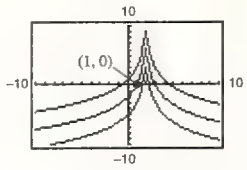
$$= -\frac{1}{2}\ln|\csc 2x + \cot 2x| + C$$

33. $\displaystyle\int \frac{\cos t}{1+\sin t}\,dt = \ln|1+\sin t| + C$

35. $\displaystyle\int \frac{\sec x \tan x}{\sec x - 1}\,dx = \ln|\sec x - 1| + C$

37. $y = \displaystyle\int \frac{3}{2-x}\,dx$

$$= -3\int \frac{1}{x-2}\,dx$$
$$= -3\ln|x-2| + C$$

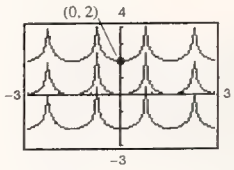


$(1, 0)$: $0 = -3\ln|1-2| + C \implies C = 0$

$y = -3\ln|x-2|$

39. $s = \displaystyle\int \tan(2\theta)\,d\theta$

$$= \frac{1}{2}\int \tan(2\theta)(2\,d\theta)$$
$$= -\frac{1}{2}\ln|\cos 2\theta| + C$$

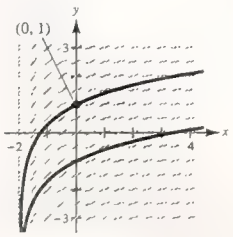


$(0, 2)$: $2 = -\dfrac{1}{2}\ln|\cos(0)| + C \implies C = 2$

$s = -\dfrac{1}{2}\ln|\cos 2\theta| + 2$

41. $\dfrac{dy}{dx} = \dfrac{1}{x+2}$, $(0, 1)$

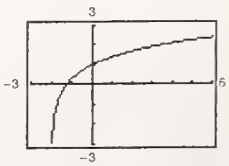
(a)



(b) $\displaystyle y = \int \frac{1}{x+2}\,dx = \ln|x+2| + C$

$y(0) = 1 \implies 1 = \ln 2 + C \implies C = 1 - \ln 2$

Hence, $y = \ln|x+2| + 1 - \ln 2 = \ln\left|\dfrac{x+2}{2}\right| + 1$.



43. $\displaystyle\int_0^4 \frac{5}{3x+1}\,dx = \left[\frac{5}{3}\ln|3x+1|\right]_0^4$

$= \dfrac{5}{3}\ln 13 \approx 4.275$

45. $u = 1 + \ln x$, $du = \dfrac{1}{x}\,dx$

$\displaystyle\int_1^e \frac{(1+\ln x)^2}{x}\,dx = \left[\frac{1}{3}(1+\ln x)^3\right]_1^e = \frac{7}{3}$

47. $\displaystyle\int_0^2 \frac{x^2-2}{x+1}\,dx = \int_0^2 \left(x - 1 - \frac{1}{x+1}\right)dx$

$= \left[\dfrac{1}{2}x^2 - x - \ln|x+1|\right]_0^2 = -\ln 3$

49. $\displaystyle\int_1^2 \frac{1-\cos\theta}{\theta-\sin\theta}\,d\theta = \left[\ln|\theta-\sin\theta|\right]_1^2$

$= \ln\left|\dfrac{2-\sin 2}{1-\sin 1}\right| \approx 1.929$

51. $-\ln|\cos x| + C = \ln\left|\dfrac{1}{\cos x}\right| + C = \ln|\sec x| + C$

53. $\ln|\sec x + \tan x| + C = \ln\left|\dfrac{(\sec x+\tan x)(\sec x-\tan x)}{(\sec x-\tan x)}\right| + C = \ln\left|\dfrac{\sec^2 x - \tan^2 x}{\sec x - \tan x}\right| + C$

$= \ln\left|\dfrac{1}{\sec x - \tan x}\right| + C = -\ln|\sec x - \tan x| + C$

55. $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx = 2(1+\sqrt{x}) - 2\ln(1+\sqrt{x}) + C_1$

$= 2\left[\sqrt{x} - \ln(1+\sqrt{x})\right] + C$ where $C = C_1 + 2$.

57. $\displaystyle\int \cos(1-x)\,dx = -\sin(1-x) + C$

59. $\displaystyle\int_{\pi/4}^{\pi/2} (\csc x - \sin x)\,dx = \left[-\ln|\csc x + \cot x| + \cos x\right]_{\pi/4}^{\pi/2} = \ln(\sqrt{2}+1) - \frac{\sqrt{2}}{2} \approx 0.174$

Note: In Exercises 61 and 63, you can use the Second Fundamental Theorem of Calculus or integrate the function.

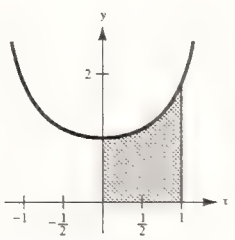
61. $F(x) = \displaystyle\int_1^x \frac{1}{t}\,dt$

$F'(x) = \dfrac{1}{x}$

63. $F(x) = \displaystyle\int_x^{3x} \frac{1}{t}\,dt = \int_1^{3x} \frac{1}{t}\,dt - \int_1^x \frac{1}{t}\,dt$

$F'(x) = \dfrac{3}{3x} - \dfrac{1}{x} = 0$

65.



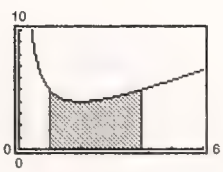
$A \approx 1.25$

Matches (d)

67. $A = \displaystyle\int_1^4 \frac{x^2+4}{x}\,dx = \int_1^4 \left(x + \frac{4}{x}\right)dx$

$= \left[\dfrac{x^2}{2} + 4\ln x\right]_1^4 = (8 + 4\ln 4) - \dfrac{1}{2}$

$= \dfrac{15}{2} + 8\ln 2 \approx 13.045$ square units

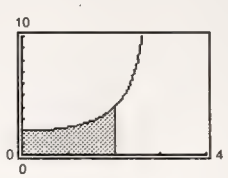


69. $\displaystyle\int_0^2 2\sec\frac{\pi x}{6}\,dx = \frac{12}{\pi}\int_0^2 \sec\left(\frac{\pi x}{6}\right)\frac{\pi}{6}\,dx$

$$= \left[\frac{12}{\pi}\ln\left|\sec\frac{\pi x}{6} + \tan\frac{\pi x}{6}\right|\right]_0^2$$

$$= \frac{12}{\pi}\ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \frac{12}{\pi}\ln|1 + 0|$$

$$= \frac{12}{\pi}\ln\left(2 + \sqrt{3}\right) \approx 5.03041$$



71. Power Rule

73. Substitution: $(u = x^2 + 4)$ and Log Rule

75. Divide the polynomials:

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$$

77. Average value $= \dfrac{1}{4-2}\displaystyle\int_2^4 \frac{8}{x^2}dx = 4\int_2^4 x^{-2}dx$

$$= \left[-4\frac{1}{x}\right]_2^4$$

$$= -4\left(\frac{1}{4} - \frac{1}{2}\right) = 1$$

79. Average value $= \dfrac{1}{e-1}\displaystyle\int_1^e \frac{\ln x}{x}dx = \frac{1}{e-1}\left[\frac{(\ln x)^2}{2}\right]_1^e$

$$= \frac{1}{e-1}\left(\frac{1}{2}\right)$$

$$= \frac{1}{2e-2} \approx 0.291$$

81. $P(t) = \displaystyle\int \frac{3000}{1 + 0.25t}\,dt = (3000)(4)\int \frac{0.25}{1 + 0.25t}\,dt = 12{,}000\ln|1 + 0.25t| + C$

$P(0) = 12{,}000\ln|1 + 0.25(0)| + C = 1000$

$C = 1000$

$P(t) = 12{,}000\ln|1 + 0.25t| + 1000 = 1000[12\ln|1 + 0.25t| + 1]$

$P(3) = 1000[12(\ln 1.75) + 1] \approx 7715$

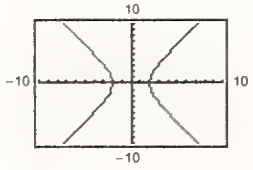
83. $\dfrac{1}{50-40}\displaystyle\int_{40}^{50} \frac{90{,}000}{400 + 3x}dx = \left[3000\ln|400 + 3x|\right]_{40}^{50} \approx \$168.27$

85. (a) $2x^2 - y^2 = 8$

$y^2 = 2x^2 - 8$

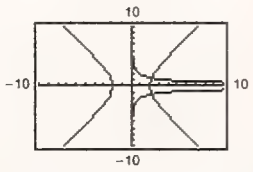
$y_1 = \sqrt{2x^2 - 8}$

$y_2 = -\sqrt{2x^2 - 8}$



(b) $y^2 = e^{-\int(1/x)dx} = e^{-\ln x + C} = e^{\ln(1/x)}(e^C) = \dfrac{1}{x}k$

Let $k = 4$ and graph $y^2 = \dfrac{4}{x}$ $\left(\begin{matrix} y_1 = 2/\sqrt{x} \\ y_2 = -2/\sqrt{x} \end{matrix}\right)$



(c) In part (a), $2x^2 - y^2 = 8$

$4x - 2yy' = 0$

$y' = \dfrac{2x}{y}.$

In part (b), $y^2 = \dfrac{4}{x} = 4x^{-1}$

$2yy' = \dfrac{-4}{x^2}$

$y' = \dfrac{-2}{yx^2} = \dfrac{-2y}{y^2x^2} = \dfrac{-2y}{4x} = \dfrac{-y}{2x}.$

Using a graphing utility the graphs intersect at (2.214, 1.344). The slopes are 3.295 and $-0.304 = (-1)/3.295$, respectively.

87. False

$$\frac{1}{2}(\ln x) = \ln(x^{1/2}) \neq (\ln x)^{1/2}$$

89. True

$$\int \frac{1}{x}dx = \ln|x| + C_1$$

$$= \ln|x| + \ln|C| = \ln|Cx|, \; C \neq 0$$

Section 5.3 Inverse Functions

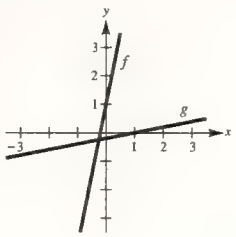
1. (a) $f(x) = 5x + 1$

$$g(x) = \frac{x-1}{5}$$

$$f(g(x)) = f\left(\frac{x-1}{5}\right) = 5\left(\frac{x-1}{5}\right) + 1 = x$$

$$g((f(x)) = g(5x+1) = \frac{(5x+1)-1}{5} = x$$

(b)



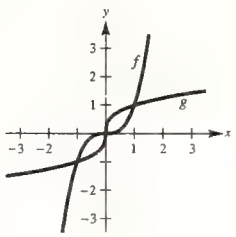
3. (a) $f(x) = x^3$

$$g(x) = \sqrt[3]{x}$$

$$f(g(x)) = f\left(\sqrt[3]{x}\right) = \left(\sqrt[3]{x}\right)^3 = x$$

$$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$$

(b)



5. (a) $f(x) = \sqrt{x-4}$

$$g(x) = x^2 + 4, \; x \geq 0$$

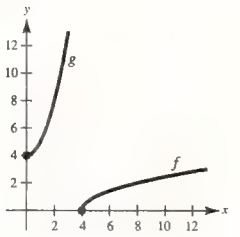
$$f(g(x)) = f(x^2+4)$$

$$= \sqrt{(x^2+4)-4} = \sqrt{x^2} = x$$

$$g(f(x)) = g\left(\sqrt{x-4}\right)$$

$$= \left(\sqrt{x-4}\right)^2 + 4 = x - 4 + 4 = x$$

(b)



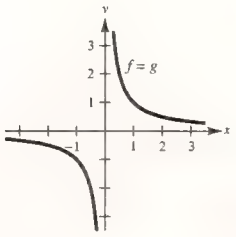
7. (a) $f(x) = \frac{1}{x}$

$$g(x) = \frac{1}{x}$$

$$f(g(x)) = \frac{1}{1/x} = x$$

$$g(f(x)) = \frac{1}{1/x} = x$$

(b)

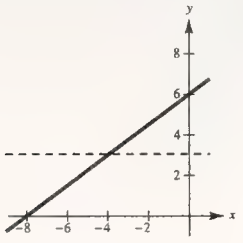


9. Matches (c)

11. Matches (a)

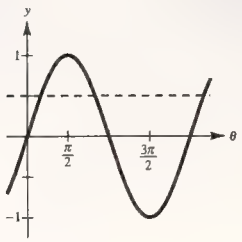
13. $f(x) = \frac{3}{4}x + 6$

One-to-one; has an inverse



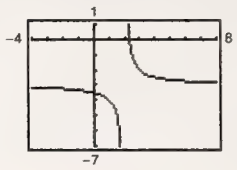
15. $f(\theta) = \sin\theta$

Not one-to-one; does not have an inverse



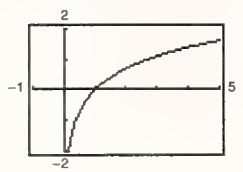
17. $h(s) = \dfrac{1}{s-2} - 3$

One-to-one; has an inverse



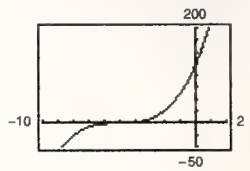
19. $f(x) = \ln x$

One-to-one; has an inverse



21. $g(x) = (x+5)^3$

One-to-one; has an inverse



23. $f(x) = (x+a)^3 + b$

$f'(x) = 3(x+a)^2 \geq 0$ for all x.

f is increasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

25. $f(x) = \dfrac{x^4}{4} - 2x^2$

$f'(x) = x^3 - 4x = 0$ when $x = 0, 2, -2$.

f is not strictly monotonic on $(-\infty, \infty)$. Therefore, f does not have an inverse.

27. $f(x) = 2 - x - x^3$

$f'(x) = -1 - 3x^2 < 0$ for all x.

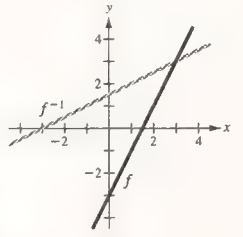
f is decreasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

29. $f(x) = 2x - 3 = y$

$$x = \frac{y+3}{2}$$

$$y = \frac{x+3}{2}$$

$$f^{-1}(x) = \frac{x+3}{2}$$

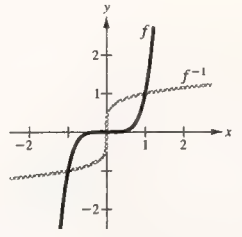


31. $f(x) = x^5 = y$

$$x = \sqrt[5]{y}$$

$$y = \sqrt[5]{x}$$

$$f^{-1}(x) = \sqrt[5]{x} = x^{1/5}$$

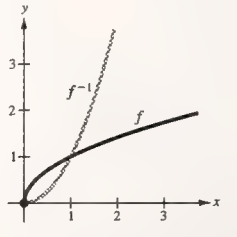


33. $f(x) = \sqrt{x} = y$

$$x = y^2$$

$$y = x^2$$

$$f^{-1}(x) = x^2, \quad x \geq 0$$

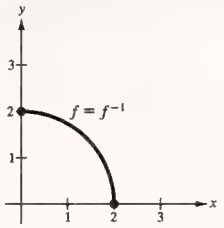


35. $f(x) = \sqrt{4 - x^2} = y, \quad 0 \le x \le 2$

$x = \sqrt{4 - y^2}$

$y = \sqrt{4 - x^2}$

$f^{-1}(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2$

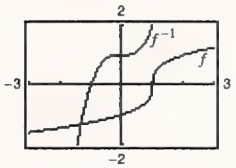


37. $f(x) = \sqrt[3]{x - 1} = y$

$x = y^3 + 1$

$y = x^3 + 1$

$f^{-1}(x) = x^3 + 1$



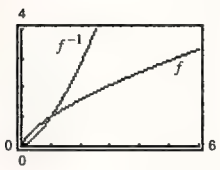
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

39. $f(x) = x^{2/3} = y, \quad x \ge 0$

$x = y^{3/2}$

$y = x^{3/2}$

$f^{-1}(x) = x^{3/2}, \quad x \ge 0$



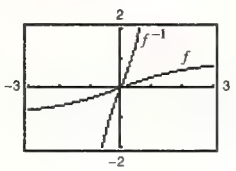
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

41. $f(x) = \dfrac{x}{\sqrt{x^2 + 7}} = y$

$x = \dfrac{\sqrt{7}y}{\sqrt{1 - y^2}}$

$y = \dfrac{\sqrt{7}x}{\sqrt{1 - x^2}}$

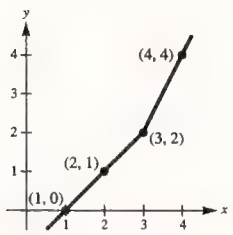
$f^{-1}(x) = \dfrac{\sqrt{7}x}{\sqrt{1 - x^2}}, \quad -1 < x < 1$



The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

43.

x	1	2	3	4
$f^{-1}(x)$	0	1	2	4



45. (a) Let x be the number of pounds of the commodity costing 1.25 per pound. Since there are 50 pounds total, the amount of the second commodity is $50 - x$. The total cost is

$$y = 1.25x + 1.60(50 - x)$$
$$= -0.35x + 80 \qquad 0 \le x \le 50.$$

(b) We find the inverse of the original function:

$$y = -0.35x + 80$$
$$0.35x = 80 - y$$
$$x = \tfrac{100}{35}(80 - y)$$

Inverse: $y = \tfrac{100}{35}(80 - x) = \tfrac{20}{7}(80 - x)$.

x represents cost and y represents pounds.

(c) Domain of inverse is $62.5 \le x \le 80$.

(d) If $x = 73$ in the inverse function,
$y = \tfrac{100}{35}(80 - 73) = \tfrac{100}{5} = 20$ pounds.

47. $f(x) = (x-4)^2$ on $[4, \infty)$

$f'(x) = 2(x-4) > 0$ on $(4, \infty)$

f is increasing on $[4, \infty)$. Therefore, f is strictly monotonic and has an inverse.

49. $f(x) = \dfrac{4}{x^2}$ on $(0, \infty)$

$f'(x) = -\dfrac{8}{x^3} < 0$ on $(0, \infty)$

f is decreasing on $(0, \infty)$. Therefore, f is strictly monotonic and has an inverse.

51. $f(x) = \cos x$ on $[0, \pi]$

$f'(x) = -\sin x < 0$ on $(0, \pi)$

f is decreasing on $[0, \pi]$. Therefore, f is strictly monotonic and has an inverse.

53. $f(x) = \dfrac{x}{x^2-4} = y$ on $(-2, 2)$

$x^2y - 4y = x$

$x^2y - x - 4y = 0$

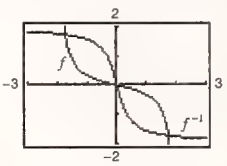
$a = y, b = -1, c = -4y$

$x = \dfrac{1 \pm \sqrt{1 - 4(y)(-4y)}}{2y} = \dfrac{1 \pm \sqrt{1 + 16y^2}}{2y}$

$y = f^{-1}(x) = \begin{cases} \left(1 - \sqrt{1 + 16x^2}\right)/2x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

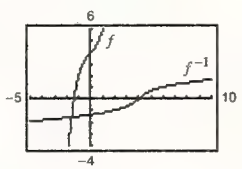
Domain: all x

Range: $-2 < y < 2$



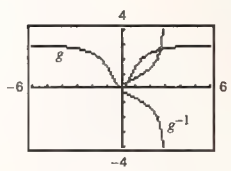
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

55. (a), (b)



(c) Yes, f is one-to-one and has an inverse. The inverse relation is an inverse function.

57. (a), (b)



(c) g is not one-to-one and does not have an inverse. The inverse relation is not an inverse function.

59. $f(x) = \sqrt{x-2}$, Domain: $x \geq 2$

$f'(x) = \dfrac{1}{2\sqrt{x-2}} > 0$ for $x > 2$.

f is one-to-one; has an inverse

$\sqrt{x-2} = y$

$x - 2 = y^2$

$x = y^2 + 2$

$y = x^2 + 2$

$f^{-1}(x) = x^2 + 2, x \geq 0$

61. $f(x) = |x-2|, x \leq 2$

$= -(x-2)$

$= 2 - x$

f is one-to-one; has an inverse

$2 - x = y$

$2 - y = x$

$f^{-1}(x) = 2 - x, x \geq 0$

63. $f(x) = (x-3)^2$ is one-to-one for $x \geq 3$.

$(x-3)^2 = y$

$x - 3 = \sqrt{y}$

$x = \sqrt{y} + 3$

$y = \sqrt{x} + 3$

$f^{-1}(x) = \sqrt{x} + 3, x \geq 0$

(Answer is not unique)

65. $f(x) = |x+3|$ is one-to-one for $x \geq -3$.

$x + 3 = y$

$x = y - 3$

$y = x - 3$

$f^{-1}(x) = x - 3, x \geq 0$

(Answer is not unique)

67. Yes, the volume is an increasing function, and hence one-to-one. The inverse function gives the time t corresponding to the volume V.

69. No, $C(t)$ is not one-to-one because long distance costs are step functions. A call lasting 2.1 minutes costs the same as one lasting 2.2 minutes.

71. $f(x) = x^3 + 2x - 1,\ f(1) = 2 = a$

$f'(x) = 3x^2 + 2$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{3(1)^2 + 2} = \frac{1}{5}$$

73. $f(x) = \sin x, f\left(\frac{\pi}{6}\right) = \frac{1}{2} = a$

$f'(x) = \cos x$

$$(f^{-1})'\left(\frac{1}{2}\right) = \frac{1}{f'(f^{-1}(1/2))} = \frac{1}{f'(\pi/6)} = \frac{1}{\cos(\pi/6)}$$

$$= \frac{1}{\sqrt{3}/2} = \frac{2\sqrt{3}}{3}$$

75. $f(x) = x^3 - \frac{4}{x},\ f(2) = 6 = a$

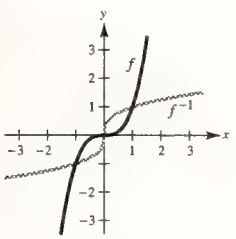
$f'(x) = 3x^2 + \frac{4}{x^2}$

$$(f^{-1})'(6) = \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(2)} = \frac{1}{3(2)^2 + (4/2^2)} = \frac{1}{13}$$

77. (a) Domain f = Domain $f^{-1} = (-\infty, \infty)$

(b) Range f = Range $f^{-1} = (-\infty, \infty)$

(c)



(d) $f(x) = x^3,\ \left(\frac{1}{2}, \frac{1}{8}\right)$

$f'(x) = 3x^2$

$f'\left(\frac{1}{2}\right) = \frac{3}{4}$

$f^{-1}(x) = \sqrt[3]{x},\ \left(\frac{1}{8}, \frac{1}{2}\right)$

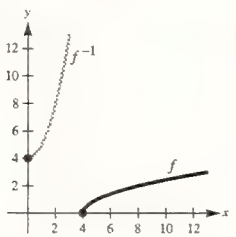
$$(f^{-1})'(x) = \frac{1}{3\sqrt[3]{x^2}}$$

$$(f^{-1})'\left(\frac{1}{8}\right) = \frac{4}{3}$$

79. (a) Domain $f = [4, \infty)$, Domain $f^{-1} = [0, \infty)$

(b) Range $f = [0, \infty)$, Range $f^{-1} = [4, \infty)$

(c)



(d) $f(x) = \sqrt{x - 4},\ (5, 1)$

$$f'(x) = \frac{1}{2\sqrt{x - 4}}$$

$f'(5) = \frac{1}{2}$

$f^{-1}(x) = x^2 + 4,\ (1, 5)$

$(f^{-1})'(x) = 2x$

$(f^{-1})'(1) = 2$

81. $x = y^3 - 7y^2 + 2$

$1 = 3y^2\frac{dy}{dx} - 14y\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{3y^2 - 14y}$. At $(-4, 1)$, $\frac{dy}{dx} = \frac{1}{3 - 14} = \frac{-1}{11}$.

Alternate solution: let $f(x) = x^3 - 7x^2 + 2$.

Then $f'(x) = 3x^2 - 14x$ and $f'(1) = -11$.

Hence, $\frac{dy}{dx} = \frac{1}{-11} = \frac{-1}{11}$.

In Exercises 83 and 85, use the following.

$f(x) = \frac{1}{8}x - 3$ **and** $g(x) = x^3$

$f^{-1}(x) = 8(x + 3)$ **and** $g^{-1}(x) = \sqrt[3]{x}$

83. $(f^{-1} \circ g^{-1})(1) = f^{-1}(g^{-1}(1)) = f^{-1}(1) = 32$

85. $(f^{-1} \circ f^{-1})(6) = f^{-1}(f^{-1}(6)) = f^{-1}(72) = 600$

In Exercises 87 and 89, use the following.

$f(x) = x + 4$ **and** $g(x) = 2x - 5$

$f^{-1}(x) = x - 4$ **and** $g^{-1}(x) = \frac{x + 5}{2}$

87. $(g^{-1} \circ f^{-1})(x) = g^{-1}(f^{-1}(x))$

$= g^{-1}(x - 4)$

$= \frac{(x - 4) + 5}{2}$

$= \frac{x + 1}{2}$

89. $(f \circ g)(x) = f(g(x))$

$= f(2x - 5)$

$= (2x - 5) + 4$

$= 2x - 1$

Hence, $(f \circ g)^{-1}(x) = \frac{x + 1}{2}$

(Note: $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$)

91. Answers will vary. See page 335 and Example 3.

93. $y = x^2$ on $(-\infty, \infty)$ does not have an inverse.

95. f is not one-to-one because many different x-values yield the same y-value.

Example: $f(0) = f(\pi) = 0$

Not continuous at $\frac{(2n - 1)\pi}{2}$, where n is an integer

97. Let $(f \circ g)(x) = y$ then $x = (f \circ g)^{-1}(y)$. Also,

$(f \circ g)(x) = y$

$f(g(x)) = y$

$g(x) = f^{-1}(y)$

$x = g^{-1}(f^{-1}(y))$

$= (g^{-1} \circ f^{-1})(y)$

Since f and g are one-to-one functions, $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

99. Suppose $g(x)$ and $h(x)$ are both inverses of $f(x)$. Then the graph of $f(x)$ contains the point (a, b) if and only if the graphs of $g(x)$ and $h(x)$ contain the point (b, a). Since the graphs of $g(x)$ and $h(x)$ are the same, $g(x) = h(x)$. Therefore, the inverse of $f(x)$ is unique.

101. False

Let $f(x) = x^2$.

103. True

105. Not true

Let $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 1 - x, & 1 < x \le 2 \end{cases}$.

f is one-to-one, but not strictly monotonic.

107. $f(x) = \int_2^x \frac{dt}{\sqrt{1 + t^4}}$, $f(2) = 0$

$$f'(x) = \frac{1}{\sqrt{1 + x^4}}$$

$$(f^{-1})'(0) = \frac{1}{f'(2)} = \frac{1}{1/\sqrt{17}} = \sqrt{17}$$

Section 5.4 Exponential Functions: Differentiation and Integration

1. $e^0 = 1$

$\ln 1 = 0$

3. $\ln 2 = 0.6931$

$e^{0.6931\ldots} = 2$

5. $e^{\ln x} = 4$

$x = 4$

7. $e^x = 12$

$x = \ln 12 \approx 2.485$

9. $9 - 2e^x = 7$

$2e^x = 7$

$e^x = 1$

$x = 0$

11. $50e^{-x} = 30$

$e^{-x} = \frac{3}{5}$

$-x = \ln\left(\frac{3}{5}\right)$

$x = \ln\left(\frac{5}{3}\right) \approx 0.511$

13. $\ln x = 2$

$x = e^2 \approx 7.3891$

15. $\ln(x - 3) = 2$

$x - 3 = e^2$

$x = 3 + e^2 \approx 10.389$

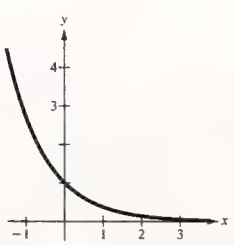
17. $\ln\sqrt{x + 2} = 1$

$\sqrt{x + 2} = e^1 = e$

$x + 2 = e^2$

$x = e^2 - 2 \approx 5.389$

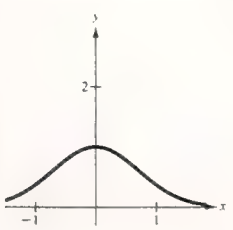
19. $y = e^{-x}$



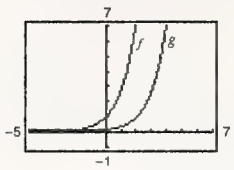
21. $y = e^{-x^2}$

Symmetric with respect to the y-axis

Horizontal asymptote: $y = 0$

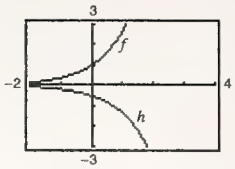


23. (a)



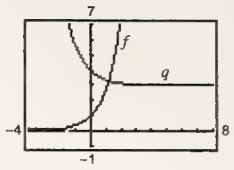
Horizontal shift 2 units to the right

(b)



A reflection in the x-axis and a vertical shrink

(c)



Vertical shift 3 units upward and a reflection in the y-axis

25. $y = Ce^{ax}$

Horizontal asymptote: $y = 0$

Matches (c)

27. $y = C(1 - e^{-ax})$

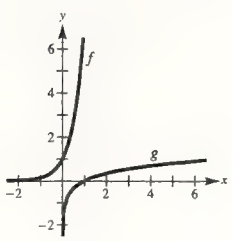
Vertical shift C units

Reflection in both the x- and y-axes

Matches (a)

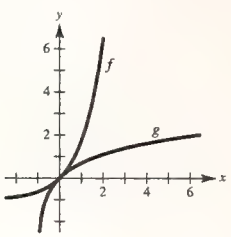
29. $f(x) = e^{2x}$

$g(x) = \ln\sqrt{x} = \frac{1}{2}\ln x$

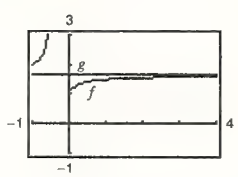


31. $f(x) = e^x - 1$

$g(x) = \ln(x + 1)$



33.



As $x \to \infty$, the graph of f approaches the graph of g.

$\lim_{x\to\infty}\left(1 + \frac{0.5}{x}\right)^x = e^{0.5}$

35. $\left(1 + \frac{1}{1{,}000{,}000}\right)^{1{,}000{,}000} \approx 2.718280469$

$e \approx 2.718281828$

$e > \left(1 + \frac{1}{1{,}000{,}000}\right)^{1{,}000{,}000}$

37. (a) $y = e^{3x}$

$y' = 3e^{3x}$

At $(0, 1)$, $y' = 3$.

(b) $y = e^{-3x}$

$y' = -3e^{-3x}$

At $(0, 1)$, $y' = -3$.

39. $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

41. $f(x) = e^{-2x+x^2}$

$\frac{dy}{dx} = 2(x - 1)e^{-2x+x^2}$

43. $y = e^{\sqrt{x}}$

$\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

45. $g(t) = (e^{-t} + e^t)^3$

$g'(t) = 3(e^{-t} + e^t)^2(e^t - e^{-t})$

47. $y = \ln e^{x^2} = x^2$

$\frac{dy}{dx} = 2x$

49. $y = \ln(1 + e^{2x})$

$\frac{dy}{dx} = \frac{2e^{2x}}{1 + e^{2x}}$

51. $y = \dfrac{2}{e^x + e^{-x}} = 2(e^x + e^{-x})^{-1}$

$$\frac{dy}{dx} = -2(e^x + e^{-x})^{-2}(e^x - e^{-x})$$

$$= \frac{-2(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

53. $y = x^2e^x - 2xe^x + 2e^x = e^x(x^2 - 2x + 2)$

$$\frac{dy}{dx} = e^x(2x - 2) + e^x(x^2 - 2x + 2) = x^2e^x$$

55. $f(x) = e^{-x} \ln x$

$$f'(x) = e^{-x}\left(\frac{1}{x}\right) - e^{-x} \ln x = e^{-x}\left(\frac{1}{x} - \ln x\right)$$

57. $y = e^x(\sin x + \cos x)$

$$\frac{dy}{dx} = e^x(\cos x - \sin x) + (\sin x + \cos x)(e^x)$$

$$= e^x(2 \cos x) = 2e^x \cos x$$

59. $xe^y - 10x + 3y = 0$

$$xe^y\frac{dy}{dx} + e^y - 10 + 3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(xe^y + 3) = 10 - e^y$$

$$\frac{dy}{dx} = \frac{10 - e^y}{xe^y + 3}$$

61. $f(x) = (3 + 2x)e^{-3x}$

$$f'(x) = (3 + 2x)(-3e^{-3x}) + 2e^{-3x}$$

$$= (-7 - 6x)e^{-3x}$$

$$f''(x) = (-7 - 6x)(-3e^{-3x}) - 6e^{-3x}$$

$$= 3(6x + 5)e^{-3x}$$

63. $y = e^x(\cos\sqrt{2}x + \sin\sqrt{2}x)$

$$y' = e^x(-\sqrt{2}\sin\sqrt{2}x + \sqrt{2}\cos\sqrt{2}x) + e^x(\cos\sqrt{2}x + \sin\sqrt{2}x)$$

$$= e^x[(1 + \sqrt{2})\cos\sqrt{2}x + (1 - \sqrt{2})\sin\sqrt{2}x]$$

$$y'' = e^x[-(\sqrt{2} + 2)\sin\sqrt{2}x + (\sqrt{2} - 2)\cos\sqrt{2}x] + e^x[(1 + \sqrt{2})\cos\sqrt{2}x + (1 - \sqrt{2})\sin\sqrt{2}x]$$

$$= e^x[(-1 - 2\sqrt{2})\sin\sqrt{2}x + (-1 + 2\sqrt{2})\cos\sqrt{2}x]$$

$$-2y' + 3y = -2e^x[(1 + \sqrt{2})\cos\sqrt{2}x + (1 - \sqrt{2})\sin\sqrt{2}x] + 3e^x[\cos\sqrt{2}x + \sin\sqrt{2}x]$$

$$= e^x[(1 - 2\sqrt{2})\cos\sqrt{2}x + (1 + 2\sqrt{2})\sin\sqrt{2}x] = -y''$$

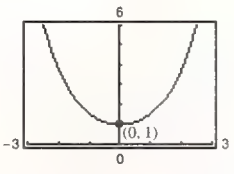
Therefore, $-2y' + 3y = -y'' \Rightarrow y'' - 2y' + 3y = 0$.

65. $f(x) = \dfrac{e^x + e^{-x}}{2}$

$f'(x) = \dfrac{e^x - e^{-x}}{2} = 0$ when $x = 0$.

$f''(x) = \dfrac{e^x + e^{-x}}{2} > 0$

Relative minimum: (0, 1)



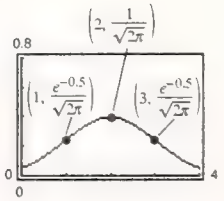
67. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-2)^2/2}$

$g'(x) = \dfrac{-1}{\sqrt{2\pi}}(x - 2)e^{-(x-2)^2/2}$

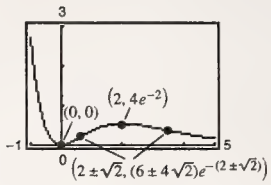
$g''(x) = \dfrac{1}{\sqrt{2\pi}}(x - 1)(x - 3)e^{-(x-2)^2/2}$

Relative maximum: $\left(2, \dfrac{1}{\sqrt{2\pi}}\right) \approx (2, 0.399)$

Points of inflection: $\left(1, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right), \left(3, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right) \approx (1, 0.242), (3, 0.242)$



69. $f(x) = x^2e^{-x}$



$f'(x) = -x^2e^{-x} + 2xe^{-x} = xe^{-x}(2 - x) = 0$ when $x = 0, 2.$

$f''(x) = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x)$

$= e^{-x}(x^2 - 4x + 2) = 0$ when $x = 2 \pm \sqrt{2}.$

Relative minimum: $(0, 0)$

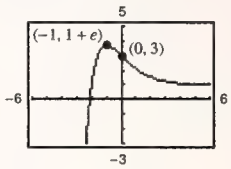
Relative maximum: $(2, 4e^{-2})$

$x = 2 \pm \sqrt{2}$

$y = (2 \pm \sqrt{2})^2e^{-(2 \pm \sqrt{2})}$

Points of inflection: (3.414, 0.384), (0.586, 0.191)

71. $g(t) = 1 + (2 + t)e^{-t}$



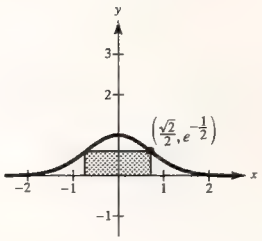
$g'(t) = (1 + t)e^{-t}$

$g''(t) = te^{-t}$

Relative maximum: $(-1, 1 + e) \approx (-1, 3.718)$

Point of inflection: $(0, 3)$

73. $A = (\text{base})(\text{height}) = 2xe^{-x^2}$



$\dfrac{dA}{dx} = -4x^2e^{-x^2} + 2e^{-x^2}$

$= 2e^{-x^2}(1 - 2x^2) = 0$ when $x = \dfrac{\sqrt{2}}{2}.$

$A = \sqrt{2}e^{-1/2}$

75. $y = \dfrac{L}{1 + ae^{-x/b}}, a > 0, b > 0, L > 0$

$$y' = \frac{-L\left(-\frac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^2} = \frac{\frac{aL}{b}e^{-x/b}}{(1 + ae^{-x/b})^2}$$

$$y'' = \frac{(1 + ae^{-x/b})^2\left(\frac{-aL}{b^2}e^{-x/b}\right) - \left(\frac{aL}{b}e^{-x/b}\right)2(1 + ae^{-x/b})\left(\frac{-a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^4}$$

$$= \frac{(1 + ae^{-x/b})\left(\frac{-aL}{b^2}e^{-x/b}\right) + 2\left(\frac{aL}{b}e^{-x/b}\right)\left(\frac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^3}$$

$$= \frac{Lae^{-x/b}[ae^{-x/b} - 1]}{(1 + ae^{-x/b})^3 b^2}$$

$y'' = 0$ if $ae^{-x/b} = 1 \Rightarrow \dfrac{-x}{b} = \ln\left(\dfrac{1}{a}\right) \Rightarrow x = b \ln a$

$$y(b \ln a) = \frac{L}{1 + ae^{-(b \ln a)/b}} = \frac{L}{1 + a(1/a)} = \frac{L}{2}$$

Therefore, the y-coordinate of the inflection point is $L/2$.

77. $e^{-x} = x \Rightarrow f(x) = x - e^{-x}$

$f'(x) = 1 + e^{-x}$

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n - e^{-x_n}}{1 + e^{-x_n}}$

$x_1 = 1$

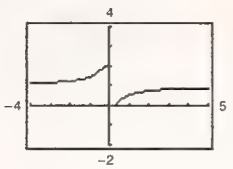
$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} \approx 0.5379$

$x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} \approx 0.5670$

$x_4 = x_3 - \dfrac{f(x_3)}{f'(x_3)} \approx 0.5671$

We approximate the root of f to be $x = 0.567$.

79. (a)

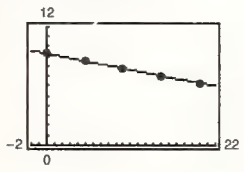


(b) When x increases without bound, $1/x$ approaches zero, and $e^{1/x}$ approaches 1. Therefore, $f(x)$ approaches $2/(1 + 1) = 1$. Thus, $f(x)$ has a horizontal asymptote at $y = 1$. As x approaches zero from the right, $1/x$ approaches ∞, $e^{1/x}$ approaches ∞ and $f(x)$ approaches zero. As x approaches zero from the left, $1/x$ approaches $-\infty$, $e^{1/x}$ approaches zero, and $f(x)$ approaches 2. The limit does not exist since the left limit does not equal the right limit. Therefore, $x = 0$ is a nonremovable discontinuity.

81.

h	0	5	10	15	20
P	10,332	5,583	2,376	1,240	517
$\ln P$	9.243	8.627	7.773	7.123	6.248

(a)



$y = -0.1499h + 9.3018$ is the regression line for data $(h, \ln P)$.

(b) $\ln P = ah + b$

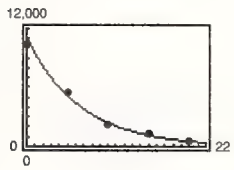
$P = e^{ah+b} = e^b e^{ah}$

$P = Ce^{ah}, C = e^b$

For our data, $a = -0.1499$ and $C = e^{9.3018} = 10{,}957.7$

$P = 10{,}957.7e^{-0.1499h}$

(c)



(d) $\dfrac{dP}{dh} = (10{,}957.71)(-0.1499)e^{-0.1499h}$

$= -1642.56e^{-0.1499h}$

For $h = 5$, $\dfrac{dP}{dh} = -776.3$. For $h = 18$, $\dfrac{dP}{dh} \approx -110.6$.

83. $f(x) = e^{x/2}, f(0) = 1$

$f'(x) = \dfrac{1}{2}e^{x/2}, f'(0) = \dfrac{1}{2}$

$f''(x) = \dfrac{1}{4}e^{x/2}, f''(0) = \dfrac{1}{4}$

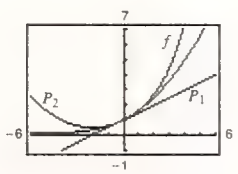
$P_1(x) = 1 + \dfrac{1}{2}(x - 0) = \dfrac{x}{2} + 1, P_1(0) = 1$

$P_1'(x) = \dfrac{1}{2}, P_1'(0) = \dfrac{1}{2}$

$P_2(x) = 1 + \dfrac{1}{2}(x - 0) + \dfrac{1}{8}(x - 0)^2 = \dfrac{x^2}{8} + \dfrac{x}{2} + 1, P_2(0) = 1$

$P_2'(x) = \dfrac{1}{4}x + \dfrac{1}{2}, P_2'(0) = \dfrac{1}{2}$

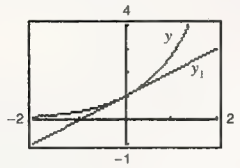
$P_2''(x) = \dfrac{1}{4}, P_2''(0) = \dfrac{1}{4}$



The values of f, P_1, P_2 and their first derivatives agree at $x = 0$. The values of the second derivatives of f and P_2 agree at $x = 0$.

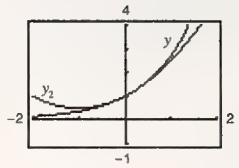
85. (a) $y = e^x$

$y_1 = 1 + x$



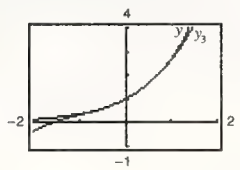
(b) $y = e^x$

$y_2 = 1 + x + \left(\frac{x^2}{2}\right)$



(c) $y = e^x$

$y_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$



87. Let $u = 5x$, $du = 5\,dx$.

$$\int e^{5x}\,5dx = e^{5x} + C$$

89. Let $u = -2x$, $du = -2\,dx$.

$$\int_0^1 e^{-2x}dx = -\frac{1}{2}\int_0^1 e^{-2x}(-2)\,dx = \left[-\frac{1}{2}e^{-2x}\right]_0^1$$
$$= \frac{1}{2}(1 - e^{-2}) = \frac{e^2 - 1}{2e^2}$$

91. $\displaystyle\int xe^{-x^2}dx = -\frac{1}{2}\int e^{-x^2}(-2x)dx = -\frac{1}{2}e^{-x^2} + C$

93. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2\int e^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)dx = 2e^{\sqrt{x}} + C$

95. Let $u = 1 + e^{-x}$, $du = -e^{-x}\,dx$.

$$\int \frac{e^{-x}}{1 + e^{-x}}dx = -\int \frac{-e^{-x}}{1 + e^{-x}}dx = -\ln(1 + e^{-x}) + C = \ln\left(\frac{e^x}{e^x + 1}\right) + C = x - \ln(e^x + 1) + C$$

97. Let $u = \frac{3}{x}$, $du = -\frac{3}{x^2}\,dx$.

$$\int_1^3 \frac{e^{3/x}}{x^2}\,dx = -\frac{1}{3}\int_1^3 e^{3/x}\left(-\frac{3}{x^2}\right)dx$$
$$= \left[-\frac{1}{3}e^{3/x}\right]_1^3 = \frac{e}{3}(e^2 - 1)$$

99. Let $u = 1 - e^x$, $du = -e^x\,dx$.

$$\int e^x\sqrt{1 - e^x}\,dx = -\int (1 - e^x)^{1/2}(-e^x)\,dx$$
$$= -\frac{2}{3}(1 - e^x)^{3/2} + C$$

101. Let $u = e^x - e^{-x}$, $du = (e^x + e^{-x})\,dx$.

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}dx = \ln|e^x - e^{-x}| + C$$

103. $\displaystyle\int \frac{5 - e^x}{e^{2x}}dx = \int 5e^{-2x}dx - \int e^{-x}\,dx$

$$= -\frac{5}{2}e^{-2x} + e^{-x} + C$$

105. $\displaystyle\int e^{\sin \pi x}\cos \pi x\,dx = \frac{1}{\pi}\int e^{\sin \pi x}(\pi \cos \pi x)\,dx$

$$= \frac{1}{\pi}e^{\sin \pi x} + C$$

107. $\displaystyle\int e^{-x}\tan(e^{-x})dx = -\int [\tan(e^{-x})](-e^{-x})\,dx$

$$= \ln|\cos(e^{-x})| + C$$

109. Let $u = ax^2$, $du = 2ax\,dx$. (Assume $a \neq 0$)

$$y = \int xe^{ax^2}\,dx$$

$$= \frac{1}{2a}\int e^{ax^2}(2ax)\,dx = \frac{1}{2a}e^{ax^2} + C$$

111. $f'(x) = \int \frac{1}{2}(e^x + e^{-x})\,dx = \frac{1}{2}(e^x - e^{-x}) + C_1$

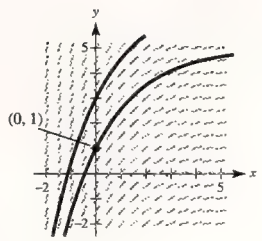
$f'(0) = C_1 = 0$

$f(x) = \int \frac{1}{2}(e^x - e^{-x})\,dx = \frac{1}{2}(e^x + e^{-x}) + C_2$

$f(0) = 1 + C_2 = 1 \Rightarrow C_2 = 0$

$f(x) = \frac{1}{2}(e^x + e^{-x})$

113. (a)



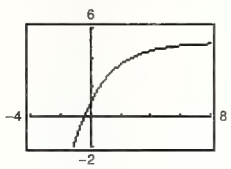
(b) $\frac{dy}{dx} = 2e^{-x/2}$, $(0, 1)$

$$y = \int 2e^{-x/2}\,dx = -4\int e^{-x/2}\left(-\frac{1}{2}dx\right)$$

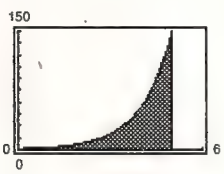
$$= -4e^{-x/2} + C$$

$(0, 1)$: $1 = -4e^0 + C = -4 + C \Rightarrow C = 5$

$y = -4e^{-x/2} + 5$

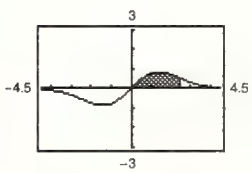


115. $\int_0^5 e^x\,dx = \left[e^x\right]_0^5 = e^5 - 1 \approx 147.413$



117. $\int_0^{\sqrt{6}} xe^{-x^2/4}dx = \left[-2e^{-x^2/4}\right]_0^{\sqrt{6}}$

$$= -2e^{-3/2} + 2 \approx 1.554$$



119. (a) $f(u - v) = e^{u-v} = (e^u)(e^{-v}) = \frac{e^u}{e^v} = \frac{f(u)}{f(v)}$

(b) $f(kx) = e^{kx} = (e^x)^k = [f(x)]^k$.

121. $0.0665\int_{48}^{60} e^{-0.0139(t-48)^2}\,dt$

Graphing Utility: $0.4772 = 47.72\%$

123. $\int_0^x e^t\,dt \geq \int_0^x 1\,dt$

$\left[e^t\right]_0^x \geq \left[t\right]_0^x$

$e^x - 1 \geq x \Rightarrow e^x \geq 1 + x$ for $x \geq 0$

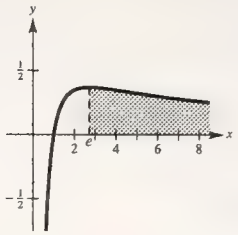
125. $f(x) = e^x$. Domain is $(-\infty, \infty)$ and range is $(0, \infty)$. f is continuous, increasing, one-to-one, and concave upwards on its entire domain.

$\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to\infty} e^x = \infty$.

127. Yes. $f(x) = Ce^x$, C a constant.

129. $e^{-x} > 0 \Rightarrow \int_0^2 e^{-x}dx > 0$.

131. $f(x) = \dfrac{\ln x}{x}$



(a) $f'(x) = \dfrac{1 - \ln x}{x^2} = 0$ when $x = e$.

On $(0, e), f'(x) > 0 \Rightarrow f$ is increasing.

On $(e, \infty), f'(x) < 0 \Rightarrow f$ is decreasing.

(b) For $e \le A < B$, we have:

$$\frac{\ln A}{A} > \frac{\ln B}{B}$$

$$B \ln A > A \ln B$$

$$\ln A^B > \ln B^A$$

$$A^B > B^A.$$

(c) Since $e < \pi$, from part (b) we have $e^\pi > \pi^e$.

Section 5.5 Bases Other than e and Applications

1. $y = \left(\dfrac{1}{2}\right)^{t/3}$

At $t_0 = 6, y = \left(\dfrac{1}{2}\right)^{6/3} = \dfrac{1}{4}$

3. $y = \left(\dfrac{1}{2}\right)^{t/7}$

At $t_0 = 10, y = \left(\dfrac{1}{2}\right)^{10/7} \approx 0.3715$

5. $\log_2 \frac{1}{8} = \log_2 2^{-3} = -3$

7. $\log_7 1 = 0$

9. (a) $2^3 = 8$

$\log_2 8 = 3$

(b) $3^{-1} = \dfrac{1}{3}$

$\log_3 \dfrac{1}{3} = -1$

11. (a) $\log_{10} 0.01 = -2$

$10^{-2} = 0.01$

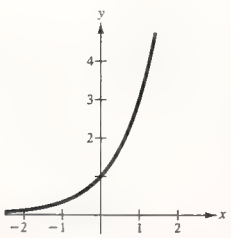
(b) $\log_{0.5} 8 = -3$

$0.5^{-3} = 8$

$\left(\frac{1}{2}\right)^{-3} = 8$

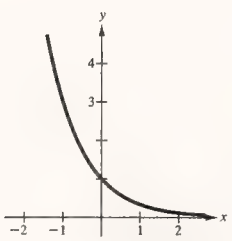
13. $y = 3^x$

x	-2	-1	0	1	2
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9



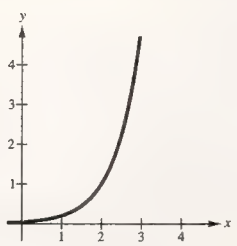
15. $y = \left(\dfrac{1}{3}\right)^x = 3^{-x}$

x	-2	-1	0	1	2
y	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$



17. $h(x) = 5^{x-2}$

x	-1	0	1	2	3
y	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5



19. (a) $\log_{10} 1000 = x$

$10^x = 1000$

$x = 3$

(b) $\log_{10} 0.1 = x$

$10^x = 0.1$

$x = -1$

21. (a) $\log_3 x = -1$

$3^{-1} = x$

$x = \frac{1}{3}$

(b) $\log_2 x = -4$

$2^{-4} = x$

$x = \frac{1}{16}$

23. (a) $x^2 - x = \log_5 25$

$x^2 - x = \log_5 5^2 = 2$

$x^2 - x - 2 = 0$

$(x + 1)(x - 2) = 0$

$x = -1$ OR $x = 2$

(b) $3x + 5 = \log_2 64$

$3x + 5 = \log_2 2^6 = 6$

$3x = 1$

$x = \frac{1}{3}$

25. $3^{2x} = 75$

$2x \ln 3 = \ln 75$

$$x = \frac{1}{2}\frac{\ln 75}{\ln 3} \approx 1.965$$

27. $2^{3-x} = 625$

$(3 - x)\ln 2 = \ln 625$

$$3 - x = \frac{\ln 625}{\ln 2}$$

$$x = 3 - \frac{\ln 625}{\ln 2} \approx -6.288$$

29. $\left(1 + \frac{0.09}{12}\right)^{12t} = 3$

$12t \ln\left(1 + \frac{0.09}{12}\right) = \ln 3$

$$t = \frac{1}{12}\frac{\ln 3}{\ln\left(1 + \frac{0.09}{12}\right)} \approx 12.253$$

31. $\log_2(x - 1) = 5$

$x - 1 = 2^5 = 32$

$x = 33$

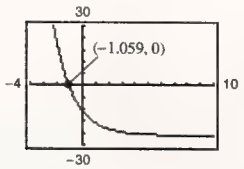
33. $\log_3 x^2 = 4.5$

$x^2 = 3^{4.5}$

$x = \pm\sqrt{3^{4.5}} \approx \pm 11.845$

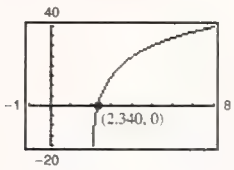
35. $g(x) = 6(2^{1-x}) - 25$

Zero: $x \approx -1.059$



37. $h(s) = 32 \log_{10}(s - 2) + 15$

Zero: $s \approx 2.340$

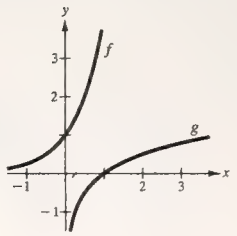


39. $f(x) = 4^x$

$g(x) = \log_4 x$

x	-2	-1	0	$\frac{1}{2}$	1
$f(x)$	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4

x	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4
$g(x)$	-2	-1	0	$\frac{1}{2}$	1



41. $f(x) = 4^x$

$f'(x) = (\ln 4)\, 4^x$

43. $y = 5^{x-2}$

$\dfrac{dy}{dx} = (\ln 5)\, 5^{x-2}$

45. $g(t) = t^2\, 2^t$

$g'(t) = t^2 (\ln 2)\, 2^t + (2t)\, 2^t$

$= t 2^t (t \ln 2 + 2)$

$= 2^t t(2 + t \ln 2)$

47. $h(\theta) = 2^{-\theta} \cos \pi\theta$

$h'(\theta) = 2^{-\theta}(-\pi \sin \pi\theta) - (\ln 2) 2^{-\theta} \cos \pi\theta$

$= -2^{-\theta}[(\ln 2) \cos \pi\theta + \pi \sin \pi\theta]$

49. $y = \log_3 x$

$\dfrac{dy}{dx} = \dfrac{1}{x \ln 3}$

51. $f(x) = \log_2 \dfrac{x^2}{x-1}$

$= 2 \log_2 x - \log_2 (x-1)$

$f'(x) = \dfrac{2}{x \ln 2} - \dfrac{1}{(x-1)\ln 2}$

$= \dfrac{x-2}{(\ln 2) x(x-1)}$

53. $y = \log_5 \sqrt{x^2-1} = \dfrac{1}{2} \log_5 (x^2-1)$

$\dfrac{dy}{dx} = \dfrac{1}{2} \cdot \dfrac{2x}{(x^2-1)\ln 5} = \dfrac{x}{(x^2-1)\ln 5}$

55. $g(t) = \dfrac{10 \log_4 t}{t} = \dfrac{10}{\ln 4}\left(\dfrac{\ln t}{t}\right)$

$g'(t) = \dfrac{10}{\ln 4}\left[\dfrac{t(1/t) - \ln t}{t^2}\right]$

$= \dfrac{10}{t^2 \ln 4}[1 - \ln t] = \dfrac{5}{t^2 \ln 2}(1 - \ln t)$

57. $y = x^{2/x}$

$\ln y = \dfrac{2}{x} \ln x$

$\dfrac{1}{y}\left(\dfrac{dy}{dx}\right) = \dfrac{2}{x}\left(\dfrac{1}{x}\right) + \ln x\left(-\dfrac{2}{x^2}\right) = \dfrac{2}{x^2}(1 - \ln x)$

$\dfrac{dy}{dx} = \dfrac{2y}{x^2}(1 - \ln x) = 2x^{(2/x)-2}(1 - \ln x)$

59. $y = (x-2)^{x+1}$

$\ln y = (x+1) \ln(x-2)$

$\dfrac{1}{y}\left(\dfrac{dy}{dx}\right) = (x+1)\left(\dfrac{1}{x-2}\right) + \ln(x-2)$

$\dfrac{dy}{dx} = y\left[\dfrac{x+1}{x-2} + \ln(x-2)\right]$

$= (x-2)^{x+1}\left[\dfrac{x+1}{x-2} + \ln(x-2)\right]$

61. $\displaystyle\int 3^x\, dx = \frac{3^x}{\ln 3} + C$

63. $\displaystyle\int_{-1}^{2} 2^x\,dx = \left[\frac{2^x}{\ln 2}\right]_{-1}^{2}$

$\displaystyle= \frac{1}{\ln 2}\left[4 - \frac{1}{2}\right]$

$\displaystyle= \frac{7}{2\ln 2} = \frac{7}{\ln 4}$

65. $\displaystyle\int x5^{-x^2}\,dx = -\frac{1}{2}\int 5^{-x^2}(-2x)\,dx$

$\displaystyle= -\left(\frac{1}{2}\right)\frac{5^{-x^2}}{\ln 5} + C$

$\displaystyle= \frac{-1}{2\ln 5}(5^{-x^2}) + C$

67. $\displaystyle\int \frac{3^{2x}}{1+3^{2x}}\,dx,\ u = 1 + 3^{2x},\ du = 2(\ln 3)3^{2x}\,dx$

$\displaystyle\frac{1}{2\ln 3}\int \frac{(2\ln 3)3^{2x}}{1+3^{2x}}\,dx = \frac{1}{2\ln 3}\ln(1 + 3^{2x}) + C$

69. $\displaystyle\frac{dy}{dx} = 0.4^{x/3}, \left(0, \frac{1}{2}\right)$

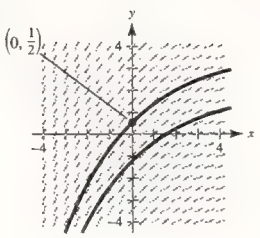
$\displaystyle y = \int 0.4^{x/3}\,dx = 3\int 0.4^{x/3}\left(\frac{1}{3}\,dx\right)$

$\displaystyle= \frac{3}{\ln 0.4}0.4^{x/3} + C = 3(\ln 2.5)(0.4)^{x/3} + C$

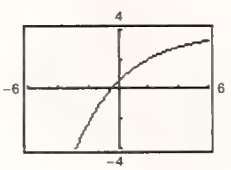
$\displaystyle y = 3\ln 2.5(0.4)^{x/3} + \frac{1}{2} - 3\ln 2.5$

$\displaystyle= \frac{3(1 - 0.4^{x/3})}{\ln 2.5} + \frac{1}{2}$

(a)

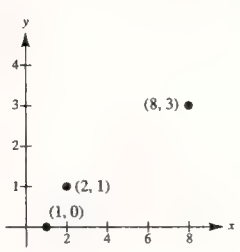


(b)



71. Answers will vary. Example: Growth and decay problems.

73.



x	1	2	8
y	0	1	3

(a) y is an exponential function of x: False

(b) y is a logarithmic function of x: True; $y = \log_2 x$

(c) x is an exponential function of y: True, $2^y = x$

(d) y is a linear function of x: False

75. $f(x) = \log_2 x \Rightarrow f'(x) = \dfrac{1}{x\ln 2}$

$g(x) = x^x \Rightarrow g'(x) = x^x(1 + \ln x)$

[**Note:** Let $y = g(x)$. Then: $\ln y = \ln x^x = x\ln x$

$\dfrac{1}{y}y' = x \cdot \dfrac{1}{x} + \ln x$

$y' = y(1 + \ln x)$

$y' = x^x(1 + \ln x) = g'(x).$]

$h(x) = x^2 \Rightarrow h'(x) = 2x$

$k(x) = 2^x \Rightarrow k'(x) = (\ln 2)2^x$

From greatest to smallest rate of growth:
$g(x), k(x), h(x), f(x)$

77. $C(t) = P(1.05)^t$

(a) $C(10) = 24.95(1.05)^{10}$

$\approx \$40.64$

(b) $\dfrac{dC}{dt} = P(\ln 1.05)(1.05)^t$

When $t = 1$: $\dfrac{dC}{dt} \approx 0.051P$

When $t = 8$: $\dfrac{dC}{dt} \approx 0.072P$

(c) $\dfrac{dC}{dt} = (\ln 1.05)[P(1.05)^t]$

$= (\ln 1.05)C(t)$

The constant of proportionality is $\ln 1.05$.

79. $P = \$1000,\ r = 3\frac{1}{2}\% = 0.035,\ t = 10$

$$A = 1000\left(1 + \frac{0.035}{n}\right)^{10n}$$

$$A = 1000e^{(0.035)(10)} = 1419.07$$

n	1	2	4	12	365	Continuous
A	1410.60	1414.78	1416.91	1418.34	1419.04	1419.07

81. $P = \$1000,\ r = 5\% = 0.05,\ t = 30$

$$A = 1000\left(1 + \frac{0.05}{n}\right)^{30n}$$

$$A = 1000e^{(0.05)30} = 4481.69$$

n	1	2	4	12	365	Continuous
A	4321.94	4399.79	4440.21	4467.74	4481.23	4481.69

83. $100{,}000 = Pe^{0.05t} \Rightarrow P = 100{,}000e^{-0.05t}$

t	1	10	20	30	40	50
P	95,122.94	60,653.07	36,787.94	22,313.02	13,583.53	8208.50

85. $100{,}000 = P\left(1 + \frac{0.05}{12}\right)^{12t} \Rightarrow P = 100{,}000\left(1 + \frac{0.05}{12}\right)^{-12t}$

t	1	10	20	30	40	50
P	95,132.82	60,716.10	36,864.45	22,382.66	13,589.88	8251.24

87. (a) $A = 20{,}000\left(1 + \frac{0.06}{365}\right)^{(365)(8)} \approx \$32{,}320.21$

(b) $A = \$30{,}000$

(c) $A = 8000\left(1 + \frac{0.06}{365}\right)^{(365)(8)} + 20{,}000\left(1 + \frac{0.06}{365}\right)^{(365)(4)}$

$\approx \$12{,}928.09 + 25{,}424.48 = \$38{,}352.57$

(d) $A = 9000\left[\left(1 + \frac{0.06}{365}\right)^{(365)(8)} + \left(1 + \frac{0.06}{365}\right)^{(365)(4)} + 1\right]$

$\approx \$34{,}985.11$

Take option (c).

89. (a) $\lim_{t\to\infty} 6.7e^{(-48.1)/t} = 6.7e^0 = 6.7$ million ft^3

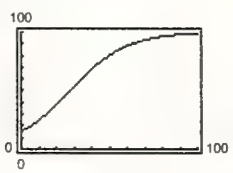
(b) $V' = \frac{322.27}{t^2}e^{-(48.1)/t}$

$V'(20) \approx 0.073$ million ft^3/yr

$V'(60) \approx 0.040$ million ft^3/yr

91. $y = \dfrac{300}{3 + 17e^{-0.0625x}}$

(a)



(b) If $x = 2$ (2000 egg masses), $y \approx 16.67 \approx 16.7\%$.

(c) If $y = 66.67\%$, then $x \approx 38.8$ or 38,800 egg masses.

(d) $y = 300(3 + 17e^{-0.0625x})^{-1}$

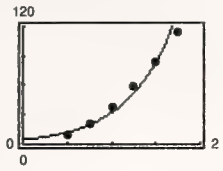
$$y' = \frac{318.75e^{-0.0625x}}{(3 + 17e^{-0.0625x})^2}$$

$$y'' = \frac{19.921875e^{-0.0625x}(17e^{-0.0625x} - 3)}{(3 + 17e^{-0.0625x})^3}$$

$17e^{-0.0625x} - 3 = 0 \Rightarrow x \approx 27.8$ or 27,800 egg masses.

93. (a) $B = 4.7539(6.7744)^d = 4.7539e^{1.9132d}$

(b)



(c) $B'(d) = 9.0952e^{1.9132d}$

$B'(0.8) \approx 42.03$ tons/inch

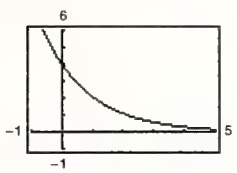
$B'(1.5) \approx 160.38$ tons/inch

95. (a) $\int_0^4 f(t)\,dt \approx 5.67$

$\int_0^4 g(t)\,dt \approx 5.67$

$\int_0^4 h(t)\,dt \approx 5.67$

(b)



(c) The functions appear to be equal: $f(t) = g(t) = h(t)$

Analytically,

$$f(t) = 4\left(\frac{3}{8}\right)^{2t/3} = 4\left[\left(\frac{3}{8}\right)^{2/3}\right]^t = 4\left(\frac{9^{1/3}}{4}\right)^t = g(t)$$

$$h(t) = 4e^{-0.653886t} = 4[e^{-0.653886}]^t = 4(0.52002)^t$$

$$g(t) = 4\left(\frac{9^{1/3}}{4}\right)^t = 4(0.52002)^t$$

No. The definite integrals over a given interval may be equal when the functions are not equal.

97. $P = \int_0^{10} 2000e^{-0.06t}\,dt$

$$= \left[\frac{2000}{-0.06}e^{-0.06t}\right]_0^{10}$$

$\approx \$15{,}039.61$

99.

t	0	1	2	3	4
y	1200	720	432	259.20	155.52

$y = C(k^t)$

When $t = 0, y = 1200 \Rightarrow C = 1200.$

$y = 1200(k^t)$

$$\frac{720}{1200} = 0.6,\ \frac{432}{720} = 0.6,\ \frac{259.20}{432} = 0.6,\ \frac{155.52}{259.20} = 0.6$$

Let $k = 0.6$.

$y = 1200(0.6)^t$

101. False. e is an irrational number.

103. True.

$$f(g(x)) = 2 + e^{\ln(x-2)}$$
$$= 2 + x - 2 = x$$
$$g(f(x)) = \ln(2 + e^x - 2)$$
$$= \ln e^x = x$$

105. True.

$$\frac{d}{dx}[e^x] = e^x \text{ and } \frac{d}{dx}[e^{-x}] = -e^{-x}$$

$e^x = e^{-x}$ when $x = 0$.

$(e^0)(-e^{-0}) = -1$

107. $\dfrac{dy}{dt} = \dfrac{8}{25}y\left(\dfrac{5}{4} - y\right), y(0) = 1$

$$\frac{dy}{y[(5/4) - y]} = \frac{8}{25}dt \Rightarrow \frac{4}{5}\int\left(\frac{1}{y} + \frac{1}{(5/4) - y}\right)dy = \int\frac{8}{25}dt \Rightarrow$$

$$\ln y - \ln\left(\frac{5}{4} - y\right) = \frac{2}{5}t + C$$

$$\ln\left(\frac{y}{(5/4) - y}\right) = \frac{2}{5}t + C$$

$$\frac{y}{(5/4) - y} = e^{(2/5)t + C} = C_1e^{(2/5)t}$$

$$y(0) = 1 \Rightarrow C_1 = 4 \Rightarrow 4e^{(2/5)t} = \frac{y}{(5/4) - y}$$

$$\Rightarrow 4e^{(2/5)t}\left(\frac{5}{4} - y\right) = y \Rightarrow 5e^{(2/5)t} = 4e^{(2/5)t}y + y = (4e^{(2/5)t} + 1)y$$

$$\Rightarrow y = \frac{5e^{(2/5)t}}{4e^{(2/5)t} + 1} = \frac{5}{4 + e^{-0.4t}} = \frac{1.25}{1 + 0.25e^{-0.4t}}$$

Section 5.6 Differential Equations: Growth and Decay

1. $\dfrac{dy}{dx} = x + 2$

$$y = \int(x + 2)dx = \frac{x^2}{2} + 2x + C$$

3. $\dfrac{dy}{dx} = y + 2$

$$\frac{dy}{y + 2} = dx$$

$$\int\frac{1}{y + 2}dy = \int dx$$

$$\ln|y + 2| = x + C_1$$

$$y + 2 = e^{x + C_1} = Ce^x$$

$$y = Ce^x - 2$$

5. $y' = \dfrac{5x}{y}$

$$yy' = 5x$$

$$\int yy'\,dx = \int 5x\,dx$$

$$\int y\,dy = \int 5x\,dx$$

$$\frac{1}{2}y^2 = \frac{5}{2}x^2 + C_1$$

$$y^2 - 5x^2 = C$$

7. $y' = \sqrt{x}\,y$

$$\frac{y'}{y} = \sqrt{x}$$

$$\int\frac{y'}{y}dx = \int\sqrt{x}\,dx$$

$$\int\frac{dy}{y} = \int\sqrt{x}\,dx$$

$$\ln y = \frac{2}{3}x^{3/2} + C_1$$

$$y = e^{(2/3)x^{3/2} + C_1}$$

$$= e^{C_1}e^{(2/3)x^{3/2}}$$

$$= Ce^{(2/3)x^{3/2}}$$

9. $(1 + x^2)y' - 2xy = 0$

$$y' = \frac{2xy}{1 + x^2}$$

$$\frac{y'}{y} = \frac{2x}{1 + x^2}$$

$$\int \frac{y'}{y}\,dx = \int \frac{2x}{1 + x^2}\,dx$$

$$\int \frac{dy}{y} = \int \frac{2x}{1 + x^2}\,dx$$

$$\ln y = \ln(1 + x^2) + C_1$$

$$\ln y = \ln(1 + x^2) + \ln C$$

$$\ln y = \ln C(1 + x^2)$$

$$y = C(1 + x^2)$$

11. $\dfrac{dQ}{dt} = \dfrac{k}{t^2}$

$$\int \frac{dQ}{dt}\,dt = \int \frac{k}{t^2}\,dt$$

$$\int dQ = -\frac{k}{t} + C$$

$$Q = -\frac{k}{t} + C$$

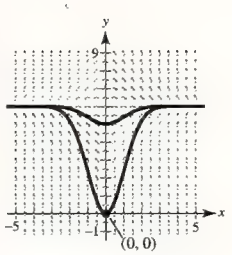
13. $\dfrac{dN}{ds} = k(250 - s)$

$$\int \frac{dN}{ds}\,ds = \int k(250 - s)\,ds$$

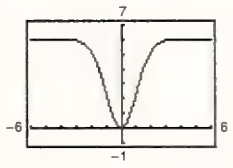
$$\int dN = -\frac{k}{2}(250 - s)^2 + C$$

$$N = -\frac{k}{2}(250 - s)^2 + C$$

15. (a)



(b) $\dfrac{dy}{dx} = x(6 - y)$, $(0, 0)$



$$\frac{dy}{y - 6} = -x$$

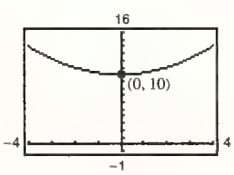
$$\ln|y - 6| = \frac{-x^2}{2} + C$$

$$y - 6 = e^{-x^2/2 + C} = C_1 e^{-x^2/2}$$

$$y = 6 + C_1 e^{-x^2/2}$$

$(0, 0)$: $0 = 6 + C_1 \Rightarrow C_1 = -6 \Rightarrow y = 6 - 6e^{-x^2/2}$

17. $\dfrac{dy}{dt} = \dfrac{1}{2}t$, $(0, 10)$



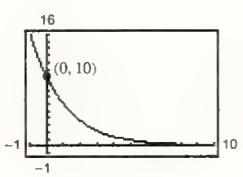
$$\int dy = \int \frac{1}{2}t\,dt$$

$$y = \frac{1}{4}t^2 + C$$

$$10 = \frac{1}{4}(0)^2 + C \Rightarrow C = 10$$

$$y = \frac{1}{4}t^2 + 10$$

19. $\dfrac{dy}{dt} = -\dfrac{1}{2}y$, $(0, 10)$



$$\int \frac{dy}{y} = \int -\frac{1}{2}\,dt$$

$$\ln y = -\frac{1}{2}t + C_1$$

$$y = e^{-(t/2) + C_1} = e^{C_1} e^{-t/2} = Ce^{-t/2}$$

$$10 = Ce^0 \Rightarrow C = 10$$

$$y = 10e^{-t/2}$$

21. $\dfrac{dy}{dx} = ky$

$y = Ce^{kx}$ (Theorem 5.16)

$(0, 4)$: $4 = Ce^0 = C$

$(3, 10)$: $10 = 4e^{3k} \Rightarrow k = \dfrac{1}{3}\ln\left(\dfrac{5}{2}\right)$

When $x = 6$, $y = 4e^{1/3\ln(5/2)(6)} = 4e^{\ln(5/2)^2}$

$$= 4\left(\frac{5}{2}\right)^2 = 25$$

23. $\dfrac{dV}{dt} = kV$

$V = Ce^{kt}$ (Theorem 5.16)

$(0, 20{,}000)$: $C = 20{,}000$

$(4, 12{,}500)$: $12{,}500 = 20{,}000e^{4k} \Rightarrow k = \dfrac{1}{4}\ln\left(\dfrac{5}{8}\right)$

When $t = 6$, $V = 20{,}000e^{1/4\ln(5/8)(6)} = 20{,}000e^{\ln(5/8)^{3/2}}$

$$= 20{,}000\left(\frac{5}{8}\right)^{3/2} \approx 9882.118$$

25. $y = Ce^{kt}$, $\left(0, \frac{1}{2}\right)$, $(5, 5)$

$C = \frac{1}{2}$

$y = \frac{1}{2}e^{kt}$

$5 = \frac{1}{2}e^{5k}$

$k = \frac{\ln 10}{5}$

$y = \frac{1}{2}e^{(\ln 10/5)t} = \frac{1}{2}(10^{t/5})$ or $y \approx \frac{1}{2}e^{0.4605t}$

27. $y = Ce^{kt}$, $(1, 1)$, $(5, 5)$

$1 = Ce^{k}$

$5 = Ce^{5k}$

$5Ce^{k} = Ce^{5k}$

$5e^{k} = e^{5k}$

$5 = e^{4k}$

$k = \frac{\ln 5}{4} \approx 0.4024$

$y = Ce^{0.4024t}$

$1 = Ce^{0.4024}$

$C \approx 0.6687 \quad (C = 5^{-1/4})$

$y \approx 0.6687e^{0.4024t}$

29. A differential equation in x and y is an equation that involves x, y and derivatives of y.

31. $\frac{dy}{dx} = \frac{1}{2}xy$

$\frac{dy}{dx} > 0$ when $xy > 0$. Quadrants I and III.

33. Since the initial quantity is 10 grams, $y = 10e^{[\ln(1/2)/1620]t}$. When $t = 1000$, $y = 10e^{[\ln(1/2)/1620](1000)} \approx 6.52$ grams. When $t = 10{,}000$, $y = 10e^{[\ln(1/2)/1620](10{,}000)} \approx 0.14$ gram.

35. Since $y = Ce^{[\ln(1/2)/1620]t}$, we have $0.5 = Ce^{[\ln(1/2)/1620](10{,}000)} \Rightarrow C \approx 36.07$.

Initial quantity: 36.07 grams.

When $t = 1000$, we have $y = Ce^{[\ln(1/2)/1620](1000)} \approx 23.51$ grams.

37. Since the initial quantity is 5 grams, we have $y = 5.0e^{[\ln(1/2)/5730]t}$.

When $t = 1000$, $y \approx 4.43$ g.

When $t = 10{,}000$, $y \approx 1.49$ g.

39. Since $y = Ce^{[\ln(1/2)/24{,}360]t}$, we have $2.1 = Ce^{[\ln(1/2)/24{,}360](1000)} \Rightarrow C \approx 2.16$. Thus, the initial quantity is 2.16 grams. When $t = 10{,}000$, $y = 2.16e^{[\ln(1/2)/24{,}360](10{,}000)} \approx 1.63$ grams.

41. Since $\frac{dy}{dx} = ky$, $y = Ce^{kt}$ or $y = y_0e^{kt}$.

$\frac{1}{2}y_0 = y_0e^{1620k}$

$k = \frac{-\ln 2}{1620}$

$y = y_0e^{-(\ln 2)t/1620}$.

When $t = 100$, $y = y_0e^{-(\ln 2)/16.2} \approx y_0(0.9581)$.
Therefore, 95.81% of the present amount still exists.

43. Since $A = 1000e^{0.06t}$, the time to double is given by $2000 = 1000e^{0.06t}$ and we have

$2 = e^{0.06t}$

$\ln 2 = 0.06t$

$t = \frac{\ln 2}{0.06} \approx 11.55$ years.

Amount after 10 years: $A = 1000e^{(0.06)(10)} \approx \$1822.12$

45. Since $A = 750e^{rt}$ and $A = 1500$ when $t = 7.75$, we have the following.

$$1500 = 750e^{7.75r}$$

$$r = \frac{\ln 2}{7.75} \approx 0.0894 = 8.94\%$$

Amount after 10 years: $A = 750e^{0.0894(10)} \approx \$1833.67$

47. Since $A = 500e^{rt}$ and $A = 1292.85$ when $t = 10$, we have the following.

$$1292.85 = 500e^{10r}$$

$$r = \frac{\ln(1292.85/500)}{10} \approx 0.0950 = 9.50\%$$

The time to double is given by

$$1000 = 500e^{0.0950t}$$

$$t = \frac{\ln 2}{0.095} \approx 7.30 \text{ years.}$$

49. $500{,}000 = P\left(1 + \dfrac{0.075}{12}\right)^{(12)(20)}$

$$P = 500{,}000\left(1 + \frac{0.075}{12}\right)^{-240}$$

$$\approx \$112{,}087.09$$

51. $500{,}000 = P\left(1 + \dfrac{0.08}{12}\right)^{(12)(35)}$

$$P = 500{,}000\left(1 + \frac{0.08}{12}\right)^{-420}$$

$$= \$30{,}688.87$$

53. (a) $2000 = 1000(1 + 0.07)^t$

$$2 = 1.07^t$$

$$\ln 2 = t \ln 1.07$$

$$t = \frac{\ln 2}{\ln 1.07} \approx 10.24 \text{ years}$$

(b) $2000 = 1000\left(1 + \dfrac{0.07}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.007}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.07}{12}\right)$$

$$t = \frac{\ln 2}{12 \ln(1 + (0.07/12))} \approx 9.93 \text{ years}$$

(c) $2000 = 1000\left(1 + \dfrac{0.07}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.07}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.07}{365}\right)$$

$$t = \frac{\ln 2}{365 \ln(1 + (0.07/365))} \approx 9.90 \text{ years}$$

(d) $2000 = 1000e^{(0.07)t}$

$$2 = e^{0.07t}$$

$$\ln 2 = 0.07t$$

$$t = \frac{\ln 2}{0.07} \approx 9.90 \text{ years}$$

55. (a) $2000 = 1000(1 + 0.085)^t$

$$2 = 1.085^t$$

$$\ln 2 = t \ln 1.085$$

$$t = \frac{\ln 2}{\ln 1.085} \approx 8.50 \text{ years}$$

(b) $2000 = 1000\left(1 + \dfrac{0.085}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.085}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.085}{12}\right)$$

$$t = \frac{1}{12} \frac{\ln 2}{\ln\left(1 + \dfrac{0.085}{12}\right)} \approx 8.18 \text{ years}$$

(c) $2000 = 1000\left(1 + \dfrac{0.085}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.085}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.085}{365}\right)$$

$$t = \frac{1}{365} \frac{\ln 2}{\ln\left(1 + \dfrac{0.085}{365}\right)} \approx 8.16 \text{ years}$$

(d) $2000 = 1000e^{0.085t}$

$$2 = e^{0.085t}$$

$$\ln 2 = 0.085t$$

$$t = \frac{\ln 2}{0.085} \approx 8.15 \text{ years}$$

57. $P = Ce^{kt} = Ce^{-0.009t}$

$P(-1) = 8.2 = Ce^{-0.009(-1)} \Rightarrow C = 8.1265$

$P = 8.1265e^{-0.009t}$

$P(10) \approx 7.43$ or 7,430,000 people in 2010

59. $P = Ce^{kt} = Ce^{0.036t}$

$P(-1) = 4.6 = Ce^{0.036(-1)} \Rightarrow C = 4.7686$

$P = 4.7686e^{0.036t}$

$P(10) \approx 6.83$ or 6,830,000 people in 2010

61. If $k < 0$, the population decreases.

If $k > 0$, the population increases.

63. $P = Ce^{kx}$, $(0, 760)$, $(1000, 672.71)$

$C = 760$

$672.71 = 760e^{1000x}$

$$x = \frac{\ln(672.71/760)}{1000} \approx -0.000122$$

$P \approx 760e^{-0.000122x}$

When $x = 3000$, $P \approx 527.06$ mm Hg.

65. (a) $19 = 30(1 - e^{20k})$

$30e^{20k} = 11$

$$k = \frac{\ln(11/30)}{20} \approx -0.0502$$

$N \approx 30(1 - e^{-0.0502t})$

(b) $25 = 30(1 - e^{-0.0502t})$

$$e^{-0.0502t} = \frac{1}{6}$$

$$t = \frac{-\ln 6}{-0.0502} \approx 36 \text{ days}$$

67. $S = Ce^{k/t}$

(a) $S = 5$ when $t = 1$

$5 = Ce^{k}$

$\lim_{t \to \infty} Ce^{k/t} = C = 30$

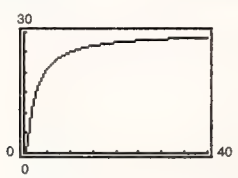
$5 = 30e^{k}$

$k = \ln \frac{1}{6} \approx -1.7918$

$S \approx 30e^{-1.7918/t}$

(b) When $t = 5$, $S \approx 20.9646$ which is 20,965 units.

(c)



69. $A(t) = V(t)e^{-0.10t} = 100{,}000e^{0.8\sqrt{t}}\, e^{-0.10t} = 100{,}000e^{0.8\sqrt{t} - 0.10t}$

$$\frac{dA}{dt} = 100{,}000\left(\frac{0.4}{\sqrt{t}} - 0.10\right)e^{0.8\sqrt{t} - 0.10t} = 0 \text{ when } 16.$$

The timber should be harvested in the year 2014, (1998 + 16). **Note:** You could also use a graphing utility to graph $A(t)$ and find the maximum of $A(t)$. Use the viewing rectangle $0 \le x \le 30$ and $0 \le y \le 600{,}000$.

71. $\beta(I) = 10 \log_{10} \frac{I}{I_0}$, $I_0 = 10^{-16}$

(a) $\beta(10^{-14}) = 10 \log_{10} \frac{10^{-14}}{10^{-16}} = 20$ decibels

(b) $\beta(10^{-9}) = 10 \log_{10} \frac{10^{-9}}{10^{-16}} = 70$ decibels

(c) $\beta(10^{-6.5}) = 10 \log_{10} \frac{10^{-6.5}}{10^{-16}} = 95$ decibels

(d) $\beta(10^{-4}) = 10 \log_{10} \frac{10^{-4}}{10^{-16}} = 120$ decibels

73. $R = \frac{\ln I - 0}{\ln 10}$, $I = e^{R \ln 10} = 10^R$

(a) $8.3 = \frac{\ln I - 0}{\ln 10}$

$I = 10^{8.3} \approx 199{,}526{,}231.5$

(b) $2R = \frac{\ln I - 0}{\ln 10}$

$I = e^{2R \ln 10} = e^{2R \ln 10} = (e^{R \ln 10})^2 = (10^R)^2$

Increases by a factor of $e^{2R \ln 10}$ or 10^R.

(c) $\frac{dR}{dI} = \frac{1}{I \ln 10}$

75. False. If $y = Ce^{kt}$, $y' = Cke^{kt} \neq$ constant.

77. True

Section 5.7 Differential Equations: Separation of Variables

1. Differential equation: $y' = 4y$

Solution: $y = Ce^{4x}$

Check: $y' = 4Ce^{4x} = 4y$

3. Differential equation: $y'' + y = 0$

Solution: $y = C_1 \cos x + C_2 \sin x$

Check: $y' = -C_1 \sin x + C_2 \cos x$

$y'' = -C_1 \cos x - C_2 \sin x$

$y'' + y = -C_1 \cos x - C_2 \sin x + C_1 \cos x + C_2 \sin x = 0$

5. $y = -\cos x \ln|\sec x + \tan x|$

$$y' = (-\cos x)\frac{1}{\sec x + \tan x}(\sec x \cdot \tan x + \sec^2 x) + \sin x \ln|\sec x + \tan x|$$

$$= \frac{(-\cos x)}{\sec x + \tan x}(\sec x)(\tan x + \sec x) + \sin x \ln|\sec x + \tan x|$$

$$= -1 + \sin x \ln|\sec x + \tan x|$$

$$y'' = (\sin x)\frac{1}{\sec x + \tan x}(\sec x \cdot \tan x + \sec^2 x) + \cos x \ln|\sec x + \tan x|$$

$$= (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x|$$

Substituting,

$$y'' + y = (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x| - \cos x \ln|\sec x + \tan x|$$

$$= \tan x.$$

In Exercises 7–11, the differential equation is $y^{(4)} - 16y = 0$.

7. $y = 3 \cos x$

$y^{(4)} = 3 \cos x$

$y^{(4)} - 16y = -45 \cos x \neq 0,$

No.

9. $y = e^{-2x}$

$y^{(4)} = 16e^{-2x}$

$y^{(4)} - 16y = 16e^{-2x} - 16e^{-2x} = 0,$

Yes.

11. $y = C_1e^{2x} + C_2e^{-2x} + C_3 \sin 2x + C_4 \cos 2x$

$y^{(4)} = 16C_1e^{2x} + 16C_2e^{-2x} + 16C_3 \sin 2x + 16C_4 \cos 2x$

$y^{(4)} - 16y = 0,$

Yes.

In 13–17, the differential equation is $xy' - 2y = x^3e^x$.

13. $y = x^2$, $y' = 2x$

$xy' - 2y = x(2x) - 2(x^2) = 0 \neq x^3e^x$

No.

15. $y = x^2(2 + e^x)$, $y' = x^2(e^x) + 2x(2 + e^x)$

$xy' - 2y = x[x^2e^x + 2xe^x + 4x] - 2[x^2e^x + 2x^2] = x^3e^x$,

Yes.

17. $y = \ln x, y' = \dfrac{1}{x}$

$xy' - 2y = x\left(\dfrac{1}{x}\right) - 2\ln x \neq x^3e^x$, No.

19. $y = Ce^{kx}$

$\dfrac{dy}{dx} = Cke^{kx}$

Since $dy/dx = 0.07y$, we have $Cke^{kx} = 0.07Ce^{kx}$.
Thus, $k = 0.07$.

21. $y^2 = Cx^3$ passes through (4, 4)

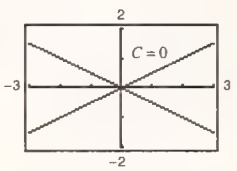
$16 = C(64) \Rightarrow C = \frac{1}{4}$

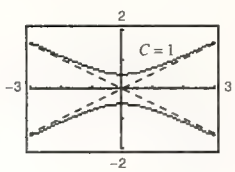
Particular solution: $y^2 = \frac{1}{4}x^3$ or $4y^2 = x^3$

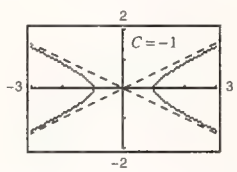
23. Differential equation: $4yy' - x = 0$

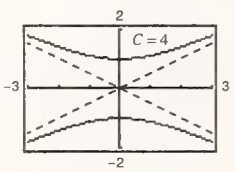
General solution: $4y^2 - x^2 = C$

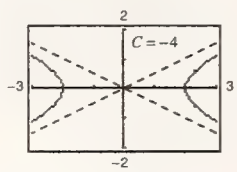
Particular solutions: $C = 0$, Two intersecting lines
$C = \pm 1$, $C = \pm 4$, Hyperbolas











25. Differential equation: $y' + 2y = 0$

General Solution: $y = Ce^{-2x}$

$y' + 2y = C(-2)e^{-2x} + 2(Ce^{-2x}) = 0$

Initial condition: $y(0) = 3, 3 = Ce^0 = C$

Particular solution: $y = 3e^{-2x}$

27. Differential equation: $y'' + 9y = 0$

General solution: $y = C_1 \sin 3x + C_2 \cos 3x$

$y' = 3C_1 \cos 3x - 3C_2 \sin 3x$,

$y'' = -9C_1 \sin 3x - 9C_2 \cos 3x$

$y'' + 9y = (-9C_1 \sin 3x - 9C_2 \cos 3x) +$
$9(C_1 \sin 3x + C_2 \cos 3x) = 0$

Initial conditions: $y\left(\dfrac{\pi}{6}\right) = 2$, $y'\left(\dfrac{\pi}{6}\right) = 1$

$2 = C_1 \sin\left(\dfrac{\pi}{2}\right) + C_2 \cos\left(\dfrac{\pi}{2}\right) \Rightarrow C_1 = 2$

$y' = 3C_1 \cos 3x - 3C_2 \sin 3x$

$1 = 3C_1 \cos\left(\dfrac{\pi}{2}\right) - 3C_2 \sin\left(\dfrac{\pi}{2}\right)$

$= -3C_2 \Rightarrow C_2 = -\dfrac{1}{3}$

Particular solution: $y = 2 \sin 3x - \dfrac{1}{3}\cos 3x$

29. Differential equation: $x^2y'' - 3xy' + 3y = 0$

General solution: $y = C_1x + C_2x^3$

$y' = C_1 + 3C_2x^2,\ y'' = 6C_2x$

$x^2y'' - 3x\,y' + 3y = x^2(6C_2x) - 3x(C_1 + 3C_2x^2) +$

$3(C_1x + C_2x^3) = 0$

Initial conditions: $y(2) = 0,\ y'(2) = 4$

$0 = 2C_1 + 8C_2$

$y' = C_1 + 3C_2x^2$

$4 = C_1 + 12C_2$

$\left.\begin{aligned} C_1 + 4C_2 &= 0 \\ C_1 + 12C_2 &= 4 \end{aligned}\right\}\ C_2 = \frac{1}{2},\ C_1 = -2$

Particular solution: $y = -2x + \frac{1}{2}x^3$

31. $\frac{dy}{dx} = 3x^2$

$y = \int 3x^2\,dx = x^3 + C$

33. $\frac{dy}{dx} = \frac{x}{1 + x^2}$

$y = \int \frac{x}{1 + x^2}\,dx = \frac{1}{2}\ln(1 + x^2) + C$

$(u = 1 + x^2,\ du = 2x\,dx)$

35. $\frac{dy}{dx} = \frac{x - 2}{x} = 1 - \frac{2}{x}$

$y = \int\left[1 - \frac{2}{x}\right]dx$

$= x - 2\ln|x| + C = x - \ln x^2 + C$

37. $\frac{dy}{dx} = \sin 2x$

$y = \int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C$

$(u = 2x,\ du = 2dx)$

39. $\frac{dy}{dx} = x\sqrt{x - 3}$ Let $u = \sqrt{x - 3}$, then $x = u^2 + 3$ and $dx = 2u\,du$.

$y = \int x\sqrt{x - 3}\,dx = \int (u^2 + 3)(u)(2u)du$

$= 2\int (u^4 + 3u^2)\,du = 2\left(\frac{u^5}{5} + u^3\right) + C = \frac{2}{5}(x - 3)^{5/2} + 2(x - 3)^{3/2} + C$

41. $\frac{dy}{dx} = xe^{x^2}$

$y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$

$(u = x^2,\ du = 2x\,dx)$

43. $\frac{dy}{dx} = \frac{x}{y}$

$\int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + C_1$

$y^2 - x^2 = C$

45. $\frac{dr}{ds} = 0.05r$

$\int \frac{dr}{r} = \int 0.05\,ds$

$\ln|r| = 0.05s + C_1$

$r = e^{0.05s + C_1} = Ce^{0.05s}$

47. $(2 + x)y' = 3y$

$\int \frac{dy}{y} = \int \frac{3}{2 + x}\,dx$

$\ln y = 3\ln(2 + x) + \ln C = \ln C(2 + x)^3$

$y = C(x + 2)^3$

49. $yy' = \sin x$

$$\int y\,dy = \int \sin x\,dx$$

$$\frac{y^2}{2} = -\cos x + C_1$$

$$y^2 = -2\cos x + C$$

51. $\sqrt{1-4x^2}\,\dfrac{dy}{dx} = x$

$$dy = \frac{x}{\sqrt{1-4x^2}}\,dx$$

$$\int dy = \int \frac{x}{\sqrt{1-4x^2}}\,dx$$

$$= -\frac{1}{8}\int (1-4x^2)^{-1/2}(-8x\,dx)$$

$$y = -\frac{1}{4}(1-4x^2)^{1/2} + C$$

53. $y\ln x - xy' = 0$

$$\int \frac{dy}{y} = \int \frac{\ln x}{x}\,dx \quad \left(u = \ln x,\ du = \frac{dx}{x}\right)$$

$$\ln y = \frac{1}{2}(\ln x)^2 + C_1$$

$$y = e^{(1/2)(\ln x)^2 + C_1} = Ce^{(\ln x)^2/2}$$

55. $yy' - e^x = 0$

$$\int y\,dy = \int e^x\,dx$$

$$\frac{y^2}{2} = e^x + C_1$$

$$y^2 = 2e^x + C$$

Initial condition: $y(0) = 4,\ 16 = 2 + C,\ C = 14$

Particular solution: $y^2 = 2e^x + 14$

57. $y(x+1) + y' = 0$

$$\int \frac{dy}{y} = -\int (x+1)\,dx$$

$$\ln y = -\frac{(x+1)^2}{2} + C_1$$

$$y = Ce^{-(x+1)^2/2}$$

Initial condition: $y(-2) = 1,\ 1 = Ce^{-1/2},\ C = e^{1/2}$

Particular solution: $y = e^{[1-(x+1)^2]/2} = e^{-(x^2+2x)/2}$

59. $y(1+x^2)\,\dfrac{dy}{dx} = x(1+y^2)$

$$\frac{y}{1+y^2}\,dy = \frac{x}{1+x^2}\,dx$$

$$\frac{1}{2}\ln(1+y^2) = \frac{1}{2}\ln(1+x^2) + C_1$$

$$\ln(1+y^2) = \ln(1+x^2) + \ln C = \ln[C(1+x^2)]$$

$$1 + y^2 = C(1+x^2)$$

$y(0) = \sqrt{3}$: $1 + 3 = C \Rightarrow C = 4$

$$1 + y^2 = 4(1+x^2)$$

$$y^2 = 3 + 4x^2$$

61. $\dfrac{du}{dv} = uv\sin v^2$

$$\int \frac{du}{u} = \int v\sin v^2\,dv$$

$$\ln u = -\frac{1}{2}\cos v^2 + C_1$$

$$u = Ce^{-(\cos v^2)/2}$$

Initial condition: $u(0) = 1,\ C = \dfrac{1}{e^{-1/2}} = e^{1/2}$

Particular solution: $u = e^{(1-\cos v^2)/2}$

63. $dP - kP\,dt = 0$

$$\int \frac{dP}{P} = k\int dt$$

$$\ln P = kt + C_1$$

$$P = Ce^{kt}$$

Initial condition: $P(0) = P_0,\ P_0 = Ce^0 = C$

Particular solution: $P = P_0e^{kt}$

65. $\dfrac{dy}{dx} = \dfrac{-9x}{16y}$

$$\int 16y\,dy = -\int 9x\,dx$$

$$8y^2 = \frac{-9}{2}x^2 + C$$

Initial condition: $y(1) = 1,\ 8 = -\dfrac{9}{2} + C,\ C = \dfrac{25}{2}$

Particular solution: $8y^2 = \dfrac{-9}{2}x^2 + \dfrac{25}{2}$,

$$16y^2 + 9x^2 = 25$$

67. $m = \dfrac{dy}{dx} = \dfrac{0 - y}{(x + 2) - x} = -\dfrac{y}{2}$

$$\int \frac{dy}{y} = \int -\frac{1}{2}\,dx$$

$$\ln y = -\frac{1}{2}x + C_1$$

$$y = Ce^{-x/2}$$

69. $f(x, y) = x^3 - 4xy^2 + y^3$

$f(tx, ty) = t^3x^3 - 4t\,xt^2y^2 + t^3y^3$

$= t^3(x^3 - 4xy^2 + y^3)$

Homogeneous of degree 3

71. $f(x, y) = \dfrac{x^2y^2}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{t^4x^2y^2}{\sqrt{t^2x^2 + t^2y^2}} = t^3\frac{x^2y^2}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 3

73. $f(x, y) = 2 \ln xy$

$f(tx, ty) = 2 \ln tx\,ty$

$= 2 \ln t^2xy = 2(\ln t^2 + \ln xy)$

Not homogeneous

75. $f(x, y) = 2 \ln \dfrac{x}{y}$

$$f(tx, ty) = 2 \ln \frac{tx}{ty} = 2 \ln \frac{x}{y}$$

Homogeneous degree 0

77. $y' = \dfrac{x + y}{2x},\ y = vx$

$$v + x\frac{dv}{dx} = \frac{x + vx}{2x}$$

$$x\frac{dv}{dx} = \frac{1 + v}{2} - v$$

$$2\int \frac{dv}{1 - v} = \int \frac{dx}{x}$$

$$-\ln(1 - v)^2 = \ln|x| + \ln C = \ln|Cx|$$

$$\frac{1}{(1 - v^2)} = |Cx|$$

$$\frac{1}{[1 - (y/x)]^2} = |Cx|$$

$$\frac{x^2}{(x - y)^2} = |Cx|$$

$$|x| = C(x - y)^2$$

79. $y' = \dfrac{x - y}{x + y},\ y = vx$

$$v + x\frac{dv}{dx} = \frac{x - xv}{x + xv}$$

$$v\,dx + x\,dv = \frac{1 - v}{1 + v}dx$$

$$\int \frac{v + 1}{v^2 + 2v - 1}\,dv = -\int \frac{dx}{x}$$

$$\frac{1}{2}\ln|v^2 + 2v - 1| = -\ln|x| + \ln C_1 = \ln\left|\frac{C_1}{x}\right|$$

$$|v^2 + 2v - 1| = \frac{C}{x^2}$$

$$\left|\frac{y^2}{x^2} + 2\frac{y}{x} - 1\right| = \frac{C}{x^2}$$

$$|y^2 + 2xy - x^2| = C$$

81. $y' = \dfrac{xy}{x^2 - y^2}, y = vx$

$$v + x\frac{dv}{dx} = \frac{x^2 v}{x^2 - x^2 v^2}$$

$$v\,dx + x\,dv = \frac{v}{1 - v^2}dx$$

$$\int \frac{1 - v^2}{v^3}dv = \int \frac{dx}{x}$$

$$-\frac{1}{2v^2} - \ln|v| = \ln|x| + \ln C_1 = \ln|C_1 x|$$

$$\frac{-1}{2v^2} = \ln|C_1 x v|$$

$$\frac{-x^2}{2y^2} = \ln|C_1 y|$$

$$y = Ce^{-x^2/2y^2}$$

83. $x\,dy - (2xe^{-y/x} + y)\,dx = 0, y = vx$

$$x(v\,dx + x\,dv) - (2xe^{-v} + vx)\,dx = 0$$

$$\int e^v\,dv = \int \frac{2}{x}dx$$

$$e^v = \ln C_1 x^2$$

$$e^{y/x} = \ln C_1 + \ln x^2$$

$$e^{y/x} = C + \ln x^2$$

Initial condition: $y(1) = 0, 1 = C$

Particular solution: $e^{y/x} = 1 + \ln x^2$

85. $\left(x \sec\dfrac{y}{x} + y\right)dx - x\,dy = 0, y = vx$

$$(x \sec v + xv)dx - x(v\,dx + x\,dv) = 0$$

$$(\sec v + v)\,dx = v\,dx + x\,dv$$

$$\int \cos v\,dv = \int \frac{dx}{x}$$

$$\sin v = \ln x + \ln C_1$$

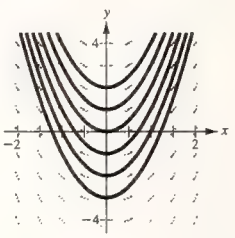
$$x = Ce^{\sin v}$$

$$y = Ce^{\sin(y/x)}$$

Initial condition: $y(1) = 0, 1 = Ce^0 = C$

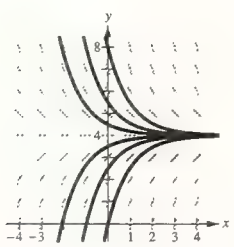
Particular solution: $x = e^{\sin(y/x)}$

87. $\dfrac{dy}{dx} = x$



$$y = \int x\,dx = \frac{1}{2}x^2 + C$$

89. $\dfrac{dy}{dx} = 4 - y$



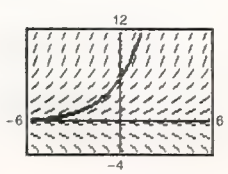
$$\int \frac{dy}{4 - y} = \int dx$$

$$\ln|4 - y| = -x + C_1$$

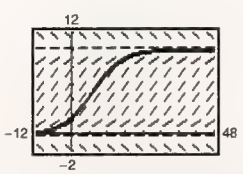
$$4 - y = e^{-x + C_1}$$

$$y = 4 + Ce^{-x}$$

91. $\dfrac{dy}{dx} = 0.5y, y(0) = 6$



93. $\dfrac{dy}{dx} = 0.02y(10 - y), y(0) = 2$



95. $\frac{dy}{dt} = ky, y = Ce^{kt}$

Initial conditions: $y(0) = y_0$

$$y(1620) = \frac{y_0}{2}$$

$$C = y_0$$

$$\frac{y_0}{2} = y_0 e^{1620k}$$

$$k = \frac{\ln(1/2)}{1620}$$

Particular solution: $y = y_0 e^{-t(\ln 2)/1620}$

When $t = 25$, $y \approx 0.989y_0$, $y = 98.9\%$ of y_0.

97. $\frac{dy}{dx} = k(y - 4)$

The direction field satisfies $(dy/dx) = 0$ along $y = 4$; but not along $y = 0$. Matches (a).

99. $\frac{dy}{dx} = ky(y - 4)$

The direction field satisfies $(dy/dx) = 0$ along $y = 0$ and $y = 4$. Matches (c).

101. $\frac{dw}{dt} = k(1200 - w)$

$$\int \frac{dw}{1200 - w} = \int k\,dt$$

$$\ln|1200 - w| = -kt + C_1$$

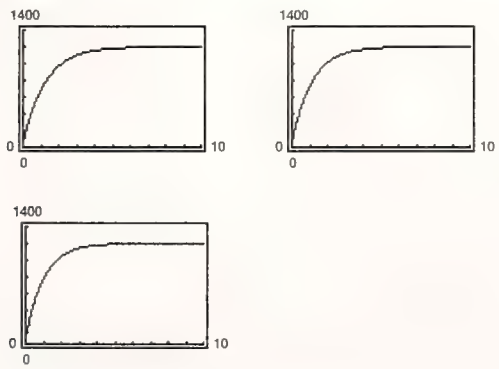
$$1200 - w = e^{-kt + C_1} = Ce^{-kt}$$

$$w = 1200 - Ce^{-kt}$$

$w(0) = 60 = 1200 - C \Rightarrow C = 1200 - 60 = 1140$

$w = 1200 - 1140e^{-kt}$

(a)



(b) $k = 0.8$: $t = 1.31$ years

$k = 0.9$: $t = 1.16$ years

$k = 1.0$: $t = 1.05$ years

(c) Maximum weight: 1200 pounds

$\lim_{t \to \infty} w = 1200$

103. (a) $\frac{dv}{dt} = k(W - v)$

$$\int \frac{dv}{W - v} = \int k\,dt$$

$$-\ln|W - v| = kt + C_1$$

$$v = W - Ce^{-kt}$$

Initial conditions:

$W = 20$, $v = 0$ when $t = 0$, and

$v = 5$ when $t = 1$.

$C = 20$, $k = -\ln(3/4)$

Particular solution:

$$v = 20(1 - e^{\ln(3/4)t}) = 20\left(1 - \left(\frac{3}{4}\right)^t\right)$$

or

$$v \approx 20(1 - e^{-0.2877t})$$

(b) $s = \int 20(1 - e^{-0.2877t})\,dt$

$$\approx 20[t + 3.4761e^{-0.2877t}] + C$$

Since $s(0) = 0$, $C \approx -69.5$ and we have $s \approx 20t + 69.5(e^{-0.2877t} - 1)$.

105. Given family (circles): $x^2 + y^2 = C$

$$2x + 2yy' = 0$$

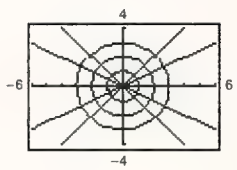
$$y' = -\frac{x}{y}$$

Orthogonal trajectory (lines): $y' = \frac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + \ln K$$

$$y = Kx$$



107. Given family (parabolas): $x^2 = Cy$

$$2x = Cy'$$

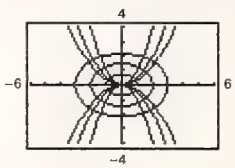
$$y' = \frac{2x}{C} = \frac{2x}{x^2/y} = \frac{2y}{x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{x}{2y}$

$$2\int y\,dy = -\int x\,dx$$

$$y^2 = -\frac{x^2}{2} + K_1$$

$$x^2 + 2y^2 = K$$



109. Given family: $y^2 = Cx^3$

$$2yy' = 3Cx^2$$

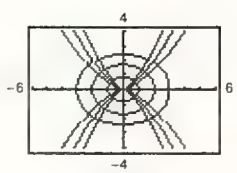
$$y' = \frac{3Cx^2}{2y} = \frac{3x^2}{2y}\left(\frac{y^2}{x^3}\right) = \frac{3y}{2x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{2x}{3y}$

$$3\int y\,dy = -2\int x\,dx$$

$$\frac{3y^2}{2} = -x^2 + K_1$$

$$3y^2 + 2x^2 = K$$



111. A general solution of order n has n arbitrary constants while in a particular solution initial conditions are given in order to solve for all these constants.

113. $M(x, y)dx + N(x, y)dy = 0$, where M and N are homogeneous functions of the same degree.

115. False. Consider Example 2. $y = x^3$ is a solution to $xy' - 3y = 0$, but $y = x^3 + 1$ is not a solution.

117. False

$$f(tx, ty) = t^2x^2 + t^2xy + 2$$

$$\neq t^2 f(x, y)$$

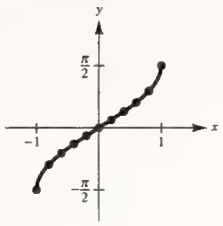
Section 5.8 Inverse Trigonometric Functions: Differentiation

1. $y = \arcsin x$

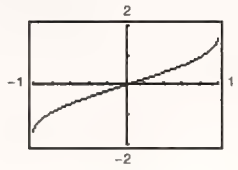
(a)

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y	-1.571	-0.927	-0.644	-0.412	-0.201	0	0.201	0.412	0.644	0.927	1.571

(b)



(c)



(d) Symmetric about origin:
$\arcsin(-x) = -\arcsin x$

Intercept: $(0, 0)$

3. False.

$$\arccos\frac{1}{2} = \frac{\pi}{3}$$

since the range is $[0, \pi]$.

5. $\arcsin\dfrac{1}{2} = \dfrac{\pi}{6}$

7. $\arccos\dfrac{1}{2} = \dfrac{\pi}{3}$

9. $\arctan\dfrac{\sqrt{3}}{3} = \dfrac{\pi}{6}$

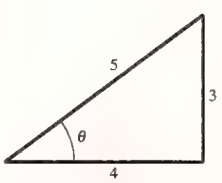
11. $\operatorname{arccsc}(-\sqrt{2}) = -\dfrac{\pi}{4}$

13. $\arccos(-0.8) \approx 2.50$

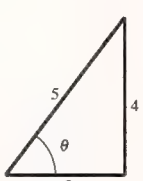
15. $\operatorname{arcsec}(1.269) = \arccos\left(\dfrac{1}{1.269}\right)$

≈ 0.66

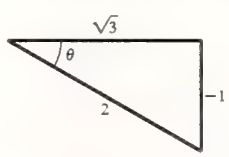
17. (a) $\sin\left(\arctan\frac{3}{4}\right) = \frac{3}{5}$



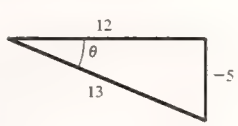
(b) $\sec\left(\arcsin\frac{4}{5}\right) = \frac{5}{3}$



19. (a) $\cot\left[\arcsin\left(-\dfrac{1}{2}\right)\right] = \cot\left(-\dfrac{\pi}{6}\right) = -\sqrt{3}$



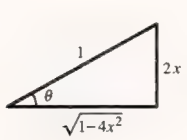
(b) $\csc\left[\arctan\left(-\dfrac{5}{12}\right)\right] = -\dfrac{13}{5}$



21. $y = \cos(\arcsin 2x)$

$\theta = \arcsin 2x$

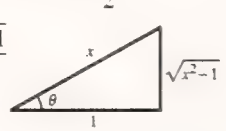
$y = \cos\theta = \sqrt{1 - 4x^2}$



23. $y = \sin(\operatorname{arcsec} x)$

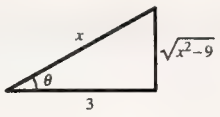
$\theta = \operatorname{arcsec} x,\ 0 \le \theta \le \pi,\ \theta \ne \dfrac{\pi}{2}$

$y = \sin\theta = \dfrac{\sqrt{x^2 - 1}}{|x|}$



The absolute value bars on x are necessary because of the restriction $0 \le \theta \le \pi$, $\theta \ne \pi/2$, and $\sin\theta$ for this domain must always be nonnegative.

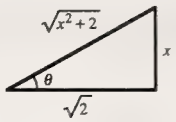
25. $y = \tan\left(\text{arcsec}\,\frac{x}{3}\right)$



$\theta = \text{arcsec}\,\frac{x}{3}$

$y = \tan\theta = \frac{\sqrt{x^2 - 9}}{3}$

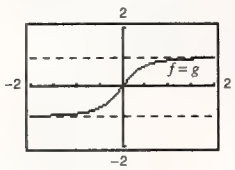
27. $y = \csc\left(\arctan\frac{x}{\sqrt{2}}\right)$



$\theta = \arctan\frac{x}{\sqrt{2}}$

$y = \csc\theta = \frac{\sqrt{x^2 + 2}}{x}$

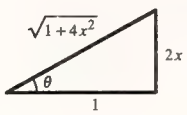
29. $\sin(\arctan 2x) = \frac{2x}{\sqrt{1 + 4x^2}}$



Asymptotes: $y = \pm 1$

$\arctan 2x = \theta$

$\tan\theta = 2x$



$\sin\theta = \frac{2x}{\sqrt{1 + 4x^2}}$

31. $\arcsin(3x - \pi) = \frac{1}{2}$

$3x - \pi = \sin\left(\frac{1}{2}\right)$

$x = \frac{1}{3}\left[\sin\left(\frac{1}{2}\right) + \pi\right] \approx 1.207$

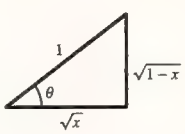
33. $\arcsin\sqrt{2x} = \arccos\sqrt{x}$

$\sqrt{2x} = \sin(\arccos\sqrt{x})$

$\sqrt{2x} = \sqrt{1 - x}, 0 \le x \le 1$

$2x = 1 - x$

$3x = 1$



$x = \frac{1}{3}$

35. (a) $\text{arccsc}\,x = \arcsin\frac{1}{x}, |x| \ge 1$

Let $y = \text{arccsc}\,x$. Then for

$$-\frac{\pi}{2} \le y < 0 \text{ and } 0 < y \le \frac{\pi}{2},$$

$\csc y = x \Rightarrow \sin y = 1/x$. Thus, $y = \arcsin(1/x)$. Therefore, $\text{arccsc}\,x = \arcsin(1/x)$.

(b) $\arctan x + \arctan\frac{1}{x} = \frac{\pi}{2}, x > 0$

Let $y = \arctan x + \arctan(1/x)$. Then,

$$\tan y = \frac{\tan(\arctan x) + \tan[\arctan(1/x)]}{1 - \tan(\arctan x)\tan[\arctan(1/x)]}$$

$$= \frac{x + (1/x)}{1 - x(1/x)}$$

$$= \frac{x + (1/x)}{0} \text{ (which is undefined).}$$

Thus, $y = \pi/2$. Therefore, $\arctan x + \arctan(1/x) = \pi/2$.

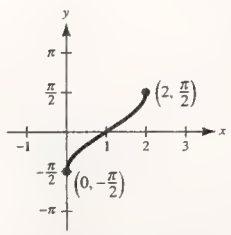
37. $f(x) = \arcsin(x - 1)$

$x - 1 = \sin y$

$x = 1 + \sin y$

Domain: $[0, 2]$

Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



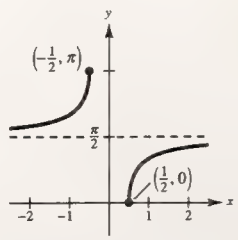
$f(x)$ is the graph of $\arcsin x$ shifted 1 unit to the right.

39. $f(x) = \text{arcsec}\,2x$

$2x = \sec y$

$x = \frac{1}{2}\sec y$

Domain: $\left(-\infty, -\frac{1}{2}\right], \left[\frac{1}{2}, \infty\right)$



Range: $\left[0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right]$

41. $f(x) = 2\arcsin(x-1)$

$$f'(x) = \frac{2}{\sqrt{1-(x-1)^2}} = \frac{2}{\sqrt{2x-x^2}}$$

43. $g(x) = 3\arccos\frac{x}{2}$

$$g'(x) = \frac{-3(1/2)}{\sqrt{1-(x^2/4)}} = \frac{-3}{\sqrt{4-x^2}}$$

45. $f(x) = \arctan\frac{x}{a}$

$$f'(x) = \frac{1/a}{1+(x^2/a^2)} = \frac{a}{a^2+x^2}$$

47. $g(x) = \dfrac{\arcsin 3x}{x}$

$$g'(x) = \frac{x\left(3/\sqrt{1-9x^2}\right) - \arcsin 3x}{x^2}$$

$$= \frac{3x - \sqrt{1-9x^2}\arcsin 3x}{x^2\sqrt{1-9x^2}}$$

49. $h(t) = \sin(\arccos t) = \sqrt{1-t^2}$

$$h'(t) = \frac{1}{2}(1-t^2)^{-1/2}(-2t) = \frac{-t}{\sqrt{1-t^2}}$$

51. $y = x\arccos x - \sqrt{1-x^2}$

$$y' = \arccos x - \frac{x}{\sqrt{1-x^2}} - \frac{1}{2}(1-x^2)^{-1/2}(-2x)$$

$$= \arccos x$$

53. $y = \dfrac{1}{2}\left(\dfrac{1}{2}\ln\dfrac{x+1}{x-1} + \arctan x\right)$

$$= \frac{1}{4}[\ln(x+1) - \ln(x-1)] + \frac{1}{2}\arctan x$$

$$\frac{dy}{dx} = \frac{1}{4}\left(\frac{1}{x+1} - \frac{1}{x-1}\right) + \frac{1/2}{1+x^2} = \frac{1}{1-x^4}$$

55. $y = x\arcsin x + \sqrt{1-x^2}$

$$\frac{dy}{dx} = x\left(\frac{1}{\sqrt{1-x^2}}\right) + \arcsin x - \frac{x}{\sqrt{1-x^2}} = \arcsin x$$

57. $y = 8\arcsin\dfrac{x}{4} - \dfrac{x\sqrt{16-x^2}}{2}$

$$y' = 2\frac{1}{\sqrt{1-(x/4)^2}} - \frac{\sqrt{16-x^2}}{2} - \frac{x}{4}(16-x^2)^{-1/2}(-2x)$$

$$= \frac{8}{\sqrt{16-x^2}} - \frac{\sqrt{16-x^2}}{2} + \frac{x^2}{2\sqrt{16-x^2}}$$

$$= \frac{16-(16-x^2)+x^2}{2\sqrt{16-x^2}} = \frac{x^2}{\sqrt{16-x^2}}$$

59. $y = \arctan x + \dfrac{x}{1+x^2}$

$$y' = \frac{1}{1+x^2} + \frac{(1+x^2)-x(2x)}{(1+x^2)^2}$$

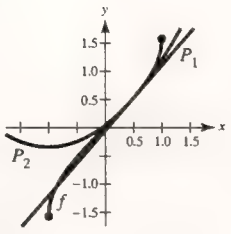
$$= \frac{(1+x^2)+(1-x^2)}{(1+x^2)^2}$$

$$= \frac{2}{(1+x^2)^2}$$

61. $f(x) = \arcsin x,\ a = \dfrac{1}{2}$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f''(x) = \frac{x}{(1-x^2)^{3/2}}$$



$$P_1(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right) = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x-\frac{1}{2}\right)$$

$$P_2(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right) + \frac{1}{2}f''\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)^2 = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x-\frac{1}{2}\right) + \frac{2\sqrt{3}}{9}\left(x-\frac{1}{2}\right)^2$$

63. $f(x) = \operatorname{arcsec} x - x$

$$f'(x) = \frac{1}{|x|\sqrt{x^2 - 1}} - 1$$

$= 0$ when $|x|\sqrt{x^2 - 1} = 1$.

$x^2(x^2 - 1) = 1$

$x^4 - x^2 - 1 = 0$ when $x^2 = \dfrac{1 + \sqrt{5}}{2}$ or

$$x = \pm\sqrt{\frac{1 + \sqrt{5}}{2}} = \pm 1.272$$

Relative maximum: $(1.272, -0.606)$

Relative minimum: $(-1.272, 3.747)$

65. $f(x) = \arctan x - \arctan(x - 4)$

$$f'(x) = \frac{1}{1 + x^2} - \frac{1}{1 + (x - 4)^2} = 0$$

$1 + x^2 = 1 + (x - 4)^2$

$0 = -8x + 16$

$x = 2$

By the First Derivative Test, (2, 2.214) is a relative maximum.

67. The trigonometric functions are not one-to-one on $(-\infty, \infty)$, sot their domains must be restricted to intervals on which they are one-to-one.

69. $y = \operatorname{arccot} x, 0 < y < \pi$

$x = \cot y$

$\tan y = \dfrac{1}{x}$

So, graph the function

$y = \arctan\left(\dfrac{1}{x}\right)$ for $x > 0$ and $y = \arctan\left(\dfrac{1}{x}\right) + \pi$ for $x < 0$.

71. (a) $\cot\theta = \dfrac{x}{5}$

$$\theta = \operatorname{arccot}\left(\frac{x}{5}\right)$$

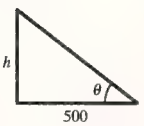
(b) $$\frac{d\theta}{dt} = \frac{-\frac{1}{5}}{1 + \left(\frac{x}{5}\right)^2}\frac{dx}{dt} = \frac{-5}{x^2 + 25}\frac{dx}{dt}$$

If $\dfrac{dx}{dt} = -400$ and $x = 10$, $\dfrac{d\theta}{dt} = 16$ rad/hr.

If $\dfrac{dx}{dt} = -400$ and $x = 3$, $\dfrac{d\theta}{dt} \approx 58.824$ rad/hr.

73. (a) $h(t) = -16t^2 + 256$

$-16t^2 + 256 = 0$ when $t = 4$ sec.



(b) $$\tan\theta = \frac{h}{500} = \frac{-16t^2 + 256}{500}$$

$$\theta = \arctan\left[\frac{16}{500}(-t^2 + 16)\right]$$

$$\frac{d\theta}{dt} = \frac{-8t/125}{1 + [(4/125)(-t^2 + 16)]^2} = \frac{-1000t}{15{,}625 + 16(16 - t^2)^2}$$

When $t = 1$, $d\theta/dt \approx -0.0520$ rad/sec.

When $t = 2$, $d\theta/dt \approx -0.1116$ rad/sec.

75. $\tan(\arctan x + \arctan y) = \dfrac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = \dfrac{x + y}{1 - xy}, xy \neq 1$

Therefore,

$$\arctan x + \arctan y = \arctan\left(\frac{x + y}{1 - xy}\right), xy \neq 1.$$

Let $x = \frac{1}{2}$ and $y = \frac{1}{3}$.

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan\frac{(1/2) + (1/3)}{1 - [(1/2)\cdot(1/3)]} = \arctan\frac{5/6}{1 - (1/6)} = \arctan\frac{5/6}{5/6} = \arctan 1 = \frac{\pi}{4}$$

77. $f(x) = kx + \sin x$

$f'(x) = k + \cos x \geq 0$ for $k \geq 1$

$f'(x) = k + \cos x \leq 0$ for $k \leq -1$

Therefore, $f(x) = kx + \sin x$ is strictly monotonic and has an inverse for $k \leq -1$ or $k \geq 1$.

79. True

$\dfrac{d}{dx}[\arctan x] = \dfrac{1}{1 + x^2} > 0$ for all x.

81. True

$\dfrac{d}{dx}[\arctan(\tan x)] = \dfrac{\sec^2 x}{1 + \tan^2 x} = \dfrac{\sec^2 x}{\sec^2 x} = 1$

Section 5.9 Inverse Trigonometric Functions: Integration

1. $\displaystyle\int \frac{5}{\sqrt{9 - x^2}}\,dx = 5\arcsin\left(\frac{x}{3}\right) + C$

3. Let $u = 3x$, $du = 3\,dx$.

$$\int_0^{1/6} \frac{1}{\sqrt{1 - 9x^2}}\,dx = \frac{1}{3}\int_0^{1/6} \frac{1}{\sqrt{1 - (3x)^2}}(3)\,dx = \left[\frac{1}{3}\arcsin(3x)\right]_0^{1/6} = \frac{\pi}{18}$$

5. $\displaystyle\int \frac{7}{16 + x^2}\,dx = \frac{7}{4}\arctan\left(\frac{x}{4}\right) + C$

7. Let $u = 2x$, $du = 2\,dx$.

$$\int_0^{\sqrt{3}/2} \frac{1}{1 + 4x^2}\,dx = \frac{1}{2}\int_0^{\sqrt{3}/2} \frac{2}{1 + (2x)^2}\,dx = \left[\frac{1}{2}\arctan(2x)\right]_0^{\sqrt{3}/2} = \frac{\pi}{6}$$

9. $\displaystyle\int \frac{1}{x\sqrt{4x^2 - 1}}\,dx = \int \frac{2}{2x\sqrt{(2x)^2 - 1}}\,dx = \operatorname{arcsec}|2x| + C$

11. $\displaystyle\int \frac{x^3}{x^2 + 1}\,dx = \int\left[x - \frac{x}{x^2 + 1}\right]dx = \int x\,dx - \frac{1}{2}\int \frac{2x}{x^2 + 1}\,dx = \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2 + 1) + C$ (Use long division.)

13. $\displaystyle\int \frac{1}{\sqrt{1 - (x + 1)^2}}\,dx = \arcsin(x + 1) + C$

15. Let $u = t^2$, $du = 2t\,dt$.

$$\int \frac{t}{\sqrt{1 - t^4}}\,dt = \frac{1}{2}\int \frac{1}{\sqrt{1 - (t^2)^2}}(2t)\,dt = \frac{1}{2}\arcsin(t^2) + C$$

17. Let $u = \arcsin x$, $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arcsin}{\sqrt{1-x^2}}\,dx = \left[\frac{1}{2}\arcsin^2 x\right]_0^{1/\sqrt{2}} = \frac{\pi^2}{32} \approx 0.308$$

19. Let $u = 1 - x^2$, $du = -2x\,dx$.

$$\int_{-1/2}^{0} \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_{-1/2}^{0} (1-x^2)^{-1/2}(-2x)\,dx$$

$$= \left[-\sqrt{1-x^2}\right]_{-1/2}^{0} = \frac{\sqrt{3}-2}{2}$$

$$\approx -0.134$$

21. Let $u = e^{2x}$, $du = 2e^{2x}\,dx$.

$$\int \frac{e^{2x}}{4+e^{4x}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{4+(e^{2x})^2}\,dx = \frac{1}{4}\arctan\frac{e^{2x}}{2} + C$$

23. Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx = -\int_{\pi/2}^{\pi} \frac{-\sin x}{1+\cos^2 x}\,dx$$

$$= \left[-\arctan(\cos x)\right]_{\pi/2}^{\pi} = \frac{\pi}{4}$$

25. $\displaystyle\int \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx.\ u = \sqrt{x},\ x = u^2,\ dx = 2u\,du$

$$\int \frac{1}{u\sqrt{1-u^2}}(2u\,du) = 2\int \frac{du}{\sqrt{1-u^2}} = 2\arcsin u + C$$

$$= 2\arcsin\sqrt{x} + C$$

27. $$\int \frac{x-3}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx - 3\int \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln(x^2+1) - 3\arctan x + C$$

29. $$\int \frac{x+5}{\sqrt{9-(x-3)^2}}\,dx = \int \frac{(x-3)}{\sqrt{9-(x-3)^2}}\,dx + \int \frac{8}{\sqrt{9-(x-3)^2}}\,dx$$

$$= -\sqrt{9-(x-3)^2} - 8\arcsin\left(\frac{x-3}{3}\right) + C$$

$$= -\sqrt{6x-x^2} + 8\arcsin\left(\frac{x}{3} - 1\right) + C$$

31. $$\int_0^2 \frac{1}{x^2-2x+2}\,dx = \int_0^2 \frac{1}{1+(x-1)^2}\,dx = \left[\arctan(x-1)\right]_0^2 = \frac{\pi}{2}$$

33. $$\int \frac{2x}{x^2+6x+13}\,dx = \int \frac{2x+6}{x^2+6x+13}dx - 6\int \frac{1}{x^2+6x+13}\,dx = \int \frac{2x+6}{x^2+6x+13}\,dx - 6\int \frac{1}{4+(x+3)^2}\,dx$$

$$= \ln|x^2+6x+13| - 3\arctan\left(\frac{x+3}{2}\right) + C$$

35. $$\int \frac{1}{\sqrt{-x^2-4x}}\,dx = \int \frac{1}{\sqrt{4-(x+2)^2}}\,dx = \arcsin\left(\frac{x+2}{2}\right) + C$$

37. Let $u = -x^2 - 4x$, $du = (-2x-4)\,dx$.

$$\int \frac{x+2}{\sqrt{-x^2-4x}}\,dx = -\frac{1}{2}\int (-x^2-4x)^{-1/2}(-2x-4)\,dx = -\sqrt{-x^2-4x} + C$$

39. $$\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}}\,dx = \int_2^3 \frac{2x-4}{\sqrt{4x-x^2}}\,dx + \int_2^3 \frac{1}{\sqrt{4x-x^2}}\,dx = -\int_2^3 (4x-x^2)^{-1/2}(4-2x)\,dx + \int_2^3 \frac{1}{\sqrt{4-(x-2)^2}}\,dx$$

$$= \left[-2\sqrt{4x-x^2} + \arcsin\left(\frac{x-2}{2}\right)\right]_2^3 = 4 - 2\sqrt{3} + \frac{\pi}{6} \approx 1.059$$

41. Let $u = x^2 + 1$, $du = 2x\,dx$.

$$\int \frac{x}{x^4 + 2x^2 + 2}\,dx = \frac{1}{2}\int \frac{2x}{(x^2 + 1)^2 + 1}\,dx = \frac{1}{2}\arctan(x^2 + 1) + C$$

43. Let $u = \sqrt{e^t - 3}$. Then $u^2 + 3 = e^t$, $2u\,du = e^t\,dt$, and $\dfrac{2u\,du}{u^2 + 3} = dt$.

$$\int \sqrt{e^t - 3}\,dt = \int \frac{2u^2}{u^2 + 3}\,du = \int 2\,du - \int 6\frac{1}{u^2 + 3}\,du$$

$$= 2u - 2\sqrt{3}\arctan\frac{u}{\sqrt{3}} + C = 2\sqrt{e^t - 3} - 2\sqrt{3}\arctan\sqrt{\frac{e^t - 3}{3}} + C$$

45. A perfect square trinomial is an expression in x with three terms that factor as a perfect square.

Example: $x^2 + 6x + 9 = (x + 3)^2$

47. (a) $\displaystyle\int \frac{1}{\sqrt{1 - x^2}}\,dx = \arcsin x + C,\ u = x$ (b) $\displaystyle\int \frac{x}{\sqrt{1 - x^2}}\,dx = -\sqrt{1 - x^2} + C,\ u = 1 - x^2$

(c) $\displaystyle\int \frac{1}{x\sqrt{1 - x^2}}\,dx$ cannot be evaluated using the basic integration rules.

49. (a) $\displaystyle\int \sqrt{x - 1}\,dx = \frac{2}{3}(x - 1)^{3/2} + C,\ u = x - 1$

(b) Let $u = \sqrt{x - 1}$. Then $x = u^2 + 1$ and $dx = 2u\,du$.

$$\int x\sqrt{x - 1}\,dx = \int (u^2 + 1)(u)(2u)\,du = 2\int (u^4 + u^2)\,du = 2\left(\frac{u^5}{5} + \frac{u^3}{3}\right) + C$$

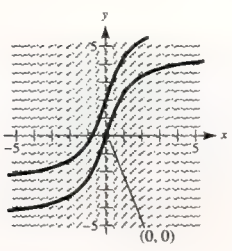
$$= \frac{2}{15}u^3(3u^2 + 5) + C = \frac{2}{15}(x - 1)^{3/2}[3(x - 1) + 5] + C = \frac{2}{15}(x - 1)^{3/2}(3x + 2) + C$$

(c) Let $u = \sqrt{x - 1}$. Then $x = u^2 + 1$ and $dx = 2u\,du$.

$$\int \frac{x}{\sqrt{x - 1}}\,dx = \int \frac{u^2 + 1}{u}(2u)\,du = 2\int (u^2 + 1)\,du = 2\left(\frac{u^3}{3} + u\right) + C = \frac{2}{3}u(u^2 + 3) + C = \frac{2}{3}\sqrt{x - 1}(x + 2) + C$$

Note: In (b) and (c), substitution was necessary *before* the basic integration rules could be used.

51. (a)

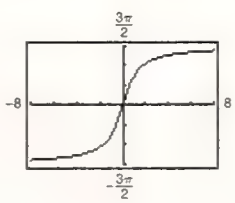


(b) $\dfrac{dy}{dx} = \dfrac{3}{1 + x^2}$, $(0, 0)$

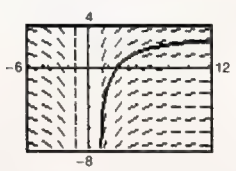
$$y = 3\int \frac{dx}{1 + x^2} = 3\arctan x + C$$

$(0, 0)$: $0 = 3\arctan(0) + C \Rightarrow C = 0$

$y = 3\arctan x$



53. $\dfrac{dy}{dx} = \dfrac{10}{x\sqrt{x^2 - 1}}$, $y(3) = 0$

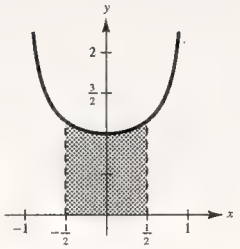


55. $A = \displaystyle\int_1^3 \frac{1}{x^2 - 2x + 1 + 4}\,dx = \int_1^3 \frac{1}{(x - 1)^2 + 2^2}\,dx$

$$= \left[\frac{1}{2}\arctan\left(\frac{x - 1}{2}\right)\right]_1^3 = \frac{1}{2}\arctan(1) = \frac{\pi}{8} \approx 0.3927$$

57. Area $\approx (1)(1) = 1$

Matches (c)



59. (a) $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx = \Big[4\arctan x\Big]_0^1 = 4\arctan 1 - 4\arctan 0 = 4\left(\frac{\pi}{4}\right) - 4(0) = \pi$

(b) Let $n = 6$.

$$4\int_0^1 \frac{4}{1+x^2}dx \approx 4\left(\frac{1}{36}\right)\left[1 + \frac{4}{1+(1/36)} + \frac{2}{1+(1/9)} + \frac{4}{1+(1/4)} + \frac{2}{1+(4/9)} + \frac{4}{1+(25/36)} + \frac{1}{2}\right] \approx 3.1415918$$

(c) 3.1415927

61. (a) $\displaystyle\frac{d}{dx}\left[\arcsin\left(\frac{u}{a}\right) + C\right] = \frac{1}{\sqrt{1-(u^2/a^2)}}\left(\frac{u'}{a}\right) = \frac{u'}{\sqrt{a^2-u^2}}$

Thus, $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C.$

(b) $\displaystyle\frac{d}{dx}\left[\frac{1}{a}\arctan\frac{u}{a} + C\right] = \frac{1}{a}\left[\frac{u'/a}{1+(u/a)^2}\right] = \frac{1}{a^2}\left[\frac{u'}{(a^2+u^2)/a^2}\right] = \frac{u'}{a^2+u^2}$

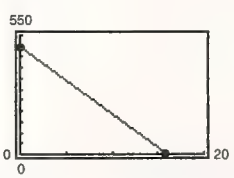
Thus, $\displaystyle\int \frac{du}{a^2+u^2} = \int \frac{u'}{a^2+u^2}\,dx = \frac{1}{a}\arctan\frac{u}{a} + C.$

(c) Assume $u > 0$.

$\displaystyle\frac{d}{dx}\left[\frac{1}{a}\operatorname{arcsec}\frac{u}{a} + C\right] = \frac{1}{a}\left[\frac{u'/a}{(u/a)\sqrt{(u/a)^2-1}}\right] = \frac{1}{a}\left[\frac{u'}{u\sqrt{u^2-a^2}/a^2}\right] = \frac{u'}{u\sqrt{u^2-a^2}}$. The case $u < 0$ is handled in a similar manner.

Thus, $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \int \frac{u'}{u\sqrt{u^2-a^2}}\,dx = \frac{1}{a}\operatorname{arsec}\frac{|u|}{a} + C.$

63. (a) $v(t) = -32t + 500$



(b) $\displaystyle s(t) = \int v(t)\,dt = \int (-32t + 500)\,dt$

$$= -16t^2 + 500t + C$$

$$s(0) = -16(0) + 500(0) + C = 0 \Rightarrow C = 0$$

$$s(t) = -16t^2 + 500t$$

When the object reaches its maximum height, $v(t) = 0$.

$$v(t) = -32t + 500 = 0$$

$$-32t = -500$$

$$t = 15.625$$

$$s(15.625) = -16(15.625)^2 + 500(15.625)$$

$$= 3906.25 \text{ ft (Maximum height)}$$

—CONTINUED—

63. —CONTINUED—

(c) $$\int \frac{1}{32 + kv^2}\,dv = -\int dt$$

$$\frac{1}{\sqrt{32k}} \arctan\left(\sqrt{\frac{k}{32}}v\right) = -t + C_1$$

$$\arctan\left(\sqrt{\frac{k}{32}}v\right) = -\sqrt{32k}\,t + C$$

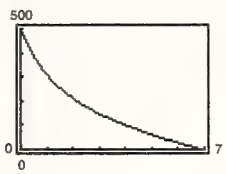
$$\sqrt{\frac{k}{32}}v = \tan\left(C - \sqrt{32k}\,t\right)$$

$$v = \sqrt{\frac{32}{k}}\tan\left(C - \sqrt{32k}\,t\right)$$

When $t = 0$, $v = 500$, $C = \arctan\left(500\sqrt{k/32}\right)$, and we have

$$v(t) = \sqrt{\frac{32}{k}}\tan\left[\arctan\left(500\sqrt{\frac{k}{32}}\right) - \sqrt{32k}\,t\right].$$

(d) When $k = 0.001$, $v(t) = \sqrt{32{,}000}\tan\left[\arctan\left(500\sqrt{0.00003125}\right) - \sqrt{0.032}\,t\right]$.



$v(t) = 0$ when $t_0 \approx 6.86$ sec.

(e) $h = \displaystyle\int_0^{6.86} \sqrt{32{,}000}\tan\left[\arctan\left(500\sqrt{0.00003125}\right) - \sqrt{0.032}\,t\right] dt$

Simpson's Rule: $n = 10$; $h \approx 1088$ feet

(f) Air resistance lowers the maximum height.

Section 5.10 Hyperbolic Functions

1. (a) $\sinh 3 = \dfrac{e^3 - e^{-3}}{2} \approx 10.018$

(b) $\tanh(-2) = \dfrac{\sinh(-2)}{\cosh(-2)} = \dfrac{e^{-2} - e^2}{e^{-2} + e^2} \approx -0.964$

3. (a) $\operatorname{csch}(\ln 2) = \dfrac{2}{e^{\ln 2} - e^{-\ln 2}} = \dfrac{2}{2 - (1/2)} = \dfrac{4}{3}$

(b) $\coth(\ln 5) = \dfrac{\cosh(\ln 5)}{\sinh(\ln 5)} = \dfrac{e^{\ln 5} + e^{-\ln 5}}{e^{\ln 5} - e^{-\ln 5}}$

$$= \frac{5 + (1/5)}{5 - (1/5)} = \frac{13}{12}$$

5. (a) $\cosh^{-1}(2) = \ln\left(2 + \sqrt{3}\right) \approx 1.317$

(b) $\operatorname{sech}^{-1}\left(\dfrac{2}{3}\right) = \ln\left(\dfrac{1 + \sqrt{1 - (4/9)}}{2/3}\right) \approx 0.962$

7. $\tanh^2 x + \operatorname{sech}^2 x = \left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 + \left(\dfrac{2}{e^x + e^{-x}}\right)^2 = \dfrac{e^{2x} - 2 + e^{-2x} + 4}{(e^x + e^{-x})^2} = \dfrac{e^{2x} + 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} = 1$

9. $\sinh x \cosh y + \cosh x \sinh y = \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$

$= \frac{1}{4}[e^{x+y} - e^{-x+y} + e^{x-y} - e^{-(x+y)} + e^{x+y} + e^{-x+y} - e^{x-y} - e^{-(x+y)}]$

$= \frac{1}{4}[2(e^{x+y} - e^{-(x+y)})] = \frac{e^{(x+y)} - e^{-(x+y)}}{2} = \sinh(x + y)$

11. $3 \sinh x + 4 \sinh^3 x = \sinh x(3 + 4 \sinh^2 x) = \left(\frac{e^x - e^{-x}}{2}\right)\left[3 + 4\left(\frac{e^x - e^{-x}}{2}\right)^2\right]$

$= \left(\frac{e^x - e^{-x}}{2}\right)[3 + e^{2x} - 2 + e^{-2x}] = \frac{1}{2}(e^x - e^{-x})(e^{2x} + e^{-2x} + 1)$

$= \frac{1}{2}[e^{3x} + e^{-x} + e^x - e^x - e^{-3x} - e^{-x}] = \frac{e^{3x} - e^{-3x}}{2} = \sinh(3x)$

13. $\sinh x = \frac{3}{2}$

$\cosh^2 x - \left(\frac{3}{2}\right)^2 = 1 \implies \cosh^2 x = \frac{13}{4} \implies \cosh x = \frac{\sqrt{13}}{2}$

$\tanh x = \frac{3/2}{\sqrt{13}/2} = \frac{3\sqrt{13}}{13}$

$\operatorname{csch} x = \frac{1}{3/2} = \frac{2}{3}$

$\operatorname{sech} x = \frac{1}{\sqrt{13}/2} = \frac{2\sqrt{13}}{13}$

$\coth x = \frac{1}{3/\sqrt{13}} = \frac{\sqrt{13}}{3}$

15. $y = \sinh(1 - x^2)$

$y' = -2x \cosh(1 - x^2)$

17. $f(x) = \ln(\sinh x)$

$f'(x) = \frac{1}{\sinh}(\cosh x) = \coth x$

19. $y = \ln\left(\tanh \frac{x}{2}\right)$

$y' = \frac{1/2}{\tanh(x/2)} \operatorname{sech}^2\left(\frac{x}{2}\right) = \frac{1}{2 \sinh(x/2) \cosh(x/2)}$

$= \frac{1}{\sinh x} = \operatorname{csch} x$

21. $h(x) = \frac{1}{4}\sinh(2x) - \frac{x}{2}$

$h'(x) = \frac{1}{2}\cosh(2x) - \frac{1}{2} = \frac{\cosh(2x) - 1}{2} = \sinh^2 x$

23. $f(t) = \arctan(\sinh t)$

$f'(t) = \frac{1}{1 + \sinh^2 t}(\cosh t)$

$= \frac{\cosh t}{\cosh^2 t} = \operatorname{sech} t$

25. Let $y = g(x)$.

$y = x^{\cosh x}$

$\ln y = \cosh x \ln x$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{\cosh x}{x} + \sinh x \ln x$

$\frac{dy}{dx} = \frac{y}{x}[\cosh x + x(\sinh x) \ln x]$

$= \frac{x^{\cosh x}}{x}[\cosh x + x(\sinh x) \ln x]$

27. $y = (\cosh x - \sinh x)^2$

$y' = 2(\cosh x - \sinh x)(\sinh x - \cosh x)$

$= -2(\cosh x - \sinh x)^2 = -2e^{-2x}$

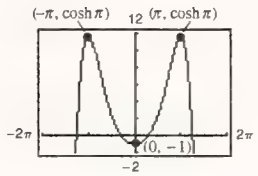
29. $f(x) = \sin x \sinh x - \cos x \cosh x, \quad -4 \le x \le 4$

$f'(x) = \sin x \cosh x + \cos x \sinh x - \cos x \sinh x + \sin x \cosh x$

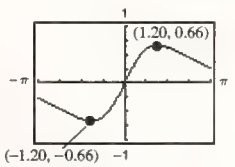
$= 2 \sin x \cosh x = 0$ when $x = 0, \pm\pi$.

Relative maxima: $(\pm\pi, \cosh \pi)$

Relative minimum: $(0, -1)$



31. $g(x) = x \operatorname{sech} x = \dfrac{x}{\cosh x}$



Relative maximum: $(1.20, 0.66)$

Relative minimum: $(-1.20, -0.66)$

33. $y = a \sinh x$

$y' = a \cosh x$

$y'' = a \sinh x$

$y''' = a \cosh x$

Therefore, $y''' - y' = 0$.

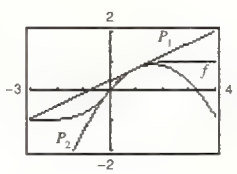
35. $f(x) = \tanh x \qquad f(1) = \tanh(1) \approx 0.7616$

$f'(x) = \operatorname{sech}^2 x \qquad f'(1) = \dfrac{1}{\cosh^2(1)} \approx 0.4200$

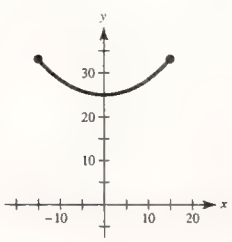
$f''(x) = -2 \operatorname{sech}^2 x \cdot \tanh x \qquad f''(1) \approx -0.6397$

$P_1(x) = f(1) + f'(1)(x - 1) = 0.7616 + 0.42(x - 1)$

$P_2(x) = 0.7616 + 0.42(x - 1) - \dfrac{0.6397}{2}(x - 1)^2$



37. (a) $y = 10 + 15 \cosh \dfrac{x}{15}, \quad -15 \le x \le 15$



(b) At $x = \pm 15$, $y = 10 + 15 \cosh(1) \approx 33.146$.

At $x = 0$, $y = 10 + 15 \cosh(1) = 25$.

(c) $y' = \sinh \dfrac{x}{15}$. At $x = 15$, $y' = \sinh(1) \approx 1.175$

39. Let $u = 1 - 2x$, $du = -2\,dx$.

$$\int \sinh(1 - 2x)\,dx = -\frac{1}{2}\int \sinh(1 - 2x)(-2)\,dx$$

$$= -\frac{1}{2}\cosh(1 - 2x) + C$$

41. Let $u = \cosh(x - 1)$, $du = \sinh(x - 1)\,dx$.

$$\int \cosh^2(x - 1)\sinh(x - 1)\,dx = \frac{1}{3}\cosh^3(x - 1) + C$$

43. Let $u = \sinh x$, $du = \cosh x\,dx$.

$$\int \frac{\cosh x}{\sinh x}\,dx = \ln|\sinh x| + C$$

45. Let $u = \frac{x^2}{2}$, $du = x\,dx$.

$$\int x \operatorname{csch}^2 \frac{x^2}{2}\,dx = \int \left(\operatorname{csch}^2 \frac{x^2}{2}\right) x\,dx = -\coth \frac{x^2}{2} + C$$

47. Let $u = \frac{1}{x}$, $du = -\frac{1}{x^2}\,dx$.

$$\int \frac{\operatorname{csch}(1/x)\coth(1/x)}{x^2}\,dx = -\int \operatorname{csch}\frac{1}{x}\coth\frac{1}{x}\left(-\frac{1}{x^2}\right)dx = \operatorname{csch}\frac{1}{x} + C$$

49. $$\int_0^4 \frac{1}{25 - x^2}\,dx = \frac{1}{10}\int \frac{1}{5 - x}\,dx + \frac{1}{10}\int \frac{1}{5 + x}\,dx = \left[\frac{1}{10}\ln\left|\frac{5 + x}{5 - x}\right|\right]_0^4 = \frac{1}{10}\ln 9 = \frac{1}{5}\ln 3$$

51. Let $u = 2x$, $du = 2\,dx$.

$$\int_0^{\sqrt{2}/4} \frac{2}{\sqrt{1 - 4x^2}}\,dx = \int_0^{\sqrt{2}/4} \frac{1}{\sqrt{1 - (2x)^2}}(2)\,dx = \left[\arcsin(2x)\right]_0^{\sqrt{2}/4} = \frac{\pi}{4}$$

53. Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{x}{x^4 + 1}\,dx = \frac{1}{2}\int \frac{2x}{(x^2)^2 + 1}\,dx = \frac{1}{2}\arctan(x^2) + C$$

55. $y = \cosh^{-1}(3x)$

$$y' = \frac{3}{\sqrt{9x^2 - 1}}$$

57. $y = \sinh^{-1}(\tan x)$

$$y' = \frac{1}{\sqrt{\tan^2 x + 1}}(\sec^2 x) = |\sec x|$$

59. $y = \tanh^{-1}(\sin 2x)$

$$y' = \frac{1}{1 - \sin^2 2x}(2\cos 2x) = 2\sec 2x$$

61. $y = 2x\sinh^{-1}(2x) - \sqrt{1 + 4x^2}$

$$y' = 2x\left(\frac{2}{\sqrt{1 + 4x^2}}\right) + 2\sinh^{-1}(2x) - \frac{4x}{\sqrt{1 + 4x^2}} = 2\sinh^{-1}(2x)$$

63. See page 395.

65. $y = a\operatorname{sech}^{-1}\frac{x}{a} - \sqrt{a^2 - x^2}$, $a > 0$

$$\frac{dy}{dx} = \frac{-1}{(x/a)\sqrt{1 - (x^2/a^2)}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{-a^2}{x\sqrt{a^2 - x^2}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{x^2 - a^2}{x\sqrt{a^2 - x^2}} = \frac{-\sqrt{a^2 - x^2}}{x}$$

67. $$\int \frac{1}{\sqrt{1 + e^{2x}}}\,dx = \int \frac{e^x}{e^x\sqrt{1 + (e^x)^2}}\,dx = -\operatorname{csch}^{-1}(e^x) + C = -\ln\left(\frac{1 + \sqrt{1 + e^{2x}}}{e^x}\right) + C$$

69. Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$.

$$\int \frac{1}{\sqrt{x}\sqrt{1 + x}}\,dx = 2\int \frac{1}{\sqrt{1 + (\sqrt{x})^2}}\left(\frac{1}{2\sqrt{x}}\right)dx = 2\sinh^{-1}\sqrt{x} + C = 2\ln\left(\sqrt{x} + \sqrt{1 + x}\right) + C$$

71. $$\int \frac{-1}{4x - x^2}\,dx = \int \frac{1}{(x - 2)^2 - 4}\,dx = \frac{1}{4}\ln\left|\frac{(x - 2) - 2}{(x - 2) + 2}\right| = \frac{1}{4}\ln\left|\frac{x - 4}{x}\right| + C$$

73. $\displaystyle\int \frac{1}{1-4x-2x^2}\,dx = \int \frac{1}{3-2(x+1)^2}\,dx = \frac{-1}{\sqrt{2}}\int \frac{\sqrt{2}}{\left[\sqrt{2}(x+1)\right]^2-\left(\sqrt{3}\right)^2}\,dx$

$$= \frac{-1}{2\sqrt{6}}\ln\left|\frac{\sqrt{2}(x+1)-\sqrt{3}}{\sqrt{2}(x+1)+\sqrt{3}}\right| + C = \frac{1}{2\sqrt{6}}\ln\left|\frac{\sqrt{2}(x+1)+\sqrt{3}}{\sqrt{2}(x+1)-\sqrt{3}}\right| + C$$

75. Let $u = 4x - 1$, $du = 4\,dx$.

$$y = \int \frac{1}{\sqrt{80+8x-16x^2}}\,dx = \frac{1}{4}\int \frac{4}{\sqrt{81-(4x-1)^2}}\,dx = \frac{1}{4}\arcsin\left(\frac{4x-1}{9}\right) + C$$

77. $\displaystyle y = \int \frac{x^3-21x}{5+4x-x^2}\,dx = \int\left(-x-4+\frac{20}{5+4x-x^2}\right)dx = \int(-x-4)\,dx + 20\int \frac{1}{3^2-(x-2)^2}\,dx$

$$= -\frac{x^2}{2} - 4x + \frac{20}{6}\ln\left|\frac{(x-2)+3}{(x-2)-3}\right| + C = -\frac{x^2}{2} - 4x + \frac{10}{3}\ln\left|\frac{x+1}{x-5}\right| + C = \frac{-x^2}{2} - 4x - \frac{10}{3}\ln\left|\frac{x-5}{x+1}\right| + C$$

79. $\displaystyle A = 2\int_0^4 \operatorname{sech}\frac{x}{2}\,dx$

$$= 2\int_0^4 \frac{2}{e^{x/2}+e^{-x/2}}\,dx$$

$$= 4\int_0^4 \frac{e^{x/2}}{(e^{x/2})^2+1}\,dx$$

$$= \Big[8\arctan(e^{x/2})\Big]_0^4$$

$$= 8\arctan(e^2) - 2\pi \approx 5.207$$

81. $\displaystyle A = \int_0^2 \frac{5x}{\sqrt{x^4+1}}\,dx$

$$= \frac{5}{2}\int_0^2 \frac{2x}{\sqrt{(x^2)^2+1}}\,dx$$

$$= \left[\frac{5}{2}\ln\left(x^2+\sqrt{x^4+1}\right)\right]_0^2$$

$$= \frac{5}{2}\ln\left(4+\sqrt{17}\right) \approx 5.237$$

83. $\displaystyle \int \frac{3k}{16}\,dt = \int \frac{1}{x^2-12x+32}\,dx$

$$\frac{3kt}{16} = \int \frac{1}{(x-6)^2-4}\,dx = \frac{1}{2(2)}\ln\left|\frac{(x-6)-2}{(x-6)+2}\right| + C = \frac{1}{4}\ln\left|\frac{x-8}{x-4}\right| + C$$

When $x = 0$: $t = 0$

$$C = -\frac{1}{4}\ln(2)$$

When $x = 1$: $t = 10$

$$\frac{30k}{16} = \frac{1}{4}\ln\left|\frac{-7}{-3}\right| - \frac{1}{4}\ln(2) = \frac{1}{4}\ln\left(\frac{7}{6}\right)$$

$$k = \frac{2}{15}\ln\left(\frac{7}{6}\right)$$

When $t = 20$: $\left(\frac{3}{16}\right)\left(\frac{2}{15}\right)\ln\left(\frac{7}{6}\right)(20) = \frac{1}{4}\ln\frac{x-8}{2x-8}$

$$\ln\left(\frac{7}{6}\right)^2 = \ln\frac{x-8}{2x-8}$$

$$\frac{49}{36} = \frac{x-8}{2x-8}$$

$$62x = 104$$

$$x = \frac{104}{62} = \frac{52}{31} \approx 1.677 \text{ kg}$$

85. As k increases, the time required for the object to reach the ground increases.

87. $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$

$y' = \dfrac{e^x - e^{-x}}{2} = \sinh x$

89. $y = \cosh^{-1} x$

$\cosh y = x$

$(\sinh y)(y') = 1$

$y' = \dfrac{1}{\sinh y} = \dfrac{1}{\sqrt{\cosh^2 y - 1}} = \dfrac{1}{\sqrt{x^2 - 1}}$

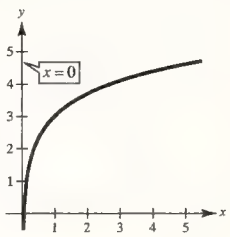
91. $y = \operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$

$y' = -2(e^x + e^{-x})^{-2}(e^x - e^{-x}) = \left(\dfrac{-2}{e^x + e^{-x}}\right)\left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right) = -\operatorname{sech} x \tanh x$

Review Exercises for Chapter 5

1. $f(x) = \ln x + 3$

Vertical shift 3 units upward
Vertical asymptote: $x = 0$



3. $\ln \sqrt[5]{\dfrac{4x^2 - 1}{4x^2 + 1}} = \dfrac{1}{5} \ln \dfrac{(2x - 1)(2x + 1)}{4x^2 + 1} = \dfrac{1}{5}[\ln(2x - 1) + \ln(2x + 1) - \ln(4x^2 + 1)]$

5. $\ln 3 + \dfrac{1}{3}\ln(4 - x^2) - \ln x = \ln 3 + \ln \sqrt[3]{4 - x^2} - \ln x = \ln\left(\dfrac{3\sqrt[3]{4 - x^2}}{x}\right)$

7. $\ln \sqrt{x + 1} = 2$

$\sqrt{x + 1} = e^2$

$x + 1 = e^4$

$x = e^4 - 1 \approx 53.598$

9. $g(x) = \ln \sqrt{x} = \dfrac{1}{2} \ln x$

$g'(x) = \dfrac{1}{2x}$

11. $f(x) = x\sqrt{\ln x}$

$f'(x) = \left(\dfrac{x}{2}\right)(\ln x)^{-1/2}\left(\dfrac{1}{x}\right) + \sqrt{\ln x}$

$= \dfrac{1}{2\sqrt{\ln x}} + \sqrt{\ln x} = \dfrac{1 + 2 \ln x}{2\sqrt{\ln x}}$

13. $y = \dfrac{1}{b^2}\left[\ln(a + bx) + \dfrac{a}{a + bx}\right]$

$\dfrac{dy}{dx} = \dfrac{1}{b^2}\left[\dfrac{b}{a + bx} - \dfrac{ab}{(a + bx)^2}\right] = \dfrac{x}{(a + bx)^2}$

15. $y = -\dfrac{1}{a}\ln\left(\dfrac{a + bx}{x}\right) = -\dfrac{1}{a}[\ln(a + bx) - \ln x]$

$\dfrac{dy}{dx} = -\dfrac{1}{a}\left(\dfrac{b}{a + bx} - \dfrac{1}{x}\right) = \dfrac{1}{x(a + bx)}$

17. $u = 7x - 2, du = 7dx$

$\displaystyle\int \frac{1}{7x - 2}\,dx = \frac{1}{7}\int \frac{1}{7x - 2}(7)\,dx = \frac{1}{7}\ln|7x - 2| + C$

19. $\displaystyle\int \frac{\sin x}{1+\cos x}\,dx = -\int \frac{-\sin x}{1+\cos x}\,dx$

$$= -\ln|1+\cos x| + C$$

21. $\displaystyle\int_1^4 \frac{x+1}{x}\,dx = \int_1^4 \left(1+\frac{1}{x}\right)dx = \Big[x + \ln|x|\Big]_1^4 = 3 + \ln 4$

23. $\displaystyle\int_0^{\pi/3} \sec\theta\,d\theta = \Big[\ln|\sec\theta + \tan\theta|\Big]_0^{\pi/3} = \ln\left(2+\sqrt{3}\right)$

25. (a) $f(x) = \frac{1}{2}x - 3$

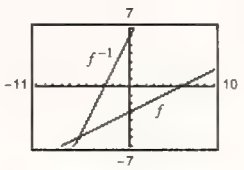
$y = \frac{1}{2}x - 3$

$2(y+3) = x$

$2(x+3) = y$

$f^{-1}(x) = 2x + 6$

(b)



(c) $f^{-1}(f(x)) = f^{-1}\left(\frac{1}{2}x - 3\right) = 2\left(\frac{1}{2}x - 3\right) + 6 = x$

$f(f^{-1}(x)) = f(2x+6) = \frac{1}{2}(2x+6) - 3 = x$

27. (a) $f(x) = \sqrt{x+1}$

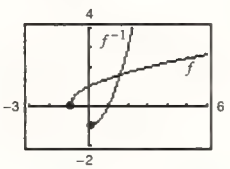
$y = \sqrt{x+1}$

$y^2 - 1 = x$

$x^2 - 1 = y$

$f^{-1}(x) = x^2 - 1, x \geq 0$

(b)



(c) $f^{-1}(f(x)) = f^{-1}\left(\sqrt{x+1}\right) = \sqrt{(x^2-1)^2} - 1 = x$

$f(f^{-1}(x)) = f(x^2-1) = \sqrt{(x^2-1)+1}$

$= \sqrt{x^2} = x$ for $x \geq 0$.

29. (a) $f(x) = \sqrt[3]{x+1}$

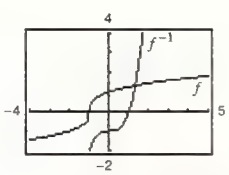
$y = \sqrt[3]{x+1}$

$y^3 - 1 = x$

$x^3 - 1 = y$

$f^{-1}(x) = x^3 - 1$

(b)



(c) $f^{-1}(f(x)) = f^{-1}\left(\sqrt[3]{x+1}\right) = \left(\sqrt[3]{x+1}\right)^3 - 1 = x$

$f(f^{-1}(x)) = f(x^3-1) = \sqrt[3]{(x^3-1)+1} = x$

31. $f(x) = x^3 + 2$

$f^{-1}(x) = (x-2)^{1/3}$

$(f^{-1})'(x) = \frac{1}{3}(x-2)^{-2/3}$

$(f^{-1})'(-1) = \frac{1}{3}(-1-2)^{-2/3} = \frac{1}{3(-3)^{2/3}}$

$= \frac{1}{3^{5/3}} \approx 0.160$

33. $f(x) = \tan x$

$f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

$f'(x) = \sec^2 x$

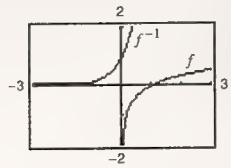
$f'\left(\frac{\pi}{6}\right) = \frac{4}{3}$

$(f^{-1})'\left(\frac{\sqrt{3}}{3}\right) = \frac{1}{f'(\pi/6)} = \frac{3}{4}$

35. (a) $f(x) = \ln \sqrt{x}$

$$y = \ln \sqrt{x}$$
$$e^y = \sqrt{x}$$
$$e^{2y} = x$$
$$e^{2x} = y$$
$$f^{-1}(x) = e^{2x}$$

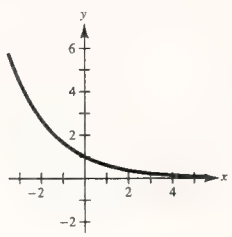
(b)



(c) $f^{-1}(f(x)) = f^{-1}(\ln \sqrt{x}) = e^{2 \ln \sqrt{x}} = e^{\ln x} = x$

$f(f^{-1}(x)) = f(e^{2x}) = \ln \sqrt{e^{2x}} = \ln e^x = x$

37. $y = e^{-x/2}$



39. $f(x) = \ln(e^{-x^2}) = -x^2$

$f'(x) = -2x$

41. $g(t) = t^2 e^t$

$g'(x) = t^2 e^t + 2te^t = te^t(t + 2)$

43. $y = \sqrt{e^{2x} + e^{-2x}}$

$$y' = \frac{1}{2}(e^{2x} + e^{-2x})^{-1/2}(2e^{2x} - 2e^{-2x}) = \frac{e^{2x} - e^{-2x}}{\sqrt{e^{2x} + e^{-2x}}}$$

45. $g(x) = \dfrac{x^2}{e^x}$

$$g'(x) = \frac{e^x(2x) - x^2e^x}{e^{2x}} = \frac{x(2 - x)}{e^x}$$

47.
$$y(\ln x) + y^2 = 0$$
$$y\left(\frac{1}{x}\right) + (\ln x)\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = 0$$
$$(2y + \ln x)\frac{dy}{dx} = \frac{-y}{x}$$
$$\frac{dy}{dx} = \frac{-y}{x(2y + \ln x)}$$

49. Let $u = -3x^2$, $du = -6x\,dx$.

$$\int xe^{-3x^2}\,dx = -\frac{1}{6}\int e^{-3x^2}(-6x)\,dx = -\frac{1}{6}e^{-3x^2} + C$$

51.
$$\int \frac{e^{4x} - e^{2x} + 1}{e^x}\,dx = \int (e^{3x} - e^x + e^{-x})\,dx$$
$$= \frac{1}{3}e^{3x} - e^x - e^{-x} + C$$
$$= \frac{e^{4x} - 3e^{2x} - 3}{3e^x} + C$$

53.
$$\int xe^{1-x^2}\,dx = -\frac{1}{2}\int e^{1-x^2}(-2x)\,dx$$
$$= -\frac{1}{2}e^{1-x^2} + C$$

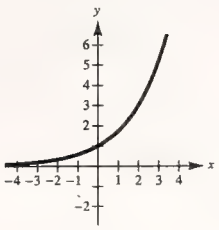
55. Let $u = e^x - 1$, $du = e^x\,dx$.

$$\int \frac{e^x}{e^x - 1}\,dx = \ln|e^x - 1| + C$$

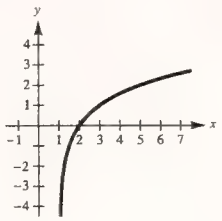
57.
$$y = e^x(a \cos 3x + b \sin 3x)$$
$$y' = e^x(-3a \sin 3x + 3b \cos 3x) + e^x(a \cos 3x + b \sin 3x)$$
$$= e^x[(-3a + b) \sin 3x + (a + 3b) \cos 3x]$$
$$y'' = e^x[3(-3a + b) \cos 3x - 3(a + 3b) \sin 3x] + e^x[(-3a + b) \sin 3x + (a + 3b) \cos 3x]$$
$$= e^x[(-6a - 8b) \sin 3x + (-8a + 6b) \cos 3x]$$
$$y'' - 2y' + 10y = e^x\{[(-6a - 8b) - 2(-3a + b) + 10b] \sin 3x + [(-8a + 6b) - 2(a + 3b) + 10a] \cos 3x\} = 0$$

59. Area $= \displaystyle\int_0^4 xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^4 = -\frac{1}{2}(e^{-16} - 1) \approx 0.500$

61. $y = 3^{3/2}$



63. $y = \log_2(x - 1)$



65. $f(x) = 3^{x-1}$

$f'(x) = 3^{x-1} \ln 3$

67. $y = x^{2x+1}$

$\ln y = (2x + 1) \ln x$

$\dfrac{y'}{y} = \dfrac{2x + 1}{x} + 2 \ln x$

$y' = y\left(\dfrac{2x + 1}{x} + 2 \ln x\right) = x^{2x+1}\left(\dfrac{2x + 1}{x} + 2 \ln x\right)$

69. $g(x) = \log_3\sqrt{1 - x} = \dfrac{1}{2}\log_3(1 - x)$

$g'(x) = \dfrac{1}{2}\dfrac{-1}{(1 - x)\ln 3} = \dfrac{1}{2(x - 1)\ln 3}$

71. $\displaystyle\int (x + 1)5^{(x+1)^2}\,dx = \frac{1}{2}\frac{1}{\ln 5}5^{(x+1)^2} + C$

73. (a) $y = x^a$

$y' = ax^{a-1}$

(b) $y = a^x$

$y' = (\ln a)a^x$

(c) $y = x^x$

$\ln y = x \ln x$

$\dfrac{1}{y}y' = x \cdot \dfrac{1}{x} + (1) \ln x$

$y' = y(1 + \ln x)$

$y' = x^x(1 + \ln x)$

(d) $y = a^a$

$y' = 0$

75. $10{,}000 = Pe^{(0.07)(15)}$

$P = \dfrac{10{,}000}{e^{1.05}} \approx \$3499.38$

77. $P(h) = 30e^{kh}$

$P(18{,}000) = 30e^{18{,}000k} = 15$

$k = \dfrac{\ln(1/2)}{18{,}000} = \dfrac{-\ln 2}{18{,}000}$

$P(h) = 30e^{-(h \ln 2)/18{,}000}$

$P(35{,}000) = 30e^{-(35{,}000 \ln 2)/18{,}000} \approx 7.79$ inches

79. $P = Ce^{0.015t}$

$2C = Ce^{0.015t}$

$2 = e^{0.015t}$

$\ln 2 = 0.015t$

$t = \dfrac{\ln 2}{0.015} \approx 46.21$ years

81. $\dfrac{dy}{dx} = \dfrac{x^2 + 3}{x}$

$\displaystyle\int dy = \int\left(x + \frac{3}{x}\right)dx$

$y = \dfrac{x^2}{2} + 3 \ln|x| + C$

83. $y' - 2xy = 0$

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{1}{y}\,dy = \int 2x\,dx$$

$$\ln|y| = x^2 + C_1$$

$$e^{x^2 + C_1} = y$$

$$y = Ce^{x^2}$$

85. $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2xy}$ (homogeneous differential equation)

$$(x^2 + y^2)\,dx - 2xy\,dy = 0$$

Let $y = vx$, $dy = x\,dv + v\,dx$.

$$(x^2 + v^2x^2)\,dx - 2x(vx)(x\,dv + v\,dx) = 0$$

$$(x^2 + v^2x^2 - 2x^2v^2)\,dx - 2x^3v\,dv = 0$$

$$(x^2 - x^2v^2)\,dx = 2x^3v\,dv$$

$$(1 - v^2)\,dx = 2x\,dv$$

$$\int \frac{dx}{x} = \int \frac{2v}{1 - v^2}\,dv$$

$$\ln|x| = -\ln|1 - v^2| + C_1 = -\ln|1 - v^2| + \ln C$$

$$x = \frac{C}{1 - v^2} = \frac{C}{1 - (y/x)^2} = \frac{Cx^2}{x^2 - y^2}$$

$$1 = \frac{Cx}{x^2 - y^2} \quad \text{or} \quad C_1 = \frac{x}{x^2 - y^2}$$

87. $y = C_1x + C_2x^3$

$y' = C_1 + 3C_2x^2$

$y'' = 6C_2x$

$$x^2y'' - 3xy' + 3y = x^2(6C_2x) - 3x(C_1 + 3C_2x^2) + (C_1x = C_2x^3)$$

$$= 6C_2x^3 - 3C_1x - 9C_2x^3 + 3C_1x + 3C_2x^3 = 0$$

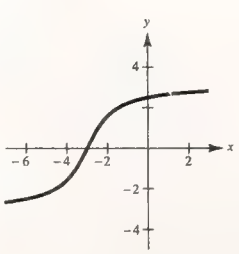
$x = 2, y = 0$: $0 = 2C_1 + 8C_2 \Rightarrow C_1 = -4C_2$

$x = 2, y' = 4$: $4 = C_1 + 12C_2$

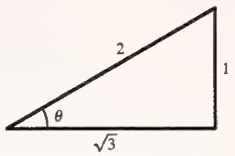
$$4 = (-4C_2) + 12C_2 = 8C_2 \Rightarrow C_2 = \frac{1}{2}, C_1 = -2$$

$$y = -2x + \frac{1}{2}x^3$$

89. $f(x) = 2\arctan(x + 3)$



91. (a) Let $\theta = \arcsin\frac{1}{2}$



$$\sin\theta = \frac{1}{2}$$

$$\sin\left(\arcsin\frac{1}{2}\right) = \sin\theta = \frac{1}{2}.$$

(b) Let $\theta = \arcsin\frac{1}{2}$

$$\sin\theta = \frac{1}{2}$$

$$\cos\left(\arcsin\frac{1}{2}\right) = \cos\theta = \frac{\sqrt{3}}{2}.$$

93. $y = \tan(\arcsin x) = \dfrac{x}{\sqrt{1-x^2}}$

$$y' = \frac{(1-x^2)^{1/2} + x^2(1-x^2)^{-1/2}}{1-x^2} = (1-x^2)^{-3/2}$$

95. $y = x\,\text{arcsec}\,x$

$$y' = \frac{x}{|x|\sqrt{x^2-1}} + \text{arcsec}\,x$$

97. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2}\arcsin x$

$$y' = \frac{2x\arcsin x}{\sqrt{1-x^2}} + (\arcsin x)^2 - 2 + \frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}} - \frac{2x}{\sqrt{1-x^2}}\arcsin x = (\arcsin x)^2$$

99. Let $u = e^{2x}$, $du = 2e^{2x}\,dx$.

$$\int \frac{1}{e^{2x}+e^{-2x}}\,dx = \int \frac{e^{2x}}{1+e^{4x}}\,dx = \frac{1}{2}\int \frac{1}{1+(e^{2x})^2}(2e^{2x})\,dx = \frac{1}{2}\arctan(e^{2x}) + C$$

101. Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1-(x^2)^2}}(2x)\,dx = \frac{1}{2}\arcsin x^2 + C$$

103. Let $u = 16 + x^2$, $du = 2x\,dx$.

$$\int \frac{x}{16+x^2}\,dx = \frac{1}{2}\int \frac{1}{16+x^2}(2x)\,dx = \frac{1}{2}\ln(16+x^2) + C$$

105. Let $u = \arctan\left(\frac{x}{2}\right)$, $du = \dfrac{2}{4+x^2}\,dx$.

$$\int \frac{\arctan(x/2)}{4+x^2}\,dx = \frac{1}{2}\int \left(\arctan\frac{x}{2}\right)\left(\frac{2}{4+x^2}\right)dx = \frac{1}{4}\left(\arctan\frac{x}{2}\right)^2 + C$$

107. $\displaystyle\int \frac{dy}{\sqrt{A^2-y^2}} = \int \sqrt{\frac{k}{m}}\,dt$

$$\arcsin\left(\frac{y}{A}\right) = \sqrt{\frac{k}{m}}\,t + C$$

Since $y = 0$ when $t = 0$, you have $C = 0$. Thus,

$$\sin\left(\sqrt{\frac{k}{m}}\,t\right) = \frac{y}{A}$$

$$y = A\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

109. $y = 2x - \cosh\sqrt{x}$

$$y' = 2 - \frac{1}{2\sqrt{x}}(\sinh\sqrt{x}) = 2 - \frac{\sinh\sqrt{x}}{2\sqrt{x}}$$

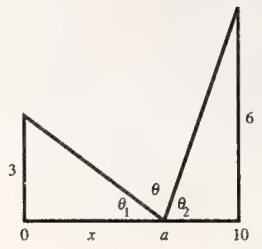
111. Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{x}{\sqrt{x^4-1}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{(x^2)^2-1}}(2x)\,dx = \frac{1}{2}\ln\left(x^2+\sqrt{x^4-1}\right) + C$$

Problem Solving for Chapter 5

1. $\tan\theta_1 = \dfrac{3}{x}$

$\tan\theta_2 = \dfrac{6}{10-x}$



Minimize $\theta_1 + \theta_2$:

$$f(x) = \theta_1 + \theta_2 = \arctan\left(\frac{3}{x}\right) + \arctan\left(\frac{6}{10-x}\right)$$

$$f'(x) = \frac{1}{1+\dfrac{9}{x^2}}\left(\frac{-3}{x^2}\right) + \frac{1}{1+\dfrac{36}{(10-x)^2}}\left(\frac{6}{(10-x)^2}\right) = 0$$

$$\frac{3}{x^2+9} = \frac{6}{(10-x)^2+36}$$

$$(10-x)^2 + 36 = 2(x^2+9)$$

$$100 - 20x + x^2 + 36 = 2x^2 + 18$$

$$x^2 + 20x - 118 = 0$$

$$x = \frac{-20 \pm \sqrt{20^2 - 4(-118)}}{2} = -10 \pm \sqrt{218}$$

$a = -10 + \sqrt{218} \approx 4.7648 \quad f(a) \approx 1.4153$

$\theta = \pi - (\theta_1 + \theta_2) \approx 1.7263$ or $98.9°$

Endpoints: $a = 0$: $\theta \approx 1.0304$

$a = 10$: $\theta \approx 1.2793$

Maximum is 1.7263 at $a = -10 + \sqrt{218} \approx 4.7648$.

3. $f(x) = \sin(\ln x)$

(a) Domain: $x > 0$ or $(0, \infty)$

(b) $f(x) = 1 = \sin(\ln x) \implies \ln x = \dfrac{\pi}{2} + 2k\pi$.

Two values are $x = e^{\pi/2}, e^{(\pi/2)+2\pi}$.

(c) $f(x) = -1 = \sin(\ln x) \implies \ln x = \dfrac{3\pi}{2} + 2k\pi$.

Two values are $x = e^{-\pi/2}, e^{3\pi/2}$.

(d) Since the range of the sine function is $[-1, 1]$, parts (b) and (c) show that the range of f is $[-1, 1]$.

(e) $f'(x) = \dfrac{1}{x}\cos(\ln x)$

$f'(x) = 0 \implies \cos(\ln x) = 0 \implies \ln x = \dfrac{\pi}{2} + k\pi \implies$

$x = e^{\pi/2}$ on $[1, 10]$

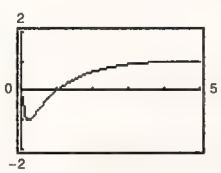
$f(e^{\pi/2}) = 1$

$f(1) = 0$

$f(10) \approx 0.7440$

} Maximum is 1 at $x = e^{\pi/2} \approx 4.8105$

(f)



$\lim_{x\to 0^+} f(x)$ seems to be $-\dfrac{1}{2}$. (This in incorrect.)

(g) For the points $x = e^{\pi/2}, e^{-3\pi/2}, e^{-7\pi/2}, \ldots$

we have $f(x) = 1$.

For the points $x = e^{-\pi/2}, e^{-5\pi/2}, e^{-9\pi/2}, \ldots$

we have $f(x) = -1$.

That is, as $x \to 0^+$, there is an infinite number of points where $f(x) = 1$, and an infinite number where $f(x) = -1$. Thus $\lim_{x\to 0+} \sin(\ln x)$ does not exist.

You can verifiy this by graphing $f(x)$ on small intervals close to the origin.

5. (a) $\dfrac{\text{Area sector}}{\text{Area circle}} = \dfrac{t}{2\pi} \Rightarrow \text{Area sector} = \dfrac{t}{2\pi}(\pi) = \dfrac{t}{2}$

(b) $\text{Area } AOP = \dfrac{1}{2}(\text{base})(\text{height}) - \displaystyle\int_1^{\cosh t} \sqrt{x^2 - 1}\, dx$

$$A(t) = \frac{1}{2}\cosh t \cdot \sinh t - \int_1^{\cosh t} \sqrt{x^2 - 1}\, dx$$

$$A'(t) = \frac{1}{2}[\cosh^2 t + \sinh^2 t] - \sqrt{\cosh^2 t - 1}\sinh t$$

$$= \frac{1}{2}[\cosh^2 t + \sinh^2 \mathrm{t}] - \sinh^2 t$$

$$= \frac{1}{2}[\cosh^2 t - \sinh^2 \mathrm{t}] = \frac{1}{2}$$

$A(t) = \dfrac{1}{2}t + C$. But, $A(0) = C = 0 \Rightarrow C = 0$

Thus, $A(t) = \dfrac{1}{2}t$ or $t = 2A(t)$.

7. $y = \ln x$

$$y' = \frac{1}{x}$$

$$y - b = \frac{1}{a}(x - a)$$

$y = \dfrac{1}{a}x + b - 1$ Tangent line

If $x = 0$, $c = b - 1$. Thus, $b - c = b - (b - 1) = 1$.

9. Let $u = 1 + \sqrt{x}$, $\sqrt{x} = u - 1$, $x = u^2 - 2u + 1$,
$dx = (2u - 2)du$.

$$\text{Area} = \int_1^4 \frac{1}{\sqrt{x} + x}\, dx = \int_2^3 \frac{2u - 2}{(u - 1) + (u^2 - 2u + 1)}\, du$$

$$= \int_2^3 \frac{2(u - 1)}{u^2 - u}\, du$$

$$= \int_2^3 \frac{2}{u}\, du$$

$$= \Big[2 \ln u\Big]_2^3$$

$$= 2 \ln 3 - 2 \ln 2 = 2 \ln\left(\frac{3}{2}\right)$$

$$\approx 0.8109$$

11. (a) $\dfrac{dy}{dt} = y^{1.01}$

$$\int y^{-1.01}\, dy = \int dt$$

$$\frac{y^{-0.01}}{-0.01} = t + C_1$$

$$\frac{1}{y^{0.01}} = -0.01t + C$$

$$y^{0.01} = \frac{1}{C - 0.01t}$$

$$y = \frac{1}{(C - 0.01t)^{100}}$$

$y(0) = 1$: $1 = \dfrac{1}{C^{100}} \Rightarrow C = 1$

Hence, $y = \dfrac{1}{(1 - 0.01t)^{100}}$.

For $T = 100$, $\lim_{t \to T^-} y = \infty$.

(b) $\displaystyle\int y^{-(1+\varepsilon)}\, dy = \int k\, dt$

$$\frac{y^{-\varepsilon}}{-\varepsilon} = kt + C_1$$

$$y^{-\varepsilon} = -\varepsilon kt + C$$

$$y = \frac{1}{(C - \varepsilon kt)^{1/\varepsilon}}$$

$y(0) = y_0 = \dfrac{1}{C^{1/\varepsilon}} \Rightarrow C^{1/\varepsilon} = \dfrac{1}{y_0} \Rightarrow C = \left(\dfrac{1}{y_0}\right)^{\varepsilon}$

Hence, $y = \dfrac{1}{\left(\dfrac{1}{y_0^{\varepsilon}} - \varepsilon kt\right)^{1/\varepsilon}}$.

For $t \to \dfrac{1}{y_0^{\varepsilon}\varepsilon k}$, $y \to \infty$.

13. Since $\frac{dy}{dt} = k(y - 20)$,

$$\int \frac{1}{y - 20}\,dy = \int k\,dt$$

$$\ln(y - 20) = kt + C$$

$$y = Ce^{kt} + 20.$$

When $t = 0$, $y = 72$. Therefore, $C = 52$.

When $t = 1$, $y = 48$. Therefore, $48 = 52e^k + 20$, $e^k = (28/52) = (7/13)$, and $k = \ln(7/13)$.
Thus, $y = 52e^{[\ln(7/13)]t} + 20$.

When $t = 5$, $y = 52e^{5\ln(7/13)} + 20 \approx 22.35°$.

15. (a) $\frac{dS}{dt} = k_1 S(L - S)$

$S = \frac{L}{1 + Ce^{-kt}}$ is a solution because

$$\frac{dS}{dt} = -L(1 + Ce^{-kt})^{-2}(-C\,ke^{-kt})$$

$$= \frac{LC\,ke^{-kt}}{(1 + Ce^{-kt})^2}$$

$$= \left(\frac{k}{L}\right)\frac{L}{1 + Ce^{-kt}} \cdot \frac{C\,Le^{-kt}}{1 + Ce^{-kt}}$$

$$= \left(\frac{k}{L}\right)\frac{L}{1 + Ce^{-kt}} \cdot \left(L - \frac{L}{1 + Ce^{-kt}}\right)$$

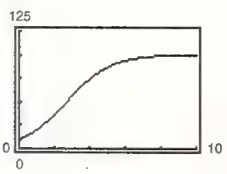
$$= k_1 S(L - S), \text{ where } k_1 = \frac{k}{L}.$$

$L = 100$. Also, $S = 10$ when $t = 0 \Rightarrow C = 9$. And,
$S = 20$ when $t = 1 \Rightarrow k = -\ln(4/9)$.

Particular Solution. $S = \frac{100}{1 + 9e^{\ln(4/9)t}}$

$$= \frac{100}{1 + 9e^{-0.8109t}}$$

(c)



(b) $\frac{dS}{dt} = \ln\left(\frac{4}{9}\right)S(100 - S)$

$$\frac{d^2S}{dt^2} = \ln\left(\frac{4}{9}\right)\left[S\left(-\frac{dS}{dt}\right) + (100 - S)\frac{dS}{dt}\right]$$

$$= \ln\left(\frac{4}{9}\right)(100 - 2S)\frac{dS}{dt}$$

$= 0$ when $S = 50$ or $\frac{dS}{dt} = 0$.

Choosing $S = 50$, we have:

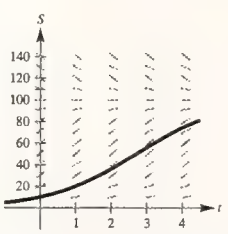
$$50 = \frac{100}{1 + 9e^{\ln(4/9)t}}$$

$$2 = 1 + 9e^{\ln(4/9)t}$$

$$\frac{\ln(1/9)}{\ln(4/9)} = t$$

$$t \approx 2.7 \text{ months}$$

(d)



(e) Sales will decrease toward the line $S = L$.

PART II

CHAPTER P
Preparation for Calculus

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CHAPTER P
Preparation for Calculus

Section P.1 Graphs and Models

Solutions to Even-Numbered Exercises

2. $y = \sqrt{9 - x^2}$

x-intercepts: $(-3, 0), (3, 0)$

y-intercept: $(0, 3)$

Matches graph (d)

4. $y = x^3 - x$

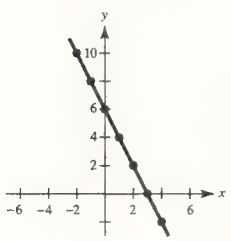
x-intercepts: $(0, 0), (-1, 0), (1, 0)$

y-intercept: $(0, 0)$

Matches graph (c)

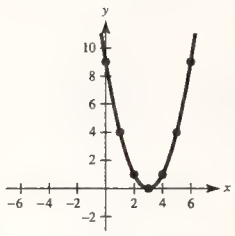
6. $y = 6 - 2x$

x	-2	-1	0	1	2	3	4
y	10	8	6	4	2	0	-2



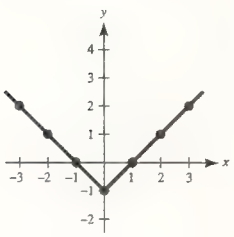
8. $y = (x - 3)^2$

x	0	1	2	3	4	5	6
y	9	4	1	0	1	4	9



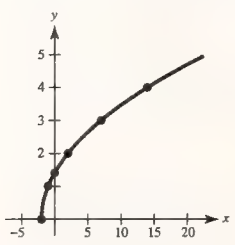
10. $y = |x| - 1$

x	-3	-2	-1	0	1	2	3
y	2	1	0	-1	0	1	2



12. $y = \sqrt{x + 2}$

x	-2	-1	0	2	7	14
y	0	1	$\sqrt{2}$	2	3	4

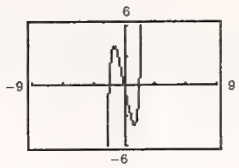


14.

```
Xmin = -30
Xmax = 30
Xscl = 5
Ymin = -10
Ymax = 50
Yscl = 5
```

Note that $y = 10$ when $x = 0$ or $x = 10$.

16.



(a) $(-0.5, y) = (-0.5, 2.47)$

(b) $(x, -4) = (-1.65, -4)$ and $(x, -4) = (1, -4)$

18. $y^2 = x^3 - 4x$

y-intercept: $y^2 = 0^3 - 4(0)$

$y = 0; (0, 0)$

x-intercepts: $0 = x^3 - 4x$

$0 = x(x - 2)(x + 2)$

$x = 0, \pm 2; (0, 0), (\pm 2, 0)$

20. $y = (x - 1)\sqrt{x^2 + 1}$

y-intercept: $y = (0 - 1)\sqrt{0^2 + 1}$

$y = -1; (0, -1)$

x-intercepts: $0 = (x - 1)\sqrt{x^2 + 1}$

$x = 1; (1, 0)$

22. $y = \dfrac{x^2 + 3x}{(3x + 1)^2}$

y-intercept: $y = \dfrac{0^2 + 3(0)}{[3(0) + 1]^2}$

$y = 0; (0, 0)$

x-intercepts: $0 = \dfrac{x^2 + 3x}{(3x + 1)^2}$

$0 = \dfrac{x(x + 3)}{(3x + 1)^2}$

$x = 0, -3; (0, 0), (-3, 0)$

24. $y = 2x - \sqrt{x^2 + 1}$

y-intercept: $y = 2(0) - \sqrt{0^2 + 1}$

$y = -1; (0, -1)$

x-intercepts: $0 = 2x - \sqrt{x^2 + 1}$

$2x = \sqrt{x^2 + 1}$

$4x^2 = x^2 + 1$

$3x^2 = 1$

$x^2 = \dfrac{1}{3}$

$x = \pm\dfrac{\sqrt{3}}{3}$

$x = \dfrac{\sqrt{3}}{3}; \left(\dfrac{\sqrt{3}}{3}, 0\right)$

Note: $x = -\sqrt{3}/3$ is an extraneous solution.

26. $y = x^2 - x$

No symmetry with respect to either axis or the origin.

28. Symmetric with respect to the origin since

$(-y) = (-x)^3 + (-x)$

$-y = -x^3 - x$

$y = x^3 + x.$

30. Symmetric with respect to the x-axis since

$x(-y)^2 = xy^2 = -10.$

32. Symmetric with respect to the origin since

$(-x)(-y) - \sqrt{4 - (-x)^2} = 0$

$xy - \sqrt{4 - x^2} = 0.$

34. $y = \dfrac{x^2}{x^2 + 1}$ is symmetric with respect to the y-axis

since $y = \dfrac{(-x)^2}{(-x)^2 + 1} = \dfrac{x^2}{x^2 + 1}$.

36. $|y| - x = 3$ is symmetric with respect to the x-axis

since $|-y| - x = 3$

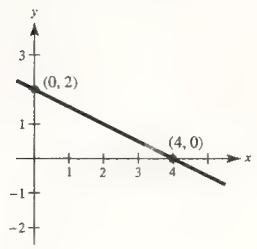
$|y| - x = 3$.

38. $y = -\dfrac{x}{2} + 2$

Intercepts:

$(4, 0), (0, 2)$

Symmetry: none

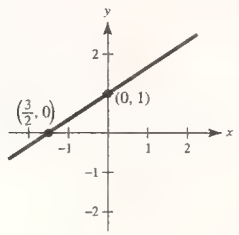


40. $y = \frac{2}{3}x + 1$

Intercepts:

$(0, 1), \left(-\frac{3}{2}, 0\right)$

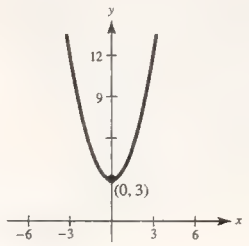
Symmetry: none



42. $y = x^2 + 3$

Intercept: $(0, 3)$

Symmetry: y-axis

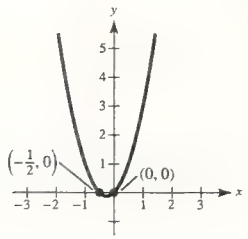


44. $y = 2x^2 + x = x(2x + 1)$

Intercepts:

$(0, 0), \left(-\frac{1}{2}, 0\right)$

Symmetry: none

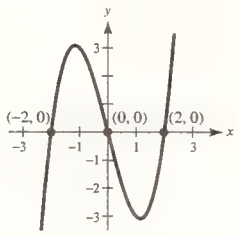


46. $y = x^3 - 4x$

Intercepts:

$(0, 0), (2, 0), (-2, 0)$

Symmetry: origin



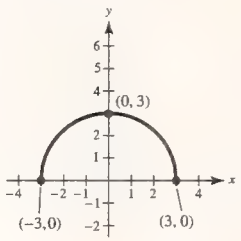
48. $y = \sqrt{9 - x^2}$

Intercepts:

$(-3, 0), (3, 0), (0, 3)$

Symmetry: y-axis

Domain: $[-3, 3]$

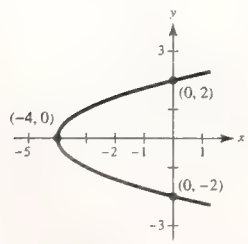


50. $x = y^2 - 4$

Intercepts:

$(0, 2), (0, -2), (-4, 0)$

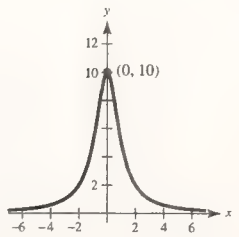
Symmetry: x-axis



52. $y = \dfrac{10}{x^2 + 1}$

Intercepts: $(0, 10)$

Symmetry: y-axis

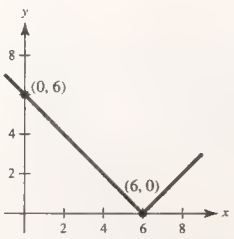


54. $y = |6 - x|$

Intercepts:

$(0, 6), (6, 0)$

Symmetry: none



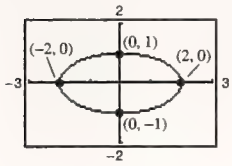
56. $x^2 + 4y^2 = 4 \Rightarrow y = \pm\dfrac{\sqrt{4 - x^2}}{2}$

Intercepts:

$(-2, 0), (2, 0), (0, -1), (0, 1)$

Symmetry: origin and both axes

Domain: $[-2, 2]$



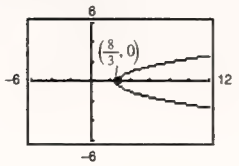
58. $3x - 4y^2 = 8$

$4y^2 = 3x - 8$

$y = \pm\sqrt{\frac{3}{4}x - 2}$

Intercept: $\left(\frac{8}{3}, 0\right)$

Symmetry: x-axis



60. $y = \left(x + \frac{5}{2}\right)(x - 2)\left(x - \frac{3}{2}\right)$ (other answers possible)

62. Some possible equations:

$x = y^2$

$x = |y|$

$x = y^4 + 1$

$x^2 + y^2 = 25$

64. $2x - 3y = 13 \Rightarrow y = \dfrac{2x - 13}{3}$

$5x + 3y = 1 \Rightarrow y = \dfrac{1 - 5x}{3}$

$\dfrac{2x - 13}{3} = \dfrac{1 - 5x}{3}$

$2x - 13 = 1 - 5x$

$7x = 14$

$x = 2$

The corresponding y-value is $y = -3$.

Point of intersection: $(2, -3)$

66. $5x - 6y = 9 \Rightarrow y = \dfrac{5x - 9}{6}$

$-7x + 3y = -18 \Rightarrow y = \dfrac{7x - 18}{3}$

$\dfrac{5x - 9}{6} = \dfrac{7x - 18}{3}$

$5x - 9 = 14x - 36$

$27 = 9x$

$x = 3$

The corresponding y-value is $y = 1$.

Point of intersection: $(3, 1)$

68. $x = 3 - y^2 \Rightarrow y^2 = 3 - x$

$y = x - 1$

$3 - x = (x - 1)^2$

$3 - x = x^2 - 2x + 1$

$0 = x^2 - x - 2 = (x + 1)(x - 2)$

$x = -1$ or $x = 2$

The corresponding y-values are $y = -2$ and $y = 1$.

Points of intersection: $(-1, -2), (2, 1)$

70. $x^2 + y^2 = 25 \Rightarrow y^2 = 25 - x^2$

$2x + y = 10 \Rightarrow y = 10 - 2x$

$25 - x^2 = (10 - 2x)^2$

$25 - x^2 = 100 - 40x + 4x^2$

$0 = 5x^2 - 40x + 75 = 5(x - 3)(x - 5)$

$x = 3$ or $x = 5$

The corresponding y-values are $y = 4$ and $y = 0$.

Points of intersection: $(3, 4), (5, 0)$

72. $y = x^3 - 4x$

$y = -(x + 2)$

$x^3 - 4x = -(x + 2)$

$x^3 - 3x + 2 = 0$

$(x - 1)^2(x + 2) = 0$

$x = 1 \quad \text{or} \quad x = -2$

The corresponding y-values are $y = -3$ and $y = 0$.

Points of intersection: $(1, -3), (-2, 0)$

74. $y = x^4 - 2x^2 + 1$

$y = 1 - x^2$

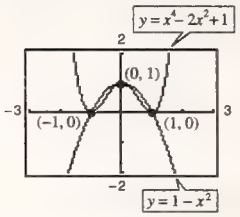
$1 - x^2 = x^4 - 2x^2 + 1$

$0 = x^4 - x^2$

$0 = x^2(x + 1)(x - 1)$

$x = -1, 0, 1$

$(-1, 0), (0, 1), (1, 0)$



76. $y = kx + 5$ matches (b).

Use $(1, 7)$: $7 = k(1) + 5 \Rightarrow k = 2$, thus, $y = 2x + 5$.

$y = x^2 + k$ matches (d).

Use $(1, -9)$: $-9 = (1)^2 + k \Rightarrow k = -10$, thus, $y = x^2 - 10$.

$y = kx^{3/2}$ matches (a).

Use $(1, 3)$: $3 = k(1)^{3/2} \Rightarrow k = 3$, thus, $y = 3x^{3/2}$.

$xy = k$ matches (c).

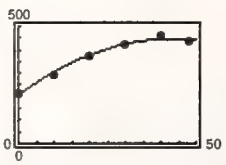
Use $(1, 36)$: $(1)(36) = k \Rightarrow k = 36$, thus, $xy = 36$.

78. (a) Using a graphing utility, you obtain

$y = -0.1283t^2 + 11.0988t + 207.1116$

(c) For the year 2004, $t = 54$ and $y \approx 432.3$ acres per farm.

(b)



80. (a) If (x, y) is on the graph, then so is $(-x, y)$ by y-axis symmetry. Since $(-x, y)$ is on the graph, then so is $(-x, -y)$ by x-axis symmetry. Hence, the graph is symmetric with respect to the origin. The converse is not true. For example, $y = x^3$ has origin symmetry but is not symmetric with respect to either the x-axis or the y-axis.

(b) Assume that the graph has x-axis and origin symmetry. If (x, y) is on the graph, so is $(x, -y)$ by x-axis symmetry. Since $(x, -y)$ is on the graph, then so is $(-x, -(-y)) = (-x, y)$ by origin symmetry. Therefore, the graph is symmetric with respect to the y-axis. The argument is similar for y-axis and origin symmetry.

82. True

84. True; the x-intercept is

$$\left(-\frac{b}{2a}, 0\right).$$

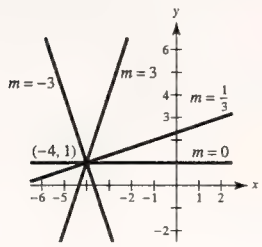
Section P.2 Linear Models and Rates of Change

2. $m = 2$

4. $m = -1$

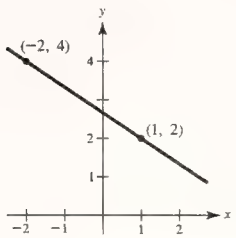
6. $m = \frac{40}{3}$

8.

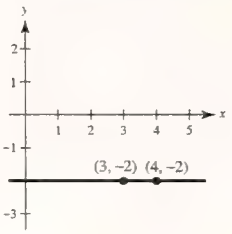


10. $m = \dfrac{4-2}{-2-1}$

$= -\dfrac{2}{3}$

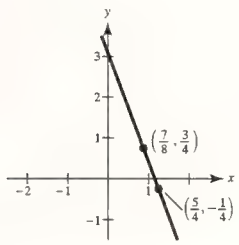


12. $m = \dfrac{-2-(-2)}{4-1} = 0$



14. $m = \dfrac{(3/4)-(-1/4)}{(7/8)-(5/4)}$

$= \dfrac{1}{-3/8} = -\dfrac{8}{3}$



16. Since the slope is undefined, the line is vertical and its equation is $x = -3$. Therefore, three additional points are $(-3, 2)$, $(-3, 3)$, and $(-3, 5)$.

18. The equation of this line is

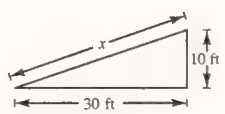
$$y + 2 = 2(x + 2)$$

$$y = 2x + 2.$$

Therefore, three additional points are $(-3, -4)$, $(-1, 0)$, and $(0, 2)$.

20. (a) Slope $= \dfrac{\Delta y}{\Delta x} = \dfrac{1}{3}$

(b)



By the Pythagorean Theorem,

$$x^2 = 30^2 + 10^2 = 1000$$

$$x \approx 31.623 \text{ feet.}$$

22. (a) $m = 400$ indicates that the revenues increase by 400 in one day.

(b) $m = 100$ indicates that the revenues increase by 100 in one day.

(c) $m = 0$ indicates that the revenues do not change from one day to the next.

24. $6x - 5y = 15$

$y = \frac{6}{5}x - 3$

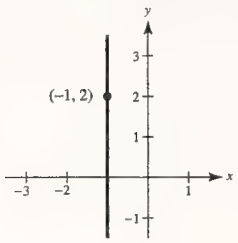
Therefore, the slope is $m = \frac{6}{5}$ and the y-intercept is $(0, -3)$.

26. $y = -1$

The line is horizontal. Therefore, the slope is $m = 0$ and the y-intercept is $(0, -1)$.

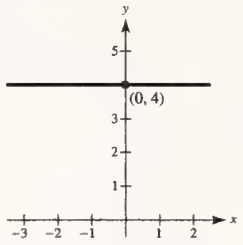
28. $x = -1$

$x + 1 = 0$



30. $y = 4$

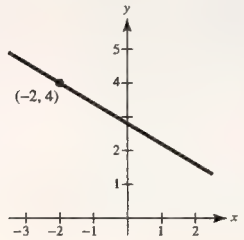
$y - 4 = 0$



32. $y - 4 = -\frac{3}{5}(x + 2)$

$5y - 20 = -3x - 6$

$3x + 5y - 14 = 0$

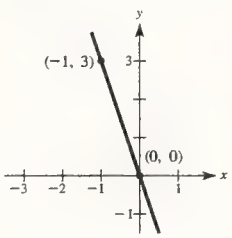


34. $m = \frac{3 - 0}{-1 - 0} = -3$

$y - 0 = -3(x - 0)$

$y = -3x$

$3x + y = 0$

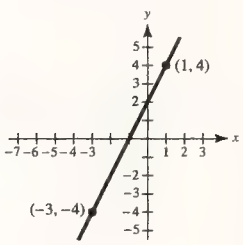


36. $m = \frac{4 - (-4)}{1 - (-3)} = \frac{8}{4} = 2$

$y - 4 = 2(x - 1)$

$y - 4 = 2x - 2$

$0 = 2x - y + 2$

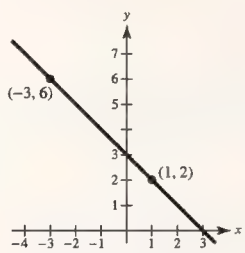


38. $m = \frac{6 - 2}{-3 - 1} = \frac{4}{-4} = -1$

$y - 2 = -1(x - 1)$

$y - 2 = -x + 1$

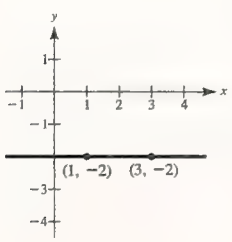
$x + y - 3 = 0$



40. $m = 0$

$y = -2$

$y + 2 = 0$



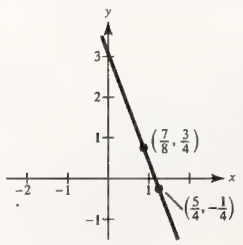
42. $m = \frac{(3/4) - (-1/4)}{(7/8) - (5/4)}$

$= \frac{1}{-3/8} = -\frac{8}{3}$

$y + \frac{1}{4} = \frac{-8}{3}\left(x - \frac{5}{4}\right)$

$12y + 3 = -32x + 40$

$32x + 12y - 37 = 0$

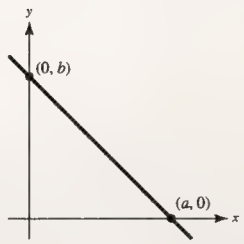


44. $m = -\frac{b}{a}$

$y = \frac{-b}{a}x + b$

$\frac{b}{a}x + y = b$

$\frac{x}{a} + \frac{y}{b} = 1$



46. $\dfrac{x}{-2/3} + \dfrac{y}{-2} = 1$

$$\frac{-3x}{2} - \frac{y}{2} = 1$$

$$3x + y = -2$$

$$3x + y + 2 = 0$$

48. $\dfrac{x}{a} + \dfrac{y}{a} = 1$

$$\frac{-3}{a} + \frac{4}{a} = 1$$

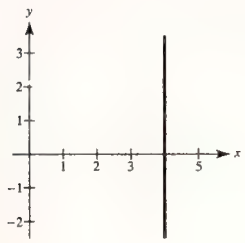
$$\frac{1}{a} = 1$$

$$a = 1 \Rightarrow x + y = 1$$

$$x + y - 1 = 0$$

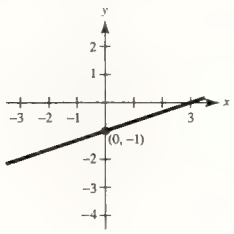
50. $x = 4$

$x - 4 = 0$



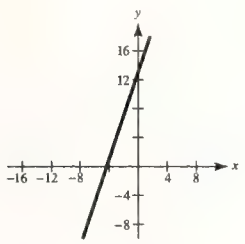
52. $y = \frac{1}{3}x - 1$

$3y - x + 3 = 0$



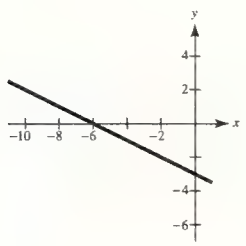
54. $y - 1 = 3(x + 4)$

$y = 3x + 13$

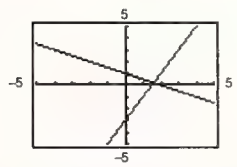


56. $x + 2y + 6 = 0$

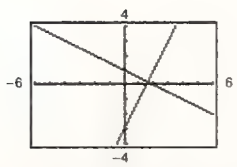
$y = -\frac{1}{2}x - 3$



58.



The lines do not appear perpendicular.



The lines appear perpendicular.

The lines are perpendicular because their slopes 2 and $-\frac{1}{2}$ are negative reciprocals of each other. You must use a square setting in order for perpendicular lines to appear perpendicular.

60. $x + y = 7$

$y = -x + 7$

$m = -1$

(a) $y - 2 = -1(x + 3)$

$y - 2 = -x - 3$

$x + y + 1 = 0$

(b) $y - 2 = 1(x + 3)$

$y - 2 = x + 3$

$x - y + 5 = 0$

62. $3x + 4y = 7$

$y = -\frac{3}{4}x + \frac{7}{4}$

$m = -\frac{3}{4}$

(a) $y - 4 = -\frac{3}{4}(x + 6)$

$4y - 16 = -3x - 18$

$3x + 4y + 2 = 0$

(b) $y - 4 = \frac{4}{3}(x + 6)$

$3y - 12 = 4x + 24$

$4x - 3y + 36 = 0$

64. (a) $y = 0$

(b) $x = -1 \Rightarrow x + 1 = 0$

66. The slope is 4.50.

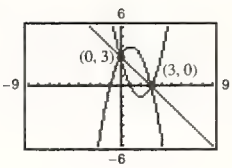
Hence, $V = 4.5(t - 1) + 156$

$= 4.5t + 151.5$

68. The slope is -5600. Hence, $V = -5600(t - 1) + 245{,}000$

$= -5600t + 250{,}600$

70.



You can use the graphing utility to determine that the points of intersection are (0, 3) and (3, 0). Analytically,

$x^2 - 4x + 3 = -x^2 + 2x + 3$

$2x^2 - 6x = 0$

$2x(x - 3) = 0$

$x = 0 \Rightarrow y = 3 \Rightarrow (0, 3)$

$x = 3 \Rightarrow y = 0 \Rightarrow (3, 0).$

The slope of the line joining (0, 3) and (3, 0) is $m = (0 - 3)/(3 - 0) = -1$. Hence, an equation of the line is

$y - 3 = -1(x - 0)$

$y = -x + 3.$

72. $m_1 = \dfrac{-6 - 4}{7 - 0} = -\dfrac{10}{7}$

$m_2 = \dfrac{11 - 4}{-5 - 0} = -\dfrac{7}{5}$

$m_1 \neq m_2$

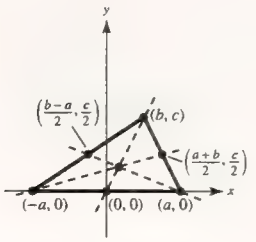
The points are not collinear.

74. Equations of medians:

$$y = \frac{c}{b}x$$

$$y = \frac{c}{3a + b}(x + a)$$

$$y = \frac{c}{-3a + b}(x - a)$$



Solving simultaneously, the point of intersection is

$$\left(\frac{b}{3}, \frac{c}{3}\right).$$

76. The slope of the line segment from $\left(\frac{b}{3}, \frac{c}{3}\right)$ to $\left(b, \frac{a^2 - b^2}{c}\right)$ is:

$$m_1 = \frac{[(a^2 - b^2)/c] - (c/3)}{b - (b/3)} = \frac{(3a^2 - 3b^2 - c^2)/(3c)}{(2b)/3} = \frac{3a^2 - 3b^2 - c^2}{2bc}$$

The slope of the line segment from $\left(\frac{b}{3}, \frac{c}{3}\right)$ to $\left(0, \frac{-a^2 + b^2 + c^2}{2c}\right)$ is:

$$m_2 = \frac{[(-a^2 + b^2 + c^2)/(2c)] - (c/3)}{0 - (b/3)} = \frac{(-3a^2 + 3b^2 + 3c^2 - 2c^2)/(6c)}{-b/3} = \frac{3a^2 - 3b^2 - c^2}{2bc}$$

$m_1 = m_2$

Therefore, the points are collinear.

78. $C = 0.34x + 150$. If $x = 137$, $C = 0.34(137) + 150 = \$196.58$

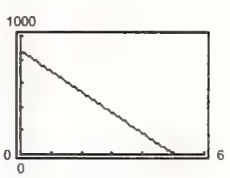
80. (a) Depreciation per year:

$\frac{875}{5} = \$175$

$y = 875 - 175x$

where $0 \le x \le 5$.

(b) $y = 875 - 175(2) = \$525$



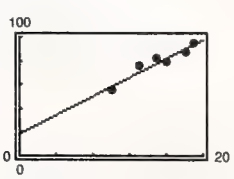
(c) $200 = 875 - 175x$

$175x = 675$

$x \approx 3.86$ years

82. (a) $y = 18.91 + 3.97x$ (x = quiz score, y = test score)

(b)



(c) If $x = 17$, $y = 18.91 + 3.97(17) = 86.4$.

(d) The slope shows the average increase in exam score for each unit increase in quiz score.

(e) The points would shift vertically upward 4 units. The new regression line would have a y-intercept 4 greater than before: $y = 22.91 + 3.97x$.

84. $4x + 3y - 10 = 0 \Rightarrow d = \frac{|4(2) + 3(3) - 10|}{\sqrt{4^2 + 3^2}} = \frac{7}{5}$

86. $x + 1 = 0 \Rightarrow d = \frac{|1(6) + (0)(2) + 1|}{\sqrt{1^2 + 0^2}} = 7$

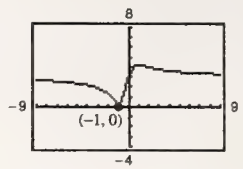
88. A point on the line $3x - 4y = 1$ is $(-1, -1)$. The distance from the point $(-1, -1)$ to $3x - 4y - 10 = 0$ is

$$d = \frac{|-3 + 4 - 10|}{5} = \frac{9}{5}.$$

90. $y = mx + 4 \Rightarrow mx + (-1)y + 4 = 0$

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|m3 + (-1)(1) + 4|}{\sqrt{m^2 + (-1)^2}}$$

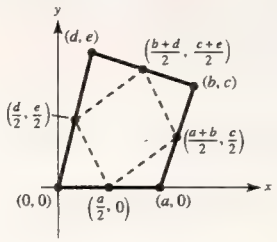
$$= \frac{|3m + 3|}{\sqrt{m^2 + 1}}$$



The distance is 0 when $m = -1$. In this case, the line $y = -x + 4$ contains the point (3, 1).

92. For simplicity, let the vertices of the quadrilateral be $(0, 0)$, $(a, 0)$, (b, c), and (d, e), as shown in the figure. The midpoints of the sides are

$$\left(\frac{a}{2}, 0\right), \left(\frac{a+b}{2}, \frac{c}{2}\right), \left(\frac{b+d}{2}, \frac{c+e}{2}\right), \text{ and } \left(\frac{d}{2}, \frac{e}{2}\right).$$



The slope of the opposite sides are equal:

$$\frac{\frac{c}{2} - 0}{\frac{a+b}{2} - \frac{a}{2}} = \frac{\frac{c+e}{2} - \frac{e}{2}}{\frac{b+d}{2} - \frac{d}{2}} = \frac{c}{b}$$

$$\frac{0 - \frac{e}{2}}{\frac{a}{2} - \frac{d}{2}} = \frac{\frac{c}{2} - \frac{c+e}{2}}{\frac{a+b}{2} - \frac{b+d}{2}} = -\frac{e}{a-d}$$

Therefore, the figure is a paralleogram.

94. If $m_1 = -1/m_2$, then $m_1 m_2 = -1$. Let L_3 be a line with slope m_3 that is perpendicular to L_1. Then $m_1 m_3 = -1$. Hence, $m_2 = m_3 \Rightarrow L_2$ and L_3 are parallel. Therefore, L_2 and L_1 are also perpendicular.

96. False; if m_1 is positive, then $m_2 = -1/m_1$ is negative.

Section P.3 Functions and Their Graphs

2. (a) $f(-2) = \sqrt{-2 + 3} = \sqrt{1} = 1$

(b) $f(6) = \sqrt{6 + 3} = \sqrt{9} = 3$

(c) $f(c) = \sqrt{c + 3}$

(d) $f(x + \Delta x) = \sqrt{x + \Delta x + 3}$

4. (a) $g(4) = 4^2(4 - 4) = 0$

(b) $g\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2\left(\frac{3}{2} - 4\right) = \frac{9}{4}\left(-\frac{5}{2}\right) = -\frac{45}{8}$

(c) $g(c) = c^2(c - 4) = c^3 - 4c^2$

(d) $g(t + 4) = (t + 4)^2(t + 4 - 4)$

$= (t + 4)^2 t = t^3 + 8t^2 + 16t$

6. (a) $f(\pi) = \sin\pi = 0$

(b) $f\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right) = \frac{-\sqrt{2}}{2}$

(c) $f\left(\frac{2\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

8. $\frac{f(x) - f(1)}{x - 1} = \frac{3x - 1 - (3 - 1)}{x - 1} = \frac{3(x - 1)}{x - 1} = 3, x \neq 1$

10. $\frac{f(x) - f(1)}{x - 1} = \frac{x^3 - x - 0}{x - 1} = \frac{x(x + 1)(x - 1)}{x - 1} = x(x + 1), x \neq 1$

12. $g(x) = x^2 - 5$

Domain: $(-\infty, \infty)$

Range: $[-5, \infty)$

14. $h(t) = \cot t$

Domain: all $t \neq k\pi$, k an integer

Range: $(-\infty, \infty)$

16. $g(x) = \dfrac{2}{x-1}$

Domain: $(-\infty, 1), (1, \infty)$

Range: $(-\infty, 0), (0, \infty)$

18. $f(x) = \begin{cases} x^2 + 2, & x \leq 1 \\ 2x^2 + 2, & x > 1 \end{cases}$

(a) $f(-2) = (-2)^2 + 2 = 6$

(b) $f(0) = 0^2 + 2 = 2$

(c) $f(1) = 1^2 + 2 = 3$

(d) $f(s^2 + 2) = 2(s^2 + 2)^2 = 2s^4 + 8s^2 + 10$

(Note: $s^2 + 2 > 1$ for all s)

Domain: $(-\infty, \infty)$

Range: $[2, \infty)$

20. $f(x) = \begin{cases} \sqrt{x+4}, & x \leq 5 \\ (x-5)^2, & x > 5 \end{cases}$

(a) $f(-3) = \sqrt{-3+4} = \sqrt{1} = 1$

(b) $f(0) = \sqrt{0+4} = 2$

(c) $f(5) = \sqrt{5+4} = 3$

(d) $f(10) = (10-5)^2 = 25$

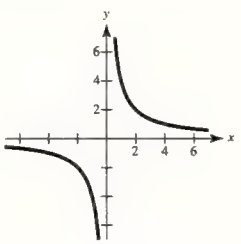
Domain: $[-4, \infty)$

Range: $[0, \infty)$

22. $g(x) = \dfrac{4}{x}$

Domain: $(-\infty, 0), (0, \infty)$

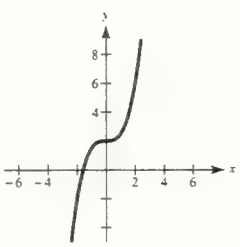
Range: $(-\infty, 0), (0, \infty)$



24. $f(x) = \frac{1}{2}x^3 + 2$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$



26. $f(x) = x + \sqrt{4 - x^2}$

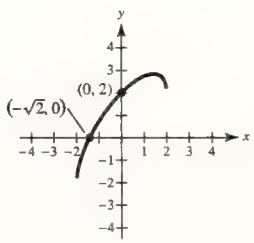
Domain: $[-2, 2]$

Range:

$[-2, 2\sqrt{2}] \approx [-2, 2.83]$

y-intercept: $(0, 2)$

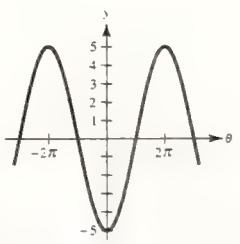
x-intercept: $(-\sqrt{2}, 0)$



28. $h(\theta) = -5\cos\dfrac{\theta}{2}$

Domain: $(-\infty, \infty)$

Range: $[-5, 5]$



30. $\sqrt{x^2 - 4} - y = 0 \Rightarrow y = \sqrt{x^2 - 4}$

y is a function of x. Vertical lines intersect the graph at most once.

32. $x^2 + y^2 = 4$

$y = \pm\sqrt{4 - x^2}$

y is not a function of x. Some vertical lines intersect the graph twice.

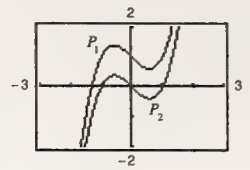
34. $x^2 + y = 4 \Rightarrow y = 4 - x^2$

y is a function of x since there is one value of y for each x.

36. $x^2y - x^2 + 4y = 0 \Rightarrow y = \dfrac{x^2}{x^2 + 4}$

y is a function of x since there is one value of y for each x.

38. $p_1(x) = x^3 - x + 1$ has one zero. $p_2(x) = x^3 - x$ has three zeros. Every cubic polynomial has at least one zero. Given $p(x) = Ax^3 + Bx^2 + Cx + D$, we have $p \to -\infty$ as $x \to -\infty$ and $p \to \infty$ as $x \to \infty$ if $A > 0$. Furthermore, $p \to \infty$ as $x \to -\infty$ and $p \to -\infty$ as $x \to \infty$ if $A < 0$. Since the graph has no breaks, the graph must cross the x-axis at least one time.

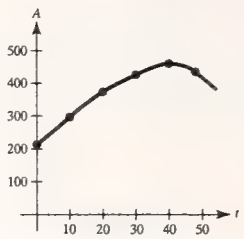


40. The function is $f(x) = cx$. Since $(1, 1/4)$ satisfies the equation, $c = 1/4$. Thus, $f(x) = (1/4)x$.

42. The function is $h(x) = c\sqrt{|x|}$. Since $(1, 3)$ satisfies the equation, $c = 3$. Thus, $h(x) = 3\sqrt{|x|}$.

44. The student travels $\dfrac{2-0}{4-0} = \dfrac{1}{2}$ mi/min during the first 4 minutes. The student is stationary for the following 2 minutes. Finally, the student travels $\dfrac{6-2}{10-6} = 1$ mi/min during the final 4 minutes.

46. (a)



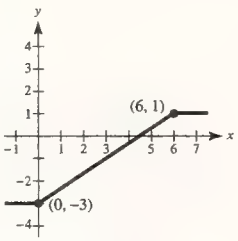
(b) $A(15) \approx 345$ acres/farm

48. (a) $g(x) = f(x - 4)$

$g(6) = f(2) = 1$

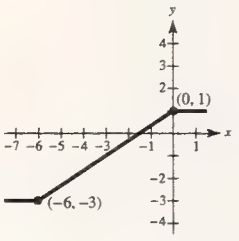
$g(0) = f(-4) = -3$

Shift f right 4 units



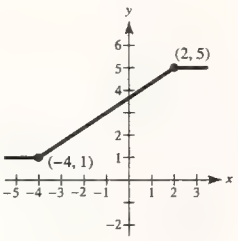
(b) $g(x) = f(x + 2)$

Shift f left 2 units



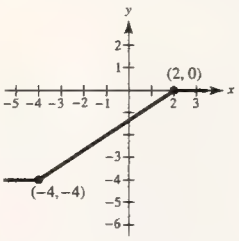
(c) $g(x) = f(x) + 4$

Vertical shift upwards 4 units



(d) $g(x) = f(x) - 1$

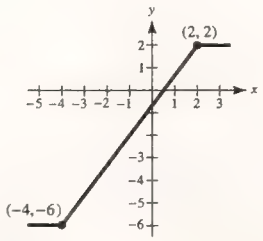
Vertical shift down 1 unit



(e) $g(x) = 2f(x)$

$g(2) = 2f(2) = 2$

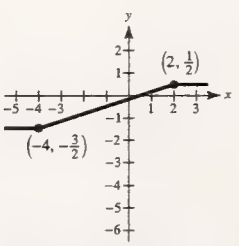
$g(-4) = 2f(-4) = -6$



(f) $g(x) = \frac{1}{2}f(x)$

$g(2) = \frac{1}{2}f(2) = \frac{1}{2}$

$g(-4) = \frac{1}{2}f(-4) = -\frac{3}{2}$



50. (a) $h(x) = \sin(x + (\pi/2)) + 1$ is a horizontal shift $\pi/2$ units to the left, followed by a vertical shift 1 unit upwards.

(b) $h(x) = -\sin(x - 1)$ is a horizontal shift 1 unit to the right followed by a reflection about the x-axis.

52. (a) $f(g(1)) = f(0) = 0$

(b) $g(f(1)) = g(1) = 0$

(c) $g(f(0)) = g(0) = -1$

(d) $f(g(-4)) = f(15) = \sqrt{15}$

(e) $f(g(x)) = f(x^2 - 1) = \sqrt{x^2 - 1}$

(f) $g(f(x)) = g\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^2 - 1 = x - 1 \quad (x \geq 0)$

54. $f(x) = x^2 - 1, g(x) = \cos x$

$(f \circ g)(x) = f(g(x)) = f(\cos x) = \cos^2 x - 1$

Domain: $(-\infty, \infty)$

$(g \circ f)(x) = g(x^2 - 1) = \cos(x^2 - 1)$

Domain: $(-\infty, \infty)$

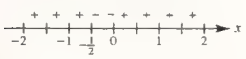
No, $f \circ g \neq g \circ f$.

56. $(f \circ g)(x) = f\left(\sqrt{x + 2}\right) = \dfrac{1}{\sqrt{x + 2}}$

Domain: $(-2, \infty)$

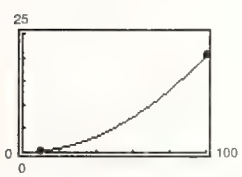
$(g \circ f)(x) = g\left(\dfrac{1}{x}\right) = \sqrt{\dfrac{1}{x} + 2} = \sqrt{\dfrac{1 + 2x}{x}}$

You can find the domain of $g \circ f$ by determining the intervals where $(1 + 2x)$ and x are both positive, or both negative.



Domain: $\left(-\infty, -\frac{1}{2}\right], (0, \infty)$

58. (a)



(b) $H\left(\dfrac{x}{1.6}\right) = 0.002\left(\dfrac{x}{1.6}\right)^2 + 0.005\left(\dfrac{x}{1.6}\right) - 0.029$

$= 0.00078125x^2 + 0.003125x - 0.029$

60. $f(-x) = \sqrt[3]{-x} = -\sqrt[3]{x} = -f(x)$

Odd

62. $f(-x) = \sin^2(-x) = \sin(-x)\sin(-x) = (-\sin x)(-\sin x) = \sin^2 x$

Even

64. (a) If f is even, then $(-4, 9)$ is on the graph.

(b) If f is odd, then $(-4, -9)$ is on the graph.

66. $f(-x) = a_{2n}(-x)^{2n} + a_{2n-2}(-x)^{2n-2} + \cdots + a_2(-x)^2 + a_0$

$= a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdots + a_2x^2 + a_0$

$= f(x)$

Even

68. Let $F(x) = f(x)g(x)$ where f is even and g is odd. Then

$F(-x) = f(-x)g(-x) = f(x)[-g(x)] = -f(x)g(x) = -F(x)$.

Thus, $F(x)$ is odd.

70. (a) Let $F(x) = f(x) \pm g(x)$ where f and g are even. Then, $F(-x) = f(-x) \pm g(-x) = f(x) \pm g(x) = F(x)$. Thus, $F(x)$ is even.

(b) Let $F(x) = f(x) \pm g(x)$ where f and g are odd. Then, $F(-x) = f(-x) \pm g(-x) = -f(x) \mp g(x) = -F(x)$. Thus, $F(x)$ is odd.

(c) Let $F(x) = f(x) \pm g(x)$ where f is odd and g is even. Then, $F(-x) = f(-x) \pm g(-x) = -f(x) \pm g(x)$. Thus, $F(x)$ is neither odd nor even.

72. By equating slopes, $\dfrac{y-2}{0-3} = \dfrac{0-2}{x-3}$

$$y - 2 = \frac{6}{x-3}$$

$$y = \frac{6}{x-3} + 2 = \frac{2x}{x-3},$$

$$L = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(\frac{2x}{x-3}\right)^2}.$$

74. True

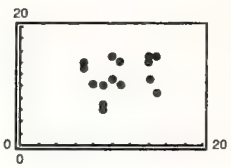
76. False; let $f(x) = x^2$. Then $f(3x) = (3x)^2 = 9x^2$ and $3f(x) = 3x^2$. Thus, $3f(x) \neq f(3x)$.

Section P.4 Fitting Models to Data

2. Trigonometric function

4. No relationship

6. (a)

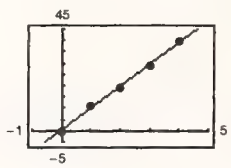


No, the relationship does not appear to be linear.

(b) Quiz scores are dependent on several variables such as study time, class attendance, etc. These variables may change from one quiz to the next.

8. (a) $s = 9.7t + 0.4$

(b)

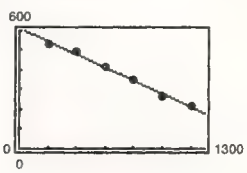


The model fits well.

(c) If $t = 2.5$, $s = 24.65$ meters/second.

10. (a) Linear model: $H = -0.3323t + 612.9333$

(b)



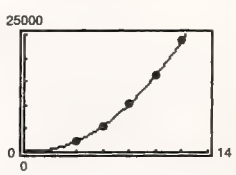
The fit is very good.

(c) When $t = 500$,

$$H = -0.3323(500) + 612.9333 \approx 446.78.$$

12. (a) $S = 180.89x^2 - 205.79x + 272$

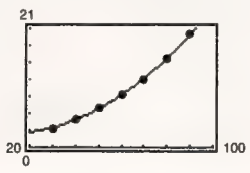
(b)



(c) When $x = 2$, $S \approx 583.98$ pounds.

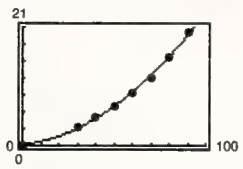
14. (a) $t = 0.00271s^2 - 0.0529s + 2.671$

(b)



(c) The curve levels off for $s < 20$.

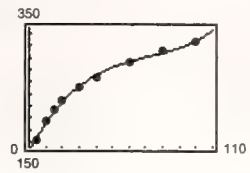
(d) $t = 0.002s^2 + 0.0346s + 0.183$



The model is better for low speeds.

16. (a) $T = 2.9856 \times 10^{-4} p^3 - 0.0641 p^2 + 5.2826p + 143.1$

(b)



(c) For $T = 300°\text{F}$, $p \approx 68.29$ pounds per square inch.

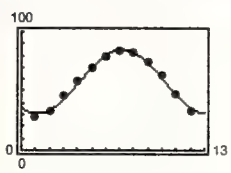
(d) The model is based on data up to 100 pounds per square inch.

18. (a) $H(t) = 84.4 + 4.28 \sin\left(\frac{\pi t}{6} + 3.86\right)$

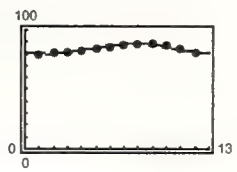
One model is

$$C(t) = 58 + 27 \sin\left(\frac{\pi t}{6} + 4.1\right).$$

(b)



(c)



(d) The average in Honolulu is 84.4.

The average in Chicago is 58.

(e) The period is 12 months (1 year).

(f) Chicago has greater variability ($27 > 4.28$).

20. Answers will vary.

Review Exercises for Chapter P

2. $y = (x - 1)(x - 3)$

$x = 0 \Rightarrow y = (0 - 1)(0 - 3) = 3 \Rightarrow (0, 3)$ y-intercept

$y = 0 \Rightarrow 0 = (x - 1)(x - 3) \Rightarrow x = 1, 3 \Rightarrow (1, 0), (3, 0)$ x-intercepts

4. $xy = 4$

$x = 0$ and $y = 0$ are both impossible. No intercepts.

6. Symmetric with respect to y-axis since

$y = (-x)^4 - (-x)^2 + 3$

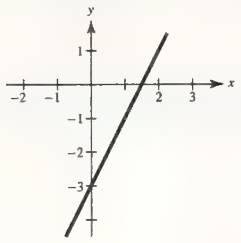
$y = x^4 - x^2 + 3.$

8. $4x - 2y = 6$

$y = 2x - 3$

Slope: 2

y-intercept: -3



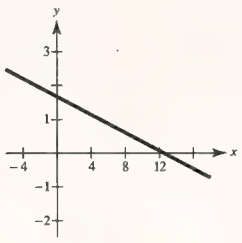
10. $0.02x + 0.15y = 0.25$

$2x + 15y = 25$

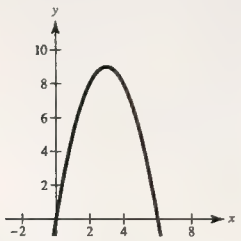
$y = -\frac{2}{15}x + \frac{5}{3}$

Slope: $-\frac{2}{15}$

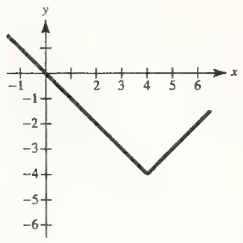
y-intercept: $\frac{5}{3}$



12. $y = x(6 - x)$



14. $y = |x - 4| - 4$



16. $y = 8\sqrt[3]{x - 6}$

```
Xmin = -40
Xmax = 40
Xscl = 10
Ymin = -40
Ymax = 40
Yscl = 10
```

18. $y = x + 1$

$(x + 1) - x^2 = 7$

$0 = x^2 - x + 6$

No real solution

No points of intersection

The graphs of $y = x + 1$ and $y = x^2 + 7$ do not intersect.

20. $y = kx^3$

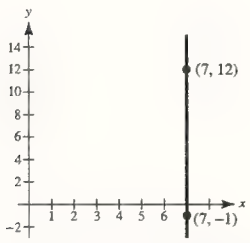
(a) $4 = k(1)^3 \Rightarrow k = 4$ and $y = 4x^3$

(b) $1 = k(-2)^3 \Rightarrow k = -\frac{1}{8}$ and $y = -\frac{1}{8}x^3$

(c) $0 = k(0)^3 \Rightarrow$ any k will do!

(d) $-1 = k(-1)^3 \Rightarrow k = 1 \Rightarrow y = x^3$

22.



The line is vertical and has no slope.

24. $\dfrac{3 - (-1)}{-3 - t} = \dfrac{3 - 6}{-3 - 8}$

$\dfrac{4}{-3 - t} = \dfrac{-3}{-11}$

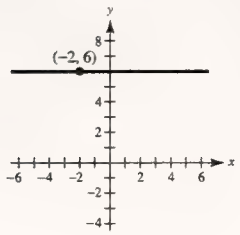
$-44 = 9 + 3t$

$-53 = 3t$

$t = -\dfrac{53}{3}$

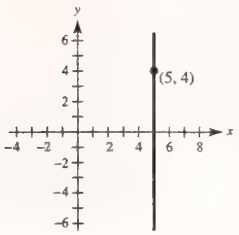
26. $y - 6 = 0(x - (-2))$

$y = 6$ Horizontal line



28. m is undefined. Line is vertical.

$x = 5$



30. (a) $y - 3 = -\frac{2}{3}(x - 1)$

$3y - 9 = -2x + 2$

$2x + 3y - 11 = 0$

(b) Slope of perpendicular line is 1.

$y - 3 = 1(x - 1)$

$y = x + 2$

$0 = x - y + 2$

(c) $m = \frac{4 - 3}{2 - 1} = 1$

$y - 3 = 1(x - 1)$

$y = x + 2$

$0 = x - y + 2$

(d) $y = 3$

$y - 3 = 0$

32. (a) $C = 9.25t + 13.50t + 36{,}500$

$= 22.75t + 36{,}500$

(b) $R = 30t$

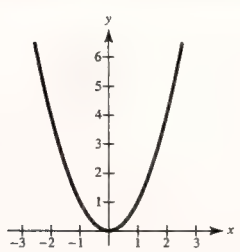
(c) $30t = 22.75t + 36{,}500$

$7.25t = 36{,}000$

$t \approx 5034.48$ hours to break even.

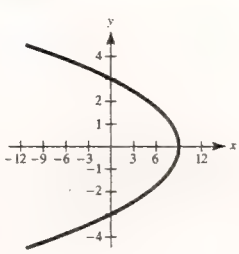
34. $x^2 - y = 0$

Function of x since there is one value for y for each x.



36. $x = 9 - y^2$

Not a function of x since there are two values of y for some x.



38. (a) $f(-4) = (-4)^2 + 2 = 18$ (because $-4 < 0$)

(b) $f(0) = |0 - 2| = 2$

(c) $f(1) = |1 - 2| = 1$

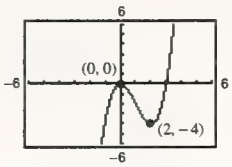
40. $f(x) = 1 - x^2$ and $g(x) = 2x + 1$

(a) $f(x) - g(x) = (1 - x^2) - (2x + 1) = -x^2 - 2x$

(b) $f(x)g(x) = (1 - x^2)(2x + 1) = -2x^3 - x^2 + 2x + 1$

(c) $g(f(x)) = g(1 - x^2) = 2(1 - x^2) + 1 = 3 - 2x^2$

42. $f(x) = x^3 - 3x^2$



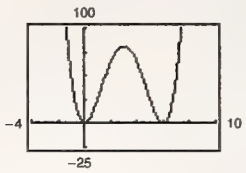
(a) The graph of g is obtained from f by a vertical shift down 1 unit, followed by a reflection in the x-axis:

$$g(x) = -[f(x) - 1]$$
$$= -x^3 + 3x^2 + 1$$

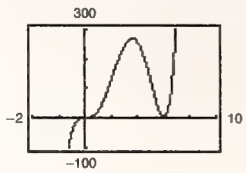
(b) The graph of g is obtained from f by a vertical shift upwards of 1 and a horizontal shift of 2 to the right.

$$g(x) = f(x - 2) + 1$$
$$= (x - 2)^3 - 3(x - 2)^2 + 1$$

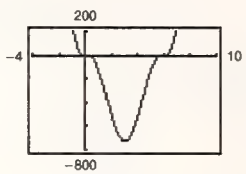
44. (a) $f(x) = x^2(x - 6)^2$



(b) $g(x) = x^3(x - 6)^2$



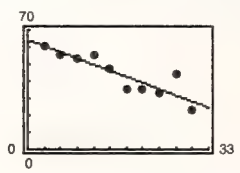
(c) $h(x) = x^3(x - 6)^3$



46. For company (a) the profit rose rapidly for the first year, and then leveled off. For the second company (b), the profit dropped, and then rose again later.

48. (a) $y = -1.204x + 64.2667$

(b)



(c) The data point (27, 44) is probably an error. Without this point, the new model is

$$y = -1.4344x + 66.4387.$$

Problem Solving for Chapter P

2. Let $y = mx + 1$ be a tangent line to the circle from the point (0, 1). Then

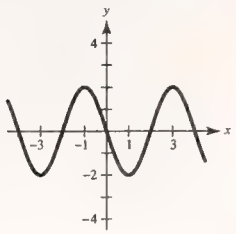
$$x^2 + (y + 1)^2 = 1$$
$$x^2 + (mx + 1 + 1)^2 = 1$$
$$(m^2 + 1)x^2 + 4mx + 3 = 0$$

Setting the discriminant $b^2 - 4ac$ equal to zero,

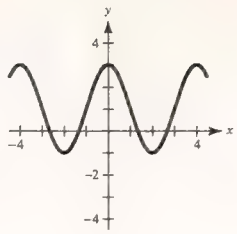
$$16m^2 - 4(m^2 + 1)(3) = 0$$
$$16m^2 - 12m^2 = 12$$
$$4m^2 = 12$$
$$m = \pm\sqrt{3}$$

Tangent lines: $y = \sqrt{3}x + 1$ and $y = -\sqrt{3}x + 1$.

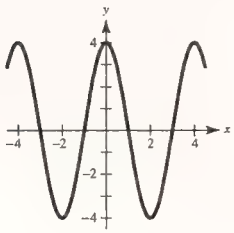
4. (a) $f(x + 1)$



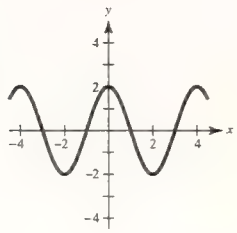
(b) $f(x) + 1$



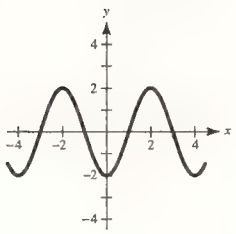
(c) $2f(x)$



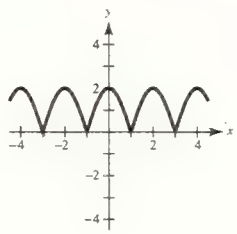
(d) $f(-x)$



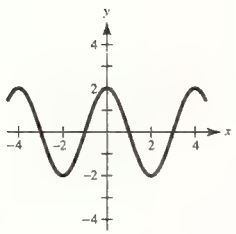
(e) $-f(x)$



(f) $|f(x)|$



(g) $f(|x|)$

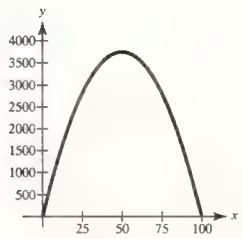


6. (a) $4y + 3x = 300 \implies y = \dfrac{300 - 3x}{4}$

$$A(x) = x(2y) = x\left(\frac{300 - 3x}{2}\right) = \frac{-3x^2 + 300x}{2}$$

Domain: $0 < x < 100$

(b)



Maximum of 3750 ft^2 at $x = 50$ ft, $y = 37.5$ ft.

(c) $A(x) = -\dfrac{3}{2}(x^2 - 100x)$

$$= -\frac{3}{2}(x^2 - 100x + 2500) + 3750$$

$$= -\frac{3}{2}(x - 50)^2 + 3750$$

$A(50) = 3750$ square feet is the maximum area, where $x = 50$ ft and $y = 37.5$ ft.

8. Let d be the distance from the starting point to the beach.

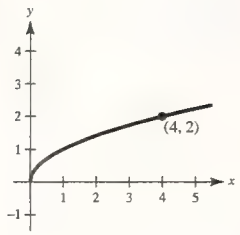
$$\text{Average velocity} = \frac{\text{distance}}{\text{time}}$$

$$= \frac{2d}{\frac{d}{120} + \frac{d}{60}}$$

$$= \frac{2}{\frac{1}{120} + \frac{1}{60}}$$

$$= 80 \text{ km/hr}$$

10.



(a) Slope $= \frac{3-2}{9-4} = \frac{1}{5}$. Slope of tangent line is greater than $\frac{1}{5}$.

(b) Slope $= \frac{2-1}{4-1} = \frac{1}{3}$. Slope of tangent line is less than $\frac{1}{3}$.

(c) Slope $= \frac{2.1-2}{4.1-4} = \frac{10}{41}$. Slope of tangent line is greater than $\frac{10}{41}$.

(d) Slope $= \frac{f(4+h) - f(4)}{(4+h) - 4}$

$$= \frac{\sqrt{4+h} - 2}{h}$$

(e) $$\frac{\sqrt{4+h} - 2}{h} = \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2}$$

$$= \frac{(4+h) - 4}{h\left(\sqrt{4+h} + 2\right)}$$

$$= \frac{1}{\sqrt{4+h} + 2}, \; h \neq 0$$

As h gets closer to 0, the slope gets closer to $\frac{1}{4}$. The slope is $\frac{1}{4}$ at the point $(4, 2)$.

12. (a) $\dfrac{I}{\sqrt{x^2+y^2}} = \dfrac{kI}{\sqrt{(x-4)^2+y^2}}$

$(x-4)^2 + y^2 = k^2(x^2+y^2)$

$(k^2-1)x^2 + (k^2-1)y^2 + 8x = 16$

If $k = 1$, then $x = 2$ is a vertical line. So, assume $k^2 - 1 \neq 0$. Then

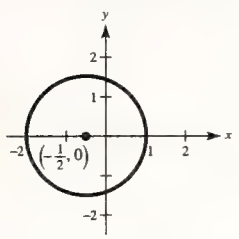
$$x^2 + y^2 + \frac{8x}{k^2-1} = \frac{16}{k^2-1}$$

$$\left(x + \frac{4}{k^2-1}\right)^2 + y^2 = \frac{16}{k^2-1} + \frac{16}{(k^2-1)^2}$$

$$\left(x + \frac{4}{k^2-1}\right)^2 + y^2 = \left(\frac{4k}{k^2-1}\right)^2, \text{ Circle}$$

(b) If $k = 3$, $\left(x + \frac{1}{2}\right)^2 + y^2 = \left(\frac{3}{2}\right)^2$

(c) For large k, the center of the circle is near $(0, 0)$, and the radius becomes smaller.



14. $f(x) = y = \dfrac{1}{1-x}$

(a) Domain: all $x \neq 1$

Range: all $y \neq 0$

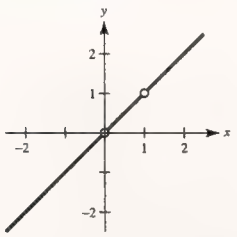
(b) $f(f(x)) = f\left(\dfrac{1}{1-x}\right) = \dfrac{1}{1-\left(\dfrac{1}{1-x}\right)} = \dfrac{1}{\dfrac{1-x-1}{1-x}} = \dfrac{1-x}{-x} = \dfrac{x-1}{x}$

Domain: all $x \neq 0, 1$

(c) $f(f(f(x))) = f\left(\dfrac{x-1}{x}\right) = \dfrac{1}{1-\left(\dfrac{x-1}{x}\right)} = \dfrac{1}{\dfrac{1}{x}} = x$

Domain: all $x \neq 0, 1$

(d) The graph is not a line. It has holes at $(0, 0)$ and $(1, 1)$.



CHAPTER 1
Limits and Their Properties

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CHAPTER 1
Limits and Their Properties

Section 1.1 A Preview of Calculus

Solutions to Even-Numbered Exercises

2. Calculus: velocity is not constant

Distance $\approx$ (20 ft/sec)(15 seconds) = 300 feet

4. Precalculus: rate of change = slope = 0.08

6. Precalculus: Area $= \pi(\sqrt{2})^2$

$= 2\pi$

8. Precalculus: Volume $= \pi(3)^2 6 = 54\pi$

10. (a) Area $\approx 5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} \approx 10.417$

$$\text{Area} \approx \frac{1}{2}\left(5 + \frac{5}{1.5} + \frac{5}{2} + \frac{5}{2.5} + \frac{5}{3} + \frac{5}{3.5} + \frac{5}{4} + \frac{5}{4.5}\right) \approx 9.145$$

(b) You could improve the approximation by using more rectangles.

Section 1.2 Finding Limits Graphically and Numerically

2.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.2564	0.2506	0.2501	0.2499	0.2494	0.2439

$$\lim_{x\to 2} \frac{x-2}{x^2-4} \approx 0.25 \quad \left(\text{Actual limit is } \tfrac{1}{4}.\right)$$

4.

x	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9
$f(x)$	-0.2485	-0.2498	-0.2500	-0.2500	-0.2502	-0.2516

$$\lim_{x\to -3} \frac{\sqrt{1-x}-2}{x+3} \approx -0.25 \quad \left(\text{Actual limit is } -\tfrac{1}{4}.\right)$$

6.

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$	0.0408	0.0401	0.0400	0.0400	0.0399	0.0392

$$\lim_{x\to 4} \frac{[x/(x+1)] - (4/5)}{x-4} \approx 0.04 \quad \left(\text{Actual limit is } \tfrac{1}{25}.\right)$$

8.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.0500	0.0050	0.0005	-0.0005	-0.0050	-0.0500

$$\lim_{x\to 0} \frac{\cos x - 1}{x} \approx 0.0000$$ (Actual limit is 0.) (Make sure you use radian mode.)

10. $\lim_{x\to 1}(x^2 + 2) = 3$

12. $\lim_{x\to 1} f(x) = \lim_{x\to 1}(x^2 + 2) = 3$

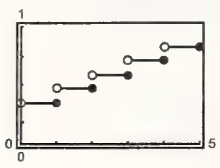
14. $\lim_{x\to 3}\frac{1}{x-3}$ does not exist since the function increases and decreases without bound as x approaches 3.

16. $\lim_{x\to 0}\sec x = 1$

18. $\lim_{x\to 1}\sin(\pi x) = 0$

20. $C(t) = 0.35 - 0.12[\![-(t-1)]\!]$

(a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
$C(t)$	0.59	0.71	0.71	0.71	0.71	0.71	0.71

$\lim_{t\to 3.5} C(t) = 0.71$

(c)

t	3	2.5	2.9	3	3.1	3.5	4
$C(t)$	0.47	0.59	0.59	0.59	0.71	0.71	0.71

$\lim_{t\to 3.5} C(t)$ does not exist. The values of C jump from 0.59 to 0.71 at $t = 3$.

22. You need to find δ such that $0 < |x-2| < \delta$ implies $|f(x) - 3| = |x^2 - 1 - 3| = |x^2 - 4| < 0.2$. That is,

$$\begin{aligned} -0.2 < x^2 - 4 &< 0.2 \\ 4 - 0.2 < x^2 &< 4 + 0.2 \\ 3.8 < x^2 &< 4.2 \\ \sqrt{3.8} < x &< \sqrt{4.2} \\ \sqrt{3.8} - 2 < x - 2 &< \sqrt{4.2} - 2 \end{aligned}$$

So take $\delta = \sqrt{4.2} - 2 \approx 0.0494$.

Then $0 < |x-2| < \delta$ implies

$$-(\sqrt{4.2} - 2) < x - 2 < \sqrt{4.2} - 2$$

$$\sqrt{3.8} - 2 < x - 2 < \sqrt{4.2} - 2.$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 3| = |x^2 - 4| < 0.2.$$

24. $\lim_{x\to 4}\left(4 - \frac{x}{2}\right) = 2$

$$\left|\left(4 - \frac{x}{2}\right) - 2\right| < 0.01$$

$$\left|2 - \frac{x}{2}\right| < 0.01$$

$$\left|-\frac{1}{2}(x-4)\right| < 0.01$$

$$0 < |x-4| < 0.02 = \delta$$

Hence, if $0 < |x-4| < \delta = 0.02$, you have

$$\left|-\frac{1}{2}(x-4)\right| < 0.01$$

$$\left|2 - \frac{x}{2}\right| < 0.01$$

$$\left|\left(4 - \frac{x}{2}\right) - 2\right| < 0.01$$

$$|f(x) - L| < 0.01$$

26. $\lim_{x\to 5}(x^2+4)=29$

$$|(x^2+4)-29|<0.01$$
$$|x^2-25|<0.01$$
$$|(x+5)(x-5)|<0.01$$
$$|x-5|<\frac{0.01}{|x+5|}$$

If we assume $4<x<6$, then $\delta=0.01/11\approx 0.0009$.

Hence, if $0<|x-5|<\delta=\frac{0.01}{11}$, you have

$$|x-5|<\frac{0.01}{11}<\frac{1}{|x+5|}(0.01)$$
$$|x-5||x+5|<0.01$$
$$|x^2-25|<0.01$$
$$|(x^2+4)-29|<0.01$$
$$|f(x)-L|<0.01$$

28. $\lim_{x\to -3}(2x+5)=-1$

Given $\epsilon>0$:

$$|(2x+5)-(-1)|<\epsilon$$
$$|2x+6|<\epsilon$$
$$2|x+3|<\epsilon$$
$$|x+3|<\frac{\epsilon}{2}=\delta$$

Hence, let $\delta=\epsilon/2$.

Hence, if $0<|x+3|<\delta=\frac{\epsilon}{2}$, you have

$$|x+3|<\frac{\epsilon}{2}$$
$$|2x+6|<\epsilon$$
$$|(2x+5)-(-1)|<\epsilon$$
$$|f(x)-L|<\epsilon$$

30. $\lim_{x\to 1}\left(\frac{2}{3}x+9\right)=\frac{2}{3}(1)+9=\frac{29}{3}$

Given $\epsilon>0$:

$$\left|\left(\tfrac{2}{3}x+9\right)-\tfrac{29}{3}\right|<\epsilon$$
$$\left|\tfrac{2}{3}x-\tfrac{2}{3}\right|<\epsilon$$
$$\tfrac{2}{3}|x-1|<\epsilon$$
$$|x-1|<\tfrac{3}{2}\epsilon$$

Hence, let $\delta=(3/2)\epsilon$.

Hence, if $0<|x-1|<\delta=\frac{3}{2}\epsilon$, you have

$$|x-1|<\tfrac{3}{2}\epsilon$$
$$\left|\tfrac{2}{3}x-\tfrac{2}{3}\right|<\epsilon$$
$$\left|\left(\tfrac{2}{3}x+9\right)-\tfrac{29}{3}\right|<\epsilon$$
$$|f(x)-L|<\epsilon$$

32. $\lim_{x\to 2}(-1)=-1$

Given $\epsilon>0$: $|-1-(-1)|<\epsilon$

$$0<\epsilon$$

Hence, any $\delta>0$ will work.

Hence, for any $\delta>0$, you have

$$|(-1)-(-1)|<\epsilon$$
$$|f(x)-L|<\epsilon$$

34. $\lim_{x\to 4}\sqrt{x}=\sqrt{4}=2$

Given $\epsilon>0$: $\left|\sqrt{x}-2\right|<\epsilon$

$$\left|\sqrt{x}-2\right|\left|\sqrt{x}+2\right|<\epsilon\left|\sqrt{x}+2\right|$$
$$\left|x-4\right|<\epsilon\left|\sqrt{x}+2\right|$$

Assuming $1<x<9$, you can choose $\delta=3\epsilon$. Then,

$$0<|x-4|<\delta=3\epsilon \Rightarrow |x-4|<\epsilon\left|\sqrt{x}+2\right|$$
$$\Rightarrow \left|\sqrt{x}-2\right|<\epsilon.$$

36. $\lim_{x\to 3}|x-3|=0$

Given $\epsilon>0$:

$$|(x-3)-0|<\epsilon$$
$$|x-3|<\epsilon=\delta$$

Hence, let $\delta=\epsilon$.

Hence for $0<|x-3|<\delta=\epsilon$, you have

$$|x-3|<\epsilon$$
$$||x-3|-0|<\epsilon$$
$$|f(x)-L|<\epsilon$$

38. $\lim_{x\to -3}(x^2 + 3x) = 0$

Given $\epsilon > 0$:

$$|(x^2 + 3x) - 0| < \epsilon$$
$$|x(x + 3)| < \epsilon$$
$$|x + 3| < \frac{\epsilon}{|x|}$$

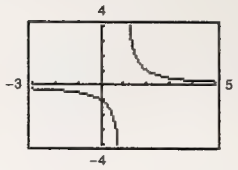
If we assume $-4 < x < -2$, then $\delta = \epsilon/4$.

Hence for $0 < |x - (-3)| < \delta = \frac{\epsilon}{4}$, you have

$$|x + 3| < \frac{1}{4}\epsilon < \frac{1}{|x|}\epsilon$$
$$|x(x + 3)| < \epsilon$$
$$|x^2 + 3x - 0| < \epsilon$$
$$|f(x) - L| < \epsilon$$

40. $f(x) = \dfrac{x - 3}{x^2 - 4x + 3}$

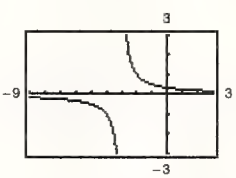
$\lim_{x\to 3} f(x) = \frac{1}{2}$



The domain is all $x \neq 1, 3$. The graphing utility does not show the hole at $\left(3, \frac{1}{2}\right)$.

42. $f(x) = \dfrac{x - 3}{x^2 - 9}$

$\lim_{x\to 3} f(x) = \frac{1}{6}$



The domain is all $x \neq \pm 3$. The graphing utility does not show the hole at $\left(3, \frac{1}{6}\right)$.

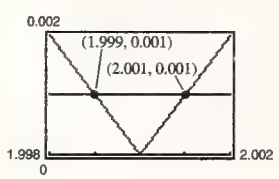
44. (a) No. The fact that $f(2) = 4$ has no bearing on the existence of the limit of $f(x)$ as x approaches 2.

(b) No. The fact that $\lim_{x\to 2} f(x) = 4$ has no bearing on the value of f at 2.

46. Let $p(x)$ be the atmospheric pressure in a plane at altitude x (in feet).

$$\lim_{x\to 0^+} p(x) = 14.7 \text{ lb/in}^2$$

48.



Using the zoom and trace feature, $\delta = 0.001$. That is, for

$$0 < |x - 2| < 0.001, \left|\frac{x^2 - 4}{x - 2} - 4\right| < 0.001.$$

50. True

52. False; let

$$f(x) = \begin{cases} x^2 - 4x, & x \neq 4 \\ 10, & x = 4 \end{cases}$$

$\lim_{x\to 4} f(x) = \lim_{x\to 4}(x^2 - 4x) = 0$ and $f(4) = 10 \neq 0$

54. $\lim_{x\to 4} \dfrac{x^2 - x - 12}{x - 4} = 7$

n	$4 + [0.1]^n$	$f(4 + [0.1]^n)$
1	4.1	7.1
2	4.01	7.01
3	4.001	7.001
4	4.0001	7.0001

n	$4 - [0.1]^n$	$f(4 - [0.1]^n)$
1	3.9	6.9
2	3.99	6.99
3	3.999	6.999
4	3.9999	6.9999

56. $f(x) = mx + b, m \neq 0$. Let $\epsilon > 0$ be given. Take $\delta = \dfrac{\epsilon}{|m|}$.

If $0 < |x - c| < \delta = \dfrac{\epsilon}{|m|}$, then

$$|m||x - c| < \epsilon$$
$$|mx - mc| < \epsilon$$
$$|(mx + b) - (mc + b)| < \epsilon$$

which shows that $\lim_{x \to c} (mx + b) = mc + b$.

58. $\lim_{xc} g(x) = L, L > 0$. Let $\epsilon = \frac{1}{2}L$. There exists $\delta > 0$ such that $0 < |x - 0| < \delta$ implies $|g(x) - L| < \epsilon = \frac{1}{2}L$.

That is,

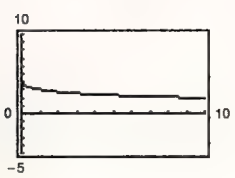
$$-\tfrac{1}{2}L < g(x) - L < \tfrac{1}{2}L$$
$$\tfrac{1}{2}L < \quad g(x) \quad < \tfrac{3}{2}L$$

Hence for x in the interval $(c - \delta, c + \delta), x \neq c$,

$$g(x) > \tfrac{1}{2}L > 0.$$

Section 1.3 Evaluating Limits Analytically

2.

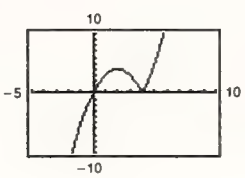


(a) $\lim_{x \to 4} g(x) = 2.4$

(b) $\lim_{x \to 0} g(x) = 4$

$$g(x) = \frac{12(\sqrt{x} - 3)}{x - 9}$$

4.



(a) $\lim_{t \to 4} f(t) = 0$

(b) $\lim_{t \to -1} f(t) = -5$

$$f(t) = t|t - 4|$$

6. $\lim_{x \to -2} x^3 = (-2)^3 = -8$

8. $\lim_{x \to -3} (3x + 2) = 3(-3) + 2 = -7$

10. $\lim_{x \to 1} (-x^2 + 1) = -(1)^2 + 1 = 0$

12. $\lim_{x \to 1} (3x^3 - 2x^2 + 4) = 3(1)^3 - 2(1)^2 + 4 = 5$

14. $\lim_{x \to -3} \dfrac{2}{x + 2} = \dfrac{2}{-3 + 2} = -2$

16. $\lim_{x \to 3} \dfrac{2x - 3}{x + 5} = \dfrac{2(3) - 3}{3 + 5} = \dfrac{3}{8}$

18. $\lim_{x \to 3} \dfrac{\sqrt{x + 1}}{x - 4} = \dfrac{\sqrt{3 + 1}}{3 - 4} = -2$

20. $\lim_{x \to 4} \sqrt[3]{x + 4} = \sqrt[3]{4 + 4} = 2$

22. $\lim_{x \to 0} (2x - 1)^3 = [2(0) - 1]^3 = -1$

24. (a) $\lim_{x \to -3} f(x) = (-3) + 7 = 4$

(b) $\lim_{x \to 4} g(x) = 4^2 = 16$

(c) $\lim_{x \to -3} g(f(x)) = g(4) = 16$

26. (a) $\lim_{x \to 4} f(x) = 2(4^2) - 3(4) + 1 = 21$

(b) $\lim_{x \to 21} g(x) = \sqrt[3]{21 + 6} = 3$

(c) $\lim_{x \to 4} g(f(x)) = g(21) = 3$

28. $\lim_{x \to \pi} \tan x = \tan \pi = 0$

30. $\lim_{x \to 1} \sin \dfrac{\pi x}{2} = \sin \dfrac{\pi}{2} = 1$

32. $\lim_{x \to \pi} \cos 3x = \cos 3\pi = -1$

34. $\lim_{x \to 5\pi/3} \cos x = \cos \dfrac{5\pi}{3} = \dfrac{1}{2}$

36. $\lim_{x \to 7} \sec\left(\dfrac{\pi x}{6}\right) = \sec \dfrac{7\pi}{6} = \dfrac{-2\sqrt{3}}{3}$

38. (a) $\lim_{x\to c}[4f(x)] = 4\lim_{x\to c} f(x) = 4\left(\frac{3}{2}\right) = 6$

(b) $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = \frac{3}{2} + \frac{1}{2} = 2$

(c) $\lim_{x\to c}[f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = \frac{3}{4}$

(d) $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{3/2}{1/2} = 3$

40. (a) $\lim_{x\to c}\sqrt[3]{f(x)} = \sqrt[3]{\lim_{x\to c} f(x)} = \sqrt[3]{27} = 3$

(b) $\lim_{x\to c}\frac{f(x)}{18} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} 18} = \frac{27}{18} = \frac{3}{2}$

(c) $\lim_{x\to c}[f(x)]^2 = \left[\lim_{x\to c} f(x)\right]^2 = (27)^2 = 729$

(d) $\lim_{x\to c}[f(x)]^{2/3} = \left[\lim_{x\to c} f(x)\right]^{2/3} = (27)^{2/3} = 9$

42. $f(x) = x - 3$ and $h(x) = \frac{x^2 - 3x}{x}$ agree except at $x = 0$.

(a) $\lim_{x\to -2} h(x) = \lim_{x\to -2} f(x) = -5$

(b) $\lim_{x\to 0} h(x) = \lim_{x\to 0} f(x) = -3$

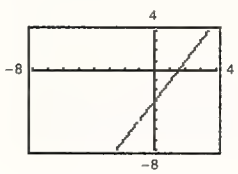
44. $g(x) = \frac{1}{x - 1}$ and $f(x) = \frac{x}{x^2 - x}$ agree except at $x = 0$.

(a) $\lim_{x\to 1} f(x)$ does not exist.

(b) $\lim_{x\to 0} f(x) = -1$

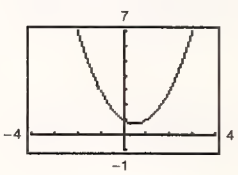
46. $f(x) = \frac{2x^2 - x - 3}{x + 1}$ and $g(x) = 2x - 3$ agree except at $x = -1$.

$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = -5$



48. $f(x) = \frac{x^3 + 1}{x + 1}$ and $g(x) = x^2 - x + 1$ agree except at $x = -1$.

$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = 3$



50. $\lim_{x\to 2}\frac{2 - x}{x^2 - 4} = \lim_{x\to 2}\frac{-(x - 2)}{(x - 2)(x + 2)}$

$= \lim_{x\to 2}\frac{-1}{x + 2} = -\frac{1}{4}$

52. $\lim_{x\to 4}\frac{x^2 - 5x + 4}{x^2 - 2x - 8} = \lim_{x\to 4}\frac{(x - 4)(x - 1)}{(x - 4)(x + 2)}$

$= \lim_{x\to 4}\frac{(x - 1)}{(x + 2)} = \frac{3}{6} = \frac{1}{2}$

54. $\lim_{x\to 0}\frac{\sqrt{2 + x} - \sqrt{2}}{x} = \lim_{x\to 0}\frac{\sqrt{2 + x} - \sqrt{2}}{x} \cdot \frac{\sqrt{2 + x} + \sqrt{2}}{\sqrt{2 + x} + \sqrt{2}}$

$= \lim_{x\to 0}\frac{2 + x - 2}{\left(\sqrt{2 + x} + \sqrt{2}\right)x} = \lim_{x\to 0}\frac{1}{\sqrt{2 + x} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

56. $\lim_{x\to 3}\frac{\sqrt{x + 1} - 2}{x - 3} = \lim_{x\to 3}\frac{\sqrt{x + 1} - 2}{x - 3} \cdot \frac{\sqrt{x + 1} + 2}{\sqrt{x + 1} + 2} = \lim_{x\to 3}\frac{x - 3}{(x - 3)\left[\sqrt{x + 1} + 2\right]} = \lim_{x\to 3}\frac{1}{\sqrt{x + 1} + 2} = \frac{1}{4}$

58. $\lim_{x\to 0}\frac{\frac{1}{x + 4} - \frac{1}{4}}{x} = \lim_{x\to 0}\frac{\frac{4 - (x + 4)}{4(x + 4)}}{x} = \lim_{x\to 0}\frac{-1}{4(x + 4)} = -\frac{1}{16}$

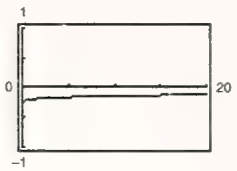
60. $\lim_{\Delta x\to 0}\frac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x\to 0}\frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta x(2x + \Delta x)}{\Delta x} = \lim_{\Delta x\to 0}(2x + \Delta x) = 2x$

62. $\lim_{\Delta x \to 0} \dfrac{(x + \Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(3x^2 + 3x\Delta x + (\Delta x)^2)}{\Delta x} = \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2$$

64. $f(x) = \dfrac{4 - \sqrt{x}}{x - 16}$

x	15.9	15.99	15.999	16	16.001	16.01	16.1
$f(x)$	−.1252	−.125	−.125	?	−.125	−.125	−.1248



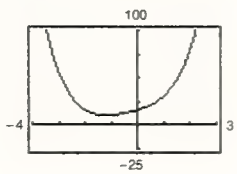
It appears that the limit is −0.125.

Analytically, $\lim_{x \to 16} \dfrac{4 - \sqrt{x}}{x - 16} = \lim_{x \to 16} \dfrac{(4 - \sqrt{x})}{(\sqrt{x} + 4)(\sqrt{x} - 4)}$

$$= \lim_{x \to 16} \frac{-1}{\sqrt{x} + 4} = -\frac{1}{8}.$$

66. $\lim_{x \to 2} \dfrac{x^5 - 32}{x - 2} = 80$

x	1.9	1.99	1.999	1.9999	2.0	2.0001	2.001	2.01	2.1
$f(x)$	72.39	79.20	79.92	79.99	?	80.01	80.08	80.80	88.41



Analytically, $\lim_{x \to 2} \dfrac{x^5 - 32}{x - 2} = \lim_{x \to 2} \dfrac{(x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16)}{x - 2}$

$$= \lim_{x \to 2} (x^4 + 2x^3 + 4x^2 + 8x + 16) = 80.$$

(*Hint:* Use long division to factor $x^5 - 32$.)

68. $\lim_{x \to 0} \dfrac{3(1 - \cos x)}{x} = \lim_{x \to 0} \left[3\left(\dfrac{1 - \cos x}{x}\right)\right] = (3)(0) = 0$

70. $\lim_{\theta \to 0} \dfrac{\cos\theta \tan\theta}{\theta} = \lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

72. $\lim_{x \to 0} \dfrac{\tan^2 x}{x} = \lim_{x \to 0} \dfrac{\sin^2 x}{x \cos^2 x} = \lim_{x \to 0} \left[\dfrac{\sin x}{x} \cdot \dfrac{\sin x}{\cos^2 x}\right]$

$$= (1)(0) = 0$$

74. $\lim_{\phi \to \pi} \phi \sec\phi = \pi(-1) = -\pi$

76. $\lim_{x \to \pi/4} \dfrac{1 - \tan x}{\sin x - \cos x} = \lim_{x \to \pi/4} \dfrac{\cos x - \sin x}{\sin x \cos x - \cos^2 x}$

$$= \lim_{x \to \pi/4} \frac{-(\sin x - \cos x)}{\cos x(\sin x - \cos x)}$$

$$= \lim_{x \to \pi/4} \frac{-1}{\cos x}$$

$$= \lim_{x \to \pi/4} (-\sec x)$$

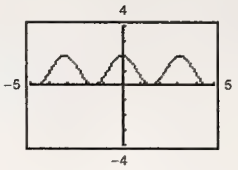
$$= -\sqrt{2}$$

78. $\lim_{x \to 0} \dfrac{\sin 2x}{\sin 3x} = \lim_{x \to 0} \left[2\left(\dfrac{\sin 2x}{2x}\right)\left(\dfrac{1}{3}\right)\left(\dfrac{3x}{\sin 3x}\right) = 2(1)\left(\dfrac{1}{3}\right)(1) = \dfrac{2}{3}\right.$

80. $f(h) = (1 + \cos 2h)$

h	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(h)$	1.98	1.9998	2	?	2	1.9998	1.98

Analytically, $\lim_{h\to 0}(1 + \cos 2h) = 1 + \cos(0) = 1 + 1 = 2$.

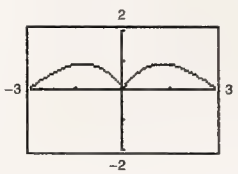


The limit appear to equal 2.

82. $f(x) = \dfrac{\sin x}{\sqrt[3]{x}}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.215	0.0464	0.01	?	0.01	0.0464	0.215

Analytically, $\lim_{x\to 0}\dfrac{\sin x}{\sqrt[3]{x}} = \lim_{x\to 0}\sqrt[3]{x^2}\left(\dfrac{\sin x}{x}\right) = (0)(1) = 0$.



The limit appear to equal 0.

84. $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$= \lim_{h\to 0}\dfrac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \lim_{h\to 0}\dfrac{1}{\sqrt{x+h}+\sqrt{x}} = \dfrac{1}{2\sqrt{x}}$

86. $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\dfrac{(x+h)^2-4(x+h)-(x^2-4x)}{h} = \lim_{h\to 0}\dfrac{x^2+2xh+h^2-4x-4h-x^2+4x}{h}$

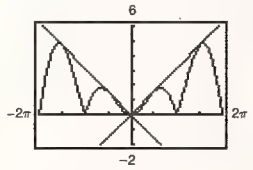
$= \lim_{h\to 0}\dfrac{h(2x+h-4)}{h} = \lim_{h\to 0}(2x+h-4) = 2x-4$

88. $\lim_{x\to a}[b - |x-a|] \le \lim_{x\to a} f(x) \le \lim_{x\to a}[b + |x-a|]$

$b \le \lim_{x\to a} f(x) \le b$

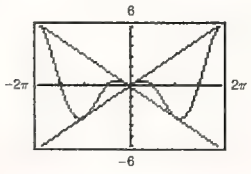
Therefore, $\lim_{x\to a} f(x) = b$.

90. $f(x) = |x \sin x|$



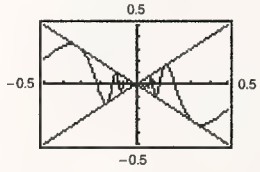
$\lim_{x\to 0}|x \sin x| = 0$

92. $f(x) = |x| \cos x$



$\lim_{x\to 0}|x| \cos x = 0$

94. $h(x) = x \cos\dfrac{1}{x}$

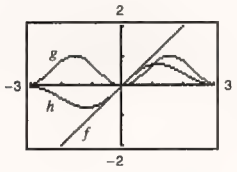


$\lim_{x\to 0}\left(x \cos\dfrac{1}{x}\right) = 0$

96. $f(x) = \dfrac{x^2 - 1}{x - 1}$ and $g(x) = x + 1$ agree at all points except $x = 1$.

98. If a function f is squeezed between two functions h and g, $h(x) \le f(x) \le g(x)$, and h and g have the same limit L as $x \to c$, then $\lim_{x \to c} f(x)$ exists and equals L.

100. $f(x) = x$, $g(x) = \sin^2 x$, $h(x) = \dfrac{\sin^2 x}{x}$



When you are "close to" 0 the magnitude of g is "smaller" than the magnitude of f and the magnitude of g is approaching zero "faster" than the magnitude of f. Thus, $|g|/|f| \approx 0$ when x is "close to" 0

102. $s(t) = -16t^2 + 1000 = 0$ when $t = \sqrt{\dfrac{1000}{16}} = \dfrac{5\sqrt{10}}{2}$ seconds

$$\lim_{t \to 5\sqrt{10}/2} \frac{s\left(\frac{5\sqrt{10}}{2}\right) - s(t)}{\frac{5\sqrt{10}}{2} - t} = \lim_{t \to 5\sqrt{10}/2} \frac{0 - (-16t^2 + 1000)}{\frac{5\sqrt{10}}{2} - t}$$

$$= \lim_{t \to 5\sqrt{10}/2} \frac{16\left(t^2 - \frac{125}{2}\right)}{\frac{5\sqrt{10}}{2} - t} = \lim_{t \to 5\sqrt{10}/2} \frac{16\left(t + \frac{5\sqrt{10}}{2}\right)\left(t - \frac{5\sqrt{10}}{2}\right)}{-\left(t - \frac{5\sqrt{10}}{2}\right)}$$

$$= \lim_{t \to 5\sqrt{10}/2} -16\left(t + \frac{5\sqrt{10}}{2}\right) = -80\sqrt{10} \text{ ft/sec} \approx -253 \text{ ft/sec}$$

104. $-4.9t^2 + 150 = 0$ when $t = \sqrt{\dfrac{150}{4.9}} = \sqrt{\dfrac{1500}{49}} \approx 5.53$ seconds.

The velocity at time $t = a$ is

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t} = \lim_{t \to a} \frac{(-4.9a^2 + 150) - (-4.9t^2 + 150)}{a - t} = \lim_{t \to a} \frac{-4.9(a - t)(a + t)}{a - t}$$

$$= \lim_{t \to a} -4.9(a + t) = -2a(4.9) = -9.8a \text{ m/sec.}$$

Hence, if $a = \sqrt{1500/49}$, the velocity is $-9.8\sqrt{1500/49} \approx -54.2$ m/sec.

106. Suppose, on the contrary, that $\lim_{x \to c} g(x)$ exists. Then, since $\lim_{x \to c} f(x)$ exists, so would $\lim_{x \to c} [f(x) + g(x)]$, which is a contradiction. Hence, $\lim_{x \to c} g(x)$ does not exist.

108. Given $f(x) = x^n$, n is a positive integer, then

$$\lim_{x \to c} x^n = \lim_{x \to c} (x x^{n-1}) = \left[\lim_{x \to c} x\right]\left[\lim_{x \to c} x^{n-1}\right]$$

$$= c\left[\lim_{x \to c} (x x^{n-2})\right] = c\left[\lim_{x \to c} x\right]\left[\lim_{x \to c} x^{n-2}\right]$$

$$= c(c)\lim_{x \to c} (x x^{n-3}) = \cdots = c^n.$$

110. Given $\lim_{x \to c} f(x) = 0$:

For every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - 0| < \epsilon$ whenever $0 < |x - c| < \delta$.

Now $|f(x) - 0| = |f(x)| = ||f(x)| - 0| < \epsilon$ for $|x - c| < \delta$. Therefore, $\lim_{x \to c} |f(x)| = 0$.

112. (a) If $\lim_{x\to c}|f(x)| = 0$, then $\lim_{x\to c}[-|f(x)|] = 0$.

$$-|f(x)| \le f(x) \le |f(x)|$$

$$\lim_{x\to c}[-|f(x)|] \le \lim_{x\to c} f(x) \le \lim_{x\to c}|f(x)|$$

$$0 \le \lim_{x\to c} f(x) \le 0$$

Therefore, $\lim_{x\to c} f(x) = 0$.

(b) Given $\lim_{x\to c} f(x) = L$:

For every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$.
Since $||f(x)| - |L|| \le |f(x) - L| < \epsilon$ for $|x - c| < \delta$, then $\lim_{x\to c}|f(x)| = |L|$.

114. True. $\lim_{x\to 0} x^3 = 0^3 = 0$

116. False. Let $f(x) = \begin{cases} x & x \neq 1 \\ 3 & x = 1 \end{cases}$, $c = 1$

Then $\lim_{x\to 1} f(x) = 1$ but $f(1) \neq 1$.

118. False. Let $f(x) = \frac{1}{2}x^2$ and $g(x) = x^2$. Then $f(x) < g(x)$ for all $x \neq 0$. But $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$.

120.
$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}$$

$$= \lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x}$$

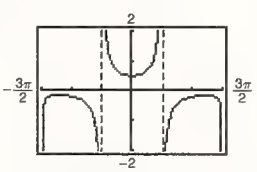
$$= \left[\lim_{x\to 0}\frac{\sin x}{x}\right]\left[\lim_{x\to 0}\frac{\sin x}{1+\cos x}\right]$$

$$= (1)(0) = 0$$

122. $f(x) = \dfrac{\sec x - 1}{x^2}$

(a) The domain of f is all $x \neq 0$, $\pi/2 + n\pi$.

(b)



The domain is not obvious. The hole at $x = 0$ is not apparent.

(c) $\lim_{x\to 0} f(x) = \dfrac{1}{2}$

(d)
$$\frac{\sec x - 1}{x^2} = \frac{\sec x - 1}{x^2}\cdot\frac{\sec x + 1}{\sec x + 1} = \frac{\sec^2 x - 1}{x^2(\sec x + 1)}$$

$$= \frac{\tan^2 x}{x^2(\sec x + 1)} = \frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{x^2}\right)\frac{1}{\sec x + 1}$$

Hence, $\lim_{x\to 0}\dfrac{\sec x - 1}{x^2} = \lim_{x\to 0}\dfrac{1}{\cos^2 x}\left(\dfrac{\sin^2 x}{x^2}\right)\dfrac{1}{\sec x + 1}$

$$= 1(1)\left(\frac{1}{2}\right) = \frac{1}{2}.$$

124. The calculator was set in degree mode, instead of radian mode.

Section 1.4 Continuity and One-Sided Limits

2. (a) $\lim_{x\to -2^+} f(x) = -2$

(b) $\lim_{x\to -2^-} f(x) = -2$

(c) $\lim_{x\to -2} f(x) = -2$

The function is continuous at $x = -2$.

4. (a) $\lim_{x\to -2^+} f(x) = 2$

(b) $\lim_{x\to -2^-} f(x) = 2$

(c) $\lim_{x\to -2} f(x) = 2$

The function is NOT continuous at $x = -2$.

6. (a) $\lim_{x\to -1^+} f(x) = 0$

(b) $\lim_{x\to -1^-} f(x) = 2$

(c) $\lim_{x\to -1} f(x)$ does not exist.

The function is NOT continuous at $x = -1$.

8. $\lim_{x\to 2^+} \dfrac{2-x}{x^2-4} = \lim_{x\to 2^+} -\dfrac{1}{x+2} = -\dfrac{1}{4}$

10. $\lim_{x\to 4^-} \dfrac{\sqrt{x}-2}{x-4} = \lim_{x\to 4^-} \dfrac{\sqrt{x}-2}{x-4} \cdot \dfrac{\sqrt{x}+2}{\sqrt{x}+2}$

$= \lim_{x\to 4^-} \dfrac{x-4}{(x-4)(\sqrt{x}+2)}$

$= \lim_{x\to 4^-} \dfrac{1}{\sqrt{x}+2} = \dfrac{1}{4}$

12. $\lim_{x\to 2^+} \dfrac{|x-2|}{x-2} = \lim_{x\to 2^+} \dfrac{x-2}{x-2} = 1$

14. $\lim_{\Delta x\to 0^+} \dfrac{(x+\Delta x)^2 + (x+\Delta x) - (x^2+x)}{\Delta x} = \lim_{\Delta x\to 0^+} \dfrac{x^2 + 2x(\Delta x) + (\Delta x)^2 + x + \Delta x - x^2 - x}{\Delta x}$

$= \lim_{\Delta x\to 0^+} \dfrac{2x(\Delta x) + (\Delta x)^2 + \Delta x}{\Delta x}$

$= \lim_{\Delta x\to 0^+} (2x + \Delta x + 1)$

$= 2x + 0 + 1 = 2x + 1$

16. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (-x^2 + 4x - 2) = 2$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 - 4x + 6) = 2$

$\lim_{x\to 2} f(x) = 2$

18. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (1 - x) = 0$

20. $\lim_{x\to \pi/2} \sec x$ does not exist since

$\lim_{x\to(\pi/2)^+} \sec x$ and $\lim_{x\to(\pi/2)^-} \sec x$ do not exist.

22. $\lim_{x\to 2^+} (2x - [\![x]\!]) = 2(2) - 2 = 2$

24. $\lim_{x\to 1} \left(1 - \left[\!\left[-\dfrac{x}{2}\right]\!\right]\right) = 1 - (-1) = 2$

26. $f(x) = \dfrac{x^2-1}{x+1}$

has a discontinuity at $x = -1$ since $f(-1)$ is not defined.

28. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$ has discontinuity at $x = 1$ since $f(1) = 2 \neq \lim_{x\to 1} f(x) = 1$.

30. $f(t) = 3 - \sqrt{9 - t^2}$ is continuous on $[-3, 3]$.

32. $g(2)$ is not defined. g is continuous on $[-1, 2)$.

34. $f(x) = \dfrac{1}{x^2 + 1}$ is continuous for all real x.

36. $f(x) = \cos\dfrac{\pi x}{2}$ is continuous for all real x.

38. $f(x) = \dfrac{x}{x^2 - 1}$ has nonremovable discontinuities at $x = 1$ and $x = -1$ since $\lim\limits_{x\to 1} f(x)$ and $\lim\limits_{x\to -1} f(x)$ do not exist.

40. $f(x) = \dfrac{x - 3}{x^2 - 9}$ has a nonremovable discontinuity at $x = -3$ since $\lim\limits_{x\to -3} f(x)$ does not exist, and has a removable discontinuity at $x = 3$ since

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{1}{x + 3} = \frac{1}{6}.$$

42. $f(x) = \dfrac{x - 1}{(x + 2)(x - 1)}$

has a nonremovable discontinuity at $x = -2$ since $\lim\limits_{x\to -2} f(x)$ does not exist, and has a removable discontinuity at $x = 1$ since

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{x + 2} = \frac{1}{3}.$$

44. $f(x) = \dfrac{|x - 3|}{x - 3}$

has a nonremovable discontinuity at $x = 3$ since $\lim\limits_{x\to 3} f(x)$ does not exist.

46. $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \geq 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1^2 = 1$

2. $\left.\begin{aligned} \lim\limits_{x\to 1^-} f(x) &= \lim\limits_{x\to 1^-} (-2x + 3) = 1 \\ \lim\limits_{x\to 1^+} f(x) &= \lim\limits_{x\to 1^+} x^2 = 1 \end{aligned}\right\} \lim\limits_{x\to 1} f(x) = 1$

3. $f(1) = \lim\limits_{x\to 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x.

48. $f(x) = \begin{cases} -2x, & x \leq 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$ has a **possible** discontinuity at $x = 2$.

1. $f(2) = -2(2) = -4$

2. $\left.\begin{aligned} \lim\limits_{x\to 2^-} f(x) &= \lim\limits_{x\to 2^-} (-2x) = -4 \\ \lim\limits_{x\to 2^+} f(x) &= \lim\limits_{x\to 2^+} (x^2 - 4x + 1) = -3 \end{aligned}\right\} \lim\limits_{x\to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

50. $f(x) = \begin{cases} \csc\dfrac{\pi x}{6}, & |x - 3| \leq 2 \\ 2, & |x - 3| > 2 \end{cases} = \begin{cases} \csc\dfrac{\pi x}{6}, & 1 \leq x \leq 5 \\ 2, & x < 1 \text{ or } x > 5 \end{cases}$ has **possible** discontinuities at $x = 1, x = 5$.

1. $f(1) = \csc\dfrac{\pi}{6} = 2$ $\qquad$ $f(5) = \csc\dfrac{5\pi}{6} = 2$

2. $\lim\limits_{x\to 1} f(x) = 2$ $\qquad$ $\lim\limits_{x\to 5} f(x) = 2$

3. $f(1) = \lim\limits_{x\to 1} f(x)$ $\qquad$ $f(5) = \lim\limits_{x\to 5} f(x)$

f is continuous at $x = 1$ and $x = 5$, therefore, f is continuous for all real x.

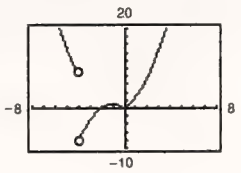
52. $f(x) = \tan\dfrac{\pi x}{2}$ has nonremovable discontinuities at each $2k + 1$, k is an integer.

54. $f(x) = 3 - [\![x]\!]$ has nonremovable discontinuities at each integer k.

56. $\lim\limits_{x\to 0^+} f(x) = 0$

$\lim\limits_{x\to 0^-} f(x) = 0$

f is not continuous at $x = -4$



58. $\lim\limits_{x\to 0^-} g(x) = \lim\limits_{x\to 0^-} \dfrac{4\sin x}{x} = 4$

$\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^+} (a - 2x) = a$

Let $a = 4$.

60. $\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} \dfrac{x^2 - a^2}{x - a}$

$= \lim\limits_{x\to a} (x + a) = 2a$

Find a such that $2a = 8 \Rightarrow a = 4$.

62. $f(g(x)) = \dfrac{1}{\sqrt{x - 1}}$

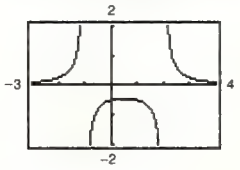
Nonremovable discontinuity at $x = 1$. Continuous for all $x > 1$.
Because $f \circ g$ is not defined for $x < 1$, it is better to say that $f \circ g$ is discontinuous from the right at $x = 1$.

64. $f(g(x)) = \sin x^2$

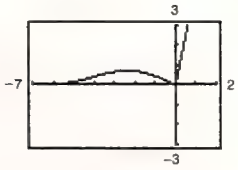
Continuous for all real x

66. $h(x) = \dfrac{1}{(x + 1)(x - 2)}$

Nonremovable discontinuity at $x = -1$ and $x = 2$.



68. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \geq 0 \end{cases}$



$f(0) = 5(0) = 0$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \dfrac{(\cos x - 1)}{x} = 0$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} (5x) = 0$

Therefore, $\lim\limits_{x\to 0} f(x) = 0 = f(0)$ and f is continuous on the entire real line. ($x = 0$ was the only possible discontinuity.)

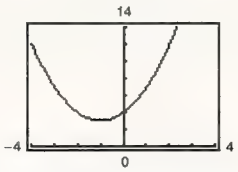
70. $f(x) = x\sqrt{x + 3}$

Continuous on $[-3, \infty]$

72. $f(x) = \dfrac{x + 1}{\sqrt{x}}$

Continuous on $(0, \infty)$

74. $f(x) = \dfrac{x^3 - 8}{x - 2}$



The graph **appears** to be continuous on the interval $[-4, 4]$. Since $f(2)$ is not defined, we know that f has a discontinuity at $x = 2$. This discontinuity is removable so it does not show up on the graph.

76. $f(x) = x^3 + 3x - 2$ is continuous on $[0, 1]$.

$f(0) = -2$ and $f(1) = 2$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1.

78. $f(x) = \dfrac{-4}{x} + \tan\dfrac{\pi x}{8}$ is continuous on $[1, 3]$.

$f(1) = -4 + \tan\dfrac{\pi}{8} < 0$ and $f(3) = -\dfrac{4}{3} + \tan\dfrac{3\pi}{8} > 0.$

By the Intermediate Value Theorem, $f(1) = 0$ for at least one value of c between 1 and 3.

80. $f(x) = x^3 + 3x - 2$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -2$ and $f(1) = 2$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1. Using a graphing utility, we find that $x \approx 0.5961$.

82. $h(\theta) = 1 + \theta - 3\tan\theta$

h is continuous on $[0, 1]$.

$h(0) = 1 > 0$ and $h(1) \approx -2.67 < 0$.

By the Intermediate Value Theorem, $h(\theta) = 0$ for at least one value θ between 0 and 1. Using a graphing utility, we find that $\theta \approx 0.4503$.

84. $f(x) = x^2 - 6x + 8$

f is continuous on $[0, 3]$.

$f(0) = 8$ and $f(3) = -1$

$-1 < 0 < 8$

The Intermediate Value Theorem applies.

$$x^2 - 6x + 8 = 0$$
$$(x - 2)(x - 4) = 0$$
$$x = 2 \text{ or } x = 4$$

$c = 2$ ($x = 4$ is not in the interval.)

Thus, $f(2) = 0$.

86. $f(x) = \dfrac{x^2 + x}{x - 1}$

f is continuous on $\left[\frac{5}{2}, 4\right]$. The nonremovable discontinuity, $x = 1$, lies outside the interval.

$f\left(\dfrac{5}{2}\right) = \dfrac{35}{6}$ and $f(4) = \dfrac{20}{3}$

$\dfrac{35}{6} < 6 < \dfrac{20}{3}$

The Intermediate Value Theorem applies.

$$\frac{x^2 + x}{x - 1} = 6$$
$$x^2 + x = 6x - 6$$
$$x^2 - 5x + 6 = 0$$
$$(x - 2)(x - 3) = 0$$
$$x = 2 \text{ or } x = 3$$

$c = 3$ ($x = 2$ is not in the interval.)

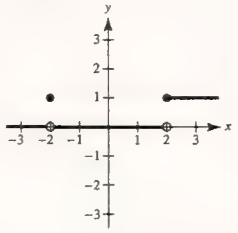
Thus, $f(3) = 6$.

88. A discontinuity at $x = c$ is removable if you can define (or redefine) the function at $x = c$ in such a way that the new function is continuous at $x = c$. Answers will vary.

(a) $f(x) = \dfrac{|x-2|}{x-2}$

(b) $f(x) = \dfrac{\sin(x+2)}{x+2}$

(c) $f(x) = \begin{cases} 1, & \text{if } x \geq 2 \\ 0, & \text{if } -2 < x < 2 \\ 1, & \text{if } x = -2 \\ 0, & \text{if } x < -2 \end{cases}$



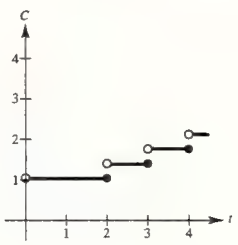
90. If f and g are continuous for all real x, then so is $f + g$ (Theorem 1.11, part 2). However, f/g might not be continuous if $g(x) = 0$. For example, let $f(x) = x$ and $g(x) = x^2 - 1$. Then f and g are continuous for all real x, but f/g is not continuous at $x = \pm 1$.

92. $C = \begin{cases} 1.04, & 0 < t \leq 2 \\ 1.04 + 0.36[\![t-1]\!], & t > 2, t \text{ is not an integer} \\ 1.04 + 0.36(t-2), & t > 2, t \text{ is an integer} \end{cases}$

Nonremovable discontinuity at each integer greater than 2.

You can also write C as

$$C = \begin{cases} 1.04, & 0 < t \leq 2 \\ 1.04 - 0.36[\![2-t]\!], & t > 2 \end{cases}.$$



94. Let $s(t)$ be the position function for the run up to the campsite. $s(0) = 0$ ($t = 0$ corresponds to 8:00 A.M., $s(20) = k$ (distance to campsite)). Let $r(t)$ be the position function for the run back down the mountain: $r(0) = k$, $r(10) = 0$. Let $f(t) = s(t) - r(t)$.

When $t = 0$ (8:00 A.M.), $f(0) = s(0) - r(0) = 0 - k < 0$.

When $t = 10$ (8:10 A.M.), $f(10) = s(10) - r(10) > 0$.

Since $f(0) < 0$ and $f(10) > 0$, then there must be a value t in the interval $[0, 10]$ such that $f(t) = 0$. If $f(t) = 0$, then $s(t) - r(t) = 0$, which gives us $s(t) = r(t)$. Therefore, at some time t, where $0 \leq t \leq 10$, the position functions for the run up and the run down are equal.

96. Suppose there exists x_1 in $[a, b]$ such that $f(x_1) > 0$ and there exists x_2 in $[a, b]$ such that $f(x_2) < 0$. Then by the Intermediate Value Theorem, $f(x)$ must equal zero for some value of x in $[x_1, x_2]$ (or $[x_2, x_1]$ if $x_2 < x_1$). Thus, f would have a zero in $[a, b]$, which is a contradiction. Therefore, $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

98. If $x = 0$, then $f(0) = 0$ and $\lim_{x \to 0} f(x) = 0$. Hence, f is continuous at $x = 0$.

If $x \neq 0$, then $\lim_{t \to x} f(t) = 0$ for x rational, whereas $\lim_{t \to x} f(t) = \lim_{t \to x} kt = kx \neq 0$ for x irrational. Hence, f is not continuous for all $x \neq 0$.

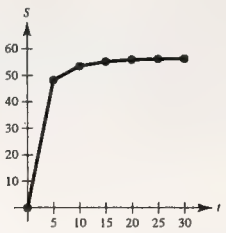
100. True

1. $f(c) = L$ is defined.
2. $\lim_{x \to c} f(x) = L$ exists.
3. $f(c) = \lim_{x \to c} f(x)$

All of the conditions for continuity are met.

102. False; a rational function can be written as $P(x)/Q(x)$ where P and Q are polynomials of degree m and n, respectively. It can have, at most, n discontinuities.

104. (a)



(b) There appears to be a limiting speed and a possible cause is air resistance.

106. Let y be a real number. If $y = 0$, then $x = 0$. If $y > 0$, then let $0 < x_0 < \pi/2$ such that $M = \tan x_0 > y$ (this is possible since the tangent function increases without bound on $[0, \pi/2)$). By the Intermediate Value Theorem, $f(x) = \tan x$ is continuous on $[0, x_0]$ and $0 < y < M$, which implies that there exists x between 0 and x_0 such that $\tan x = y$. The argument is similar if $y < 0$.

108. **1.** $f(c)$ is defined.

2. $\lim_{x \to c} f(x) = \lim_{\Delta x \to 0} f(c + \Delta x) = f(c)$ exists.
[Let $x = c + \Delta x$. As $x \to c$, $\Delta x \to 0$]

3. $\lim_{x \to c} f(x) = f(c)$.
Therefore, f is continuous at $x = c$.

110. Define $f(x) = f_2(x) - f_1(x)$. Since f_1 and f_2 are continuous on $[a, b]$, so is f.

$$f(a) = f_2(a) - f_1(a) > 0 \quad \text{and} \quad f(b) = f_2(b) - f_1(b) < 0.$$

By the Intermediate Value Theorem, there exists c in $[a, b]$ such that $f(c) = 0$.

$$f(c) = f_2(c) - f_1(c) = 0 \Rightarrow f_1(c) = f_2(c)$$

Section 1.5 Infinite Limits

2. $\lim_{x \to -2^+} \dfrac{1}{x + 2} = \infty$

$\lim_{x \to -2^-} \dfrac{1}{x + 2} = -\infty$

4. $\lim_{x \to -2^+} \sec \dfrac{\pi x}{4} = \infty$

$\lim_{x \to -2^-} \sec \dfrac{\pi x}{4} = -\infty$

6. $f(x) = \dfrac{x}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	-1.077	-5.082	-50.08	-500.1	499.9	49.92	4.915	0.9091

$\lim_{x \to -3^-} f(x) = -\infty$

$\lim_{x \to -3^+} f(x) = \infty$

8. $f(x) = \sec\dfrac{\pi x}{6}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	-3.864	-19.11	-191.0	-1910	1910	191.0	19.11	3.864

$\lim_{x\to -3^-} f(x) = -\infty$

$\lim_{x\to -3^+} f(x) = \infty$

10. $\lim_{x\to 2^+} \dfrac{4}{(x-2)^3} = \infty$

$\lim_{x\to 2^-} \dfrac{4}{(x-2)^3} = -\infty$

Therefore, $x = 2$ is a vertical asymptote.

12. $\lim_{x\to 0^-} \dfrac{2+x}{x^2(1-x)} = \lim_{x\to 0^+} \dfrac{2+x}{x^2(1-x)} = \infty$

Therefore, $x = 0$ is a vertical asymptote.

$\lim_{x\to 1^-} \dfrac{2+x}{x^2(1-x)} = \infty$

$\lim_{x\to 1^+} \dfrac{2+x}{x^2(1-x)} = -\infty$

Therefore, $x = 1$ is a vertical asymptote.

14. No vertical asymptote since the denominator is never zero.

16. $\lim_{s\to -5^-} h(s) = -\infty$ and $\lim_{s\to -5^+} h(s) = \infty$.

Therefore, $s = -5$ is a vertical asymptote.

$\lim_{s\to 5^-} h(s) = -\infty$ and $\lim_{s\to 5^+} h(s) = \infty$.

Therefore, $s = 5$ is a vertical asymptote.

18. $f(x) = \sec \pi x = \dfrac{1}{\cos \pi x}$ has vertical asymptotes at

$x = \dfrac{2n+1}{2}$, n any integer.

20. $g(x) = \dfrac{(1/2)x^3 - x^2 - 4x}{3x^2 - 6x - 24} = \dfrac{1}{6}\dfrac{x(x^2 - 2x - 8)}{x^2 - 2x - 8}$

$= \dfrac{1}{6}x,$

$x \neq -2, 4$

No vertical asymptotes. The graph has holes at $x = -2$ and $x = 4$.

22. $f(x) = \dfrac{4(x^2 + x - 6)}{x(x^3 - 2x^2 - 9x + 18)} = \dfrac{4(x+3)(x-2)}{x(x-2)(x^2-9)} = \dfrac{4}{x(x-3)}, x \neq -3, 2$

Vertical asymptotes at $x = 0$ and $x = 3$. The graph has holes at $x = -3$ and $x = 2$.

24. $h(x) = \dfrac{x^2 - 4}{x^3 + 2x^2 + x + 2} = \dfrac{(x+2)(x-2)}{(x+2)(x^2+1)}$

has no vertical asymptote since

$$\lim_{x\to -2} h(x) = \lim_{x\to -2} \frac{x-2}{x^2+1} = -\frac{4}{5}.$$

26. $h(t) = \dfrac{t(t-2)}{(t-2)(t+2)(t^2+4)} = \dfrac{t}{(t+2)(t^2+4)}, t \neq 2$

Vertical asymptote at $t = -2$. The graph has a hole at $t = 2$.

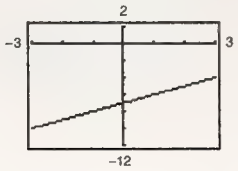
28. $g(\theta) = \dfrac{\tan\theta}{\theta} = \dfrac{\sin\theta}{\theta\cos\theta}$ has vertical asymptotes at

$$\theta = \frac{(2n+1)\pi}{2} = \frac{\pi}{2} + n\pi,\ n \text{ any integer.}$$

There is no vertical asymptote at $\theta = 0$ since

$$\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1.$$

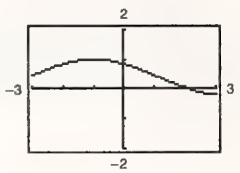
30. $\displaystyle\lim_{x\to -1}\frac{x^2-6x-7}{x+1} = \lim_{x\to -1}(x-7) = -8$



Removable discontinuity at $x = -1$

32. $\displaystyle\lim_{x\to -1}\frac{\sin(x+1)}{x+1} = 1$

Removable discontinuity at $x = -1$



34. $\displaystyle\lim_{x\to 1^+}\frac{2+x}{1-x} = -\infty$

36. $\displaystyle\lim_{x\to 4^-}\frac{x^2}{x^2+16} = \frac{1}{2}$

38. $\displaystyle\lim_{x\to -(1/2)^+}\frac{6x^2+x-1}{4x^2-4x-3} = \lim_{x\to -(1/2)^+}\frac{3x-1}{2x-3} = \frac{5}{8}$

40. $\displaystyle\lim_{x\to 3}\frac{x-2}{x^2} = \frac{1}{9}$

42. $\displaystyle\lim_{x\to 0^-}\left(x^2 - \frac{1}{x}\right) = \infty$

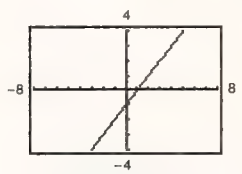
44. $\displaystyle\lim_{x\to(\pi/2)^+}\frac{-2}{\cos x} = \infty$

46. $\displaystyle\lim_{x\to 0}\frac{(x+2)}{\cot x} = \lim_{x\to 0}[(x+2)\tan x] = 0$

48. $\displaystyle\lim_{x\to(1/2)^-} x^2\tan\pi x = \infty$ and $\displaystyle\lim_{x\to(1/2)^+} x^2\tan\pi x = -\infty$.
Therefore, $\displaystyle\lim_{x\to(1/2)} x^2\tan\pi x$ does not exist.

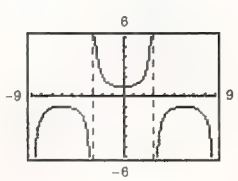
50. $f(x) = \dfrac{x^3-1}{x^2+x+1}$

$\displaystyle\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x-1) = 0$



52. $f(x) = \sec\dfrac{\pi x}{6}$

$\displaystyle\lim_{x\to 3^+} f(x) = -\infty$



54. The line $x = c$ is a vertical asymptote if the graph of f approaches $\pm\infty$ as x approaches c.

56. No. For example, $f(x) = \dfrac{1}{x^2+1}$ has no vertical asymptote.

58. $P = \dfrac{k}{V}$

$\displaystyle\lim_{V\to 0^+}\frac{k}{V} = k(\infty) = \infty$ (In this case we know that $k > 0$.)

60. (a) $r = 50\pi \sec^2 \frac{\pi}{6} = \frac{200\pi}{3}$ ft/sec

(b) $r = 50\pi \sec^2 \frac{\pi}{3} = 200\pi$ ft/sec

(c) $\lim_{\theta \to (\pi/2)^-} [50\pi \sec^2 \theta] = \infty$

62. $m = \dfrac{m_0}{\sqrt{1 - (v^2/c^2)}}$

$\lim_{v \to c^-} m = \lim_{v \to c^-} \dfrac{m_0}{\sqrt{1 - (v^2/c^2)}} = \infty$

64. (a) Average speed $= \dfrac{\text{Total distance}}{\text{Total time}}$

$$50 = \frac{2d}{(d/x) + (d/y)}$$

$$50 = \frac{2xy}{y + x}$$

$$50y + 50x = 2xy$$

$$50x = 2xy - 50y$$

$$50x = 2y(x - 25)$$

$$\frac{25x}{x - 25} = y$$

Domain: $x > 25$

(b)

x	30	40	50	60
y	150	66.667	50	42.857

(c) $\lim_{x \to 25^+} \dfrac{25x}{x - 25} = \infty$

As x gets close to 25 mph, y becomes larger and larger.

66. (a) $A = \frac{1}{2}bh - \frac{1}{2}r^2\theta = \frac{1}{2}(10)(10 \tan \theta) - \frac{1}{2}(10)^2\theta$

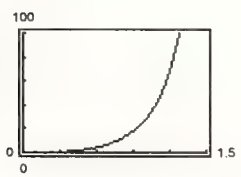
$$= 50 \tan \theta - 50\,\theta$$

Domain: $\left(0, \dfrac{\pi}{2}\right)$

(b)

ϕ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$	0.47	4.21	18.0	68.6	630.1

(c)



(d) $\lim_{\theta \to \pi/2^-} A = \infty$

68. False; for instance, let

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

The graph of f has a hole at $(1, 2)$, not a vertical asymptote.

70. True

72. Let $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{x^4}$, and $c = 0$.

$\lim_{x \to 0} \dfrac{1}{x^2} = \infty$ and $\lim_{x \to 0} \dfrac{1}{x^4} = \infty$, but

$\lim_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{x^4}\right) = \lim_{x \to 0} \left(\dfrac{x^2 - 1}{x^4}\right) = -\infty \neq 0.$

74. Given $\lim_{x \to c} f(x) = \infty$, let $g(x) = 1$. then $\lim_{x \to c} \dfrac{g(x)}{f(x)} = 0$ by Theorem 1.15.

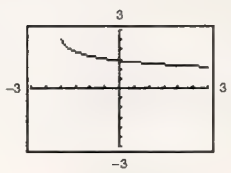
Review Exercises for Chapter 1

2. Precalculus. $L = \sqrt{(9-1)^2 + (3-1)^2} \approx 8.25$

4.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	1.432	1.416	1.414	1.414	1.413	1.397

$\lim_{x\to 0} f(x) \approx 1.414$



6. $g(x) = \dfrac{3x}{x-2}$

(a) $\lim_{x\to 2} g(x)$ does not exist.

(b) $\lim_{x\to 0} g(x) = 0$

8. $\lim_{x\to 9} \sqrt{x} = \sqrt{9} = 3.$

Let $\epsilon > 0$ be given. We need

$$|\sqrt{x} - 3| < \epsilon \Rightarrow |\sqrt{x} + 3||\sqrt{x} - 3| < \epsilon|\sqrt{x} + 3|$$
$$|x - 9| < \epsilon|\sqrt{x} + 3|$$

Assuming $4 < x < 16$, you can choose $\delta = 5\epsilon$.

Hence, for $0 < |x - 9| < \delta = 5\epsilon$, you have

$$|x - 9| < 5\epsilon < |\sqrt{x} + 3|\epsilon$$
$$|\sqrt{x} - 3| < \epsilon$$
$$|f(x) - L| < \epsilon$$

10. $\lim_{x\to 5} 9 = 9$. Let $\epsilon > 0$ be given. δ can be any positive number. Hence, for $0 < |x - 5| < \delta$, you have

$$|9 - 9| < \epsilon$$
$$|f(x) - L| < \epsilon$$

12. $\lim_{y\to 4} 3|y - 1| = 3|4 - 1| = 9$

14. $\lim_{t\to 3} \dfrac{t^2 - 9}{t - 3} = \lim_{t\to 3} (t + 3) = 6$

16. $$\lim_{x\to 0} \frac{\sqrt{4+x} - 2}{x} = \lim_{x\to 0} \frac{\sqrt{4+x} - 2}{x} \cdot \frac{\sqrt{4+x} + 2}{\sqrt{4+x} + 2}$$
$$= \lim_{x\to 0} \frac{1}{\sqrt{4+x} + 2} = \frac{1}{4}$$

18. $$\lim_{s\to 0} \frac{\left(1/\sqrt{1+s}\right) - 1}{s} = \lim_{s\to 0} \left[\frac{\left(1/\sqrt{1+s}\right) - 1}{s} \cdot \frac{\left(1/\sqrt{1+s}\right) + 1}{\left(1/\sqrt{1+s}\right) + 1}\right]$$
$$= \lim_{s\to 0} \frac{[1/(1+s)] - 1}{s\left[\left(1/\sqrt{1+s}\right) + 1\right]} = \lim_{s\to 0} \frac{-1}{(1+s)\left[\left(1/\sqrt{1+s}\right) + 1\right]} = -\frac{1}{2}$$

20. $$\lim_{x\to -2} \frac{x^2 - 4}{x^3 + 8} = \lim_{x\to -2} \frac{(x+2)(x-2)}{(x+2)(x^2 - 2x + 4)}$$
$$= \lim_{x\to -2} \frac{x-2}{x^2 - 2x + 4}$$
$$= -\frac{4}{12} = -\frac{1}{3}$$

22. $\lim_{x\to(\pi/4)} \dfrac{4x}{\tan x} = \dfrac{4(\pi/4)}{1} = \pi$

24. $\lim_{\Delta x \to 0} \dfrac{\cos(\pi + \Delta x) + 1}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{\cos \pi \cos \Delta x - \sin \pi \sin \Delta x + 1}{\Delta x}$

$$= \lim_{\Delta x \to 0} \left[-\frac{(\cos \Delta x - 1)}{\Delta x} \right] - \lim_{\Delta x \to 0} \left[\sin \pi \frac{\sin \Delta x}{\Delta x} \right]$$

$$= -0 - (0)(1) = 0$$

26. $\lim_{x \to c} [f(x) + 2g(x)] = -\frac{3}{4} + 2\left(\frac{2}{3}\right) = \frac{7}{12}$

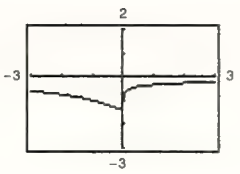
28. $f(x) = \dfrac{1 - \sqrt[3]{x}}{x - 1}$

(a)

x	1.1	1.01	1.001	1.0001
$f(x)$	−0.3228	−0.3322	−0.3332	−0.3333

$\lim_{x \to 1^+} \dfrac{1 - \sqrt[3]{x}}{x - 1} \approx -0.333$ (Actual limit is $-\frac{1}{3}$.)

(b)



(c) $\lim_{x \to 1^+} \dfrac{1 - \sqrt[3]{x}}{x - 1} = \lim_{x \to 1^+} \dfrac{1 - \sqrt[3]{x}}{x - 1} \cdot \dfrac{1 + \sqrt[3]{x} + (\sqrt[3]{x})^2}{1 + \sqrt[3]{x} + (\sqrt[3]{x})^2}$

$$= \lim_{x \to 1^+} \frac{1 - x}{(x - 1)[1 + \sqrt[3]{x} + (\sqrt[3]{x})^2]}$$

$$= \lim_{x \to 1^+} \frac{-1}{1 + \sqrt[3]{x} + (\sqrt[3]{x})^2}$$

$$= -\frac{1}{3}$$

30. $s(t) = 0 \Rightarrow -4.9t^2 + 200 = 0 \Rightarrow t^2 \approx 40.816 \Rightarrow t \approx 6.39$ sec

When $t = 6.39$, the velocity is approximately

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t} = \lim_{t \to a} -4.9(a + t)$$

$$= \lim_{t \to 6.39} -4.9(6.39 + 6.39) = -62.6 \text{ m/sec.}$$

32. $\lim_{x \to 4} [\![x - 1]\!]$ does not exist. The graph jumps from 2 to 3 at $x = 4$.

34. $\lim_{x \to 1^+} g(x) = 1 + 1 = 2.$

36. $\lim_{s \to -2} f(s) = 2$

38. $f(x) = \begin{cases} \dfrac{3x^2 - x - 2}{x - 1}, & x \neq 1 \\ 0, & x = 1 \end{cases}$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{3x^2 - x - 2}{x - 1}$$

$$= \lim_{x \to 1} (3x + 2) = 5 \neq 0$$

Removable discontinuity at $x = 1$

Continuous on $(-\infty, 1) \cup (1, \infty)$

40. $f(x) = \begin{cases} 5 - x, & x \leq 2 \\ 2x - 3, & x > 2 \end{cases}$

$\lim_{x \to 2^-} (5 - x) = 3$

$\lim_{x \to 2^+} (2x - 3) = 1$

Nonremovable discontinuity at $x = 2$
Continuous on $(-\infty, 2) \cup (2, \infty)$

42. $f(x) = \sqrt{\frac{x+1}{x}} = \sqrt{1 + \frac{1}{x}}$

$\lim_{x \to 0^+} \sqrt{1 + \frac{1}{x}} = \infty$

Domain: $(-\infty, -1], (0, \infty)$

Nonremovable discontinuity at $x = 0$
Continuous on $(-\infty, -1] \cup (0, \infty)$

44. $f(x) = \frac{x+1}{2x+2}$

$\lim_{x \to -1} \frac{x+1}{2(x+1)} = \frac{1}{2}$

Removable discontinuity at $x = -1$
Continuous on $(-\infty, -1) \cup (-1, \infty)$

46. $f(x) = \tan 2x$

Nonremovable discontinuities when

$$x = \frac{(2n+1)\pi}{4}$$

Continuous on

$$\left(\frac{(2n-1)\pi}{4}, \frac{(2n+1)\pi}{4}\right)$$

for all integers n.

48. $\lim_{x \to 1^+} (x + 1) = 2$

$\lim_{x \to 3^-} (x + 1) = 4$

Find b and c so that $\lim_{x \to 1^-} (x^2 + bx + c) = 2$ and $\lim_{x \to 3^+} (x^2 + bx + c) = 4$.

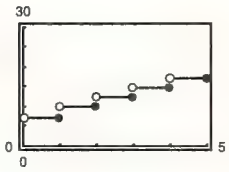
Consequently we get $1 + b + c = 2$ and $9 + 3b + c = 4$.

Solving simultaneously, $b = -3$ and $c = 4$.

50. $C = 9.80 + 2.50[-[\![-x]\!] - 1], \ x > 0$

$= 9.80 - 2.50[[\![-x]\!] + 1]$

C has a nonremovable discontinuity at each integer.



52. $f(x) = \sqrt{(x-1)x}$

(a) Domain: $(-\infty, 0] \cup [1, \infty)$

(b) $\lim_{x \to 0^-} f(x) = 0$

(c) $\lim_{x \to 1^+} f(x) = 0$

54. $h(x) = \frac{4x}{4 - x^2}$

Vertical asymptotes at $x = 2$ and $x = -2$

56. $f(x) = \csc \pi x$

Vertical asymptote at every integer k

58. $\lim_{x \to (1/2)^+} \frac{x}{2x - 1} = \infty$

60. $\lim_{x \to -1^-} \frac{x+1}{x^4 - 1} = \lim_{x \to -1^-} \frac{1}{(x^2+1)(x-1)} = -\frac{1}{4}$

62. $\lim_{x \to -1^+} \frac{x^2 - 2x + 1}{x + 1} = \infty$

64. $\lim_{x \to 2^-} \frac{1}{\sqrt[3]{x^2 - 4}} = -\infty$

66. $\lim_{x \to 0^+} \frac{\sec x}{x} = \infty$

68. $\lim_{x \to 0^-} \frac{\cos^2 x}{x} = -\infty$

70. $f(x) = \dfrac{\tan 2x}{x}$

(a)

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	2.0271	2.0003	2.0000	2.0000	2.0003	2.0271

$$\lim_{x\to 0}\frac{\tan 2x}{x} = 2$$

(b) Yes, define

$$f(x) = \begin{cases} \dfrac{\tan 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}.$$

Now $f(x)$ is continuous at $x = 0$.

Problem Solving for Chapter 1

2. (a) Area $\triangle PAO = \frac{1}{2}bh = \frac{1}{2}(1)(x) = \frac{x}{2}$

Area $\triangle PBO = \frac{1}{2}bh = \frac{1}{2}(1)(y) = \frac{y}{2} = \frac{x^2}{2}$

(b) $a(x) = \dfrac{\text{Area } \triangle PBO}{\text{Area } \triangle PAO} = \dfrac{x^2/2}{x/2} = x$

x	4	2	1	0.1	0.01
Area $\triangle PAO$	2	1	1/2	1/20	1/200
Area $\triangle PBO$	8	2	1/2	1/200	1/20,000
$a(x)$	4	2	1	1/10	1/100

(c) $\lim_{x\to 0^+} a(x) = \lim_{x\to 0^+} x = 0$

4. (a) Slope $= \dfrac{4-0}{3-0} = \dfrac{4}{3}$

(b) Slope $= -\dfrac{3}{4}$ Tangent line: $y - 4 = -\dfrac{3}{4}(x-3)$

$$y = -\frac{3}{4}x + \frac{25}{4}$$

(c) Let $Q = (x, y) = \left(x, \sqrt{25 - x^2}\right)$

$$m_x = \frac{\sqrt{25-x^2} - 4}{x - 3}$$

(d) $\lim_{x\to 3} m_x = \lim_{x\to 3} \dfrac{\sqrt{25-x^2}-4}{x-3} \cdot \dfrac{\sqrt{25-x^2}+4}{\sqrt{25-x^2}+4}$

$$= \lim_{x\to 3} \frac{25 - x^2 - 16}{(x-3)\left(\sqrt{25-x^2}+4\right)}$$

$$= \lim_{x\to 3} \frac{(3-x)(3+x)}{(x-3)\left(\sqrt{25-x^2}+4\right)}$$

$$= \lim_{x\to 3} \frac{-(3+x)}{\sqrt{25-x^2}+4} = \frac{-6}{4+4} = -\frac{3}{4}$$

This is the slope of the tangent line at P.

6. $\dfrac{\sqrt{a+bx}-\sqrt{3}}{x} = \dfrac{\sqrt{a+bx}-\sqrt{3}}{x} \cdot \dfrac{\sqrt{a+bx}+\sqrt{3}}{\sqrt{a+bx}+\sqrt{3}}$

$$= \frac{(a+bx)-3}{x\left(\sqrt{a+bx}+\sqrt{3}\right)}$$

Letting $a = 3$ simplifies the numerator.

Thus,

$$\lim_{x\to 0}\frac{\sqrt{3+bx}-\sqrt{3}}{x} = \lim_{x\to 0}\frac{bx}{x\left(\sqrt{3+bx}+\sqrt{3}\right)}$$

$$= \lim_{x\to 0}\frac{b}{\sqrt{3+bx}+\sqrt{3}}.$$

Setting $\dfrac{b}{\sqrt{3}+\sqrt{3}} = \sqrt{3}$, you obtain $b = 6$.

Thus, $a = 3$ and $b = 6$.

8. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (a^2 - 2) = a^2 - 2$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \dfrac{ax}{\tan x} = a \left(\text{because } \lim_{x\to 0}\dfrac{\tan x}{x} = 1\right)$

Thus,

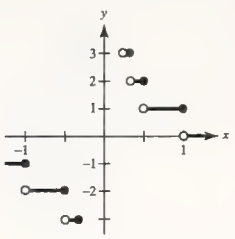
$$a^2 - 2 = a$$

$$a^2 - a - 2 = 0$$

$$(a-2)(a+1) = 0$$

$$a = -1, 2$$

10.



(a) $f\left(\frac{1}{4}\right) = [\![4]\!] = 4$

$f(3) = \left[\!\left[\frac{1}{3}\right]\!\right] = 0$

$f(1) = [\![1]\!] = 1$

(b) $\lim_{x \to 1^-} f(x) = 1$

$\lim_{x \to 1^+} f(x) = 0$

$\lim_{x \to 0^-} f(x) = -\infty$

$\lim_{x \to 0^+} f(x) = \infty$

(c) f is continuous for all real numbers except

$x = 0, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots$

12. (a) $v^2 = \dfrac{192{,}000}{r} + {v_0}^2 - 48$

$$\frac{192{,}000}{r} = v^2 - {v_0}^2 + 48$$

$$r = \frac{192{,}000}{v - {v_0}^2 + 48}$$

$$\lim_{v \to 0} r = \frac{192{,}000}{48 - {v_0}^2}$$

Let $v_0 = \sqrt{48} = 4\sqrt{3}$ feet/sec.

(b) $v^2 = \dfrac{1920}{r} + {v_0}^2 - 2.17$

$$\frac{1920}{r} = v^2 - {v_0}^2 + 2.17$$

$$r = \frac{1920}{v^2 - {v_0}^2 + 2.17}$$

$$\lim_{v \to 0} r = \frac{1920}{2.17 - {v_0}^2}$$

Let $v_0 = \sqrt{2.17}$ mi/sec $(\approx 1.47$ mi/sec).

(c) $r = \dfrac{10{,}600}{v^2 - {v_0}^2 + 6.99}$

$$\lim_{v \to 0} r = \frac{10{,}600}{6.99 - {v_0}^2}$$

Let $v_0 = \sqrt{6.99} \approx 2.64$ mi/sec.

Since this is smaller than the escape velocity for earth, the mass is less.

14. Let $a \neq 0$ and let $\epsilon > 0$ be given. There exists $\delta_1 > 0$ such that if $0 < |x - 0| < \delta$, then $|f(x) - L| < \epsilon$. Let $\delta = \delta_1/|a|$. Then for $0 < |x - 0| < \delta = \delta_1/|a|$, you have

$$|x| < \frac{\delta_1}{|a|}$$

$$|ax| < \delta_1$$

$$|f(ax) - L| < \epsilon.$$

As a counterexample, let $f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$.

Then $\lim_{x \to 0} f(x) = 1 = L$,

but $\lim_{x \to 0} f(ax) = \lim_{x \to 0} f(0) = 2$.

CHAPTER 2
Differentiation

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CHAPTER 2
Differentiation

Section 2.1 The Derivative and the Tangent Line Problem

Solutions to Even-Numbered Exercises

2. (a) $m = \frac{1}{4}$

(b) $m = 1$

4. (a) $\frac{f(4) - f(1)}{4 - 1} = \frac{5 - 2}{3} = 1$

$\frac{f(4) - f(3)}{4 - 3} \approx \frac{5 - 4.75}{1} = 0.25$

Thus, $\frac{f(4) - f(1)}{4 - 1} > \frac{f(4) - f(3)}{4 - 3}$

(b) The slope of the tangent line at $(1, 2)$ equals $f'(1)$. This slope is steeper than the slope of the line through $(1, 2)$ and $(4, 5)$. Thus,

$$\frac{f(4) - f(1)}{4 - 1} < f'(1).$$

6. $g(x) = \frac{3}{2}x + 1$ is a line. Slope $= \frac{3}{2}$

8. Slope at $(2, 1) = \lim_{\Delta x \to 0} \frac{g(2 + \Delta x) - g(2)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{5 - (2 + \Delta x)^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{5 - 4 - 4(\Delta x) - (\Delta x)^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (-4 - \Delta x) = -4$$

10. Slope at $(-2, 7) = \lim_{\Delta t \to 0} \frac{h(-2 + \Delta t) - h(-2)}{\Delta t}$

$$= \lim_{\Delta t \to 0} \frac{(-2 + \Delta t)^2 + 3 - 7}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{4 - 4(\Delta t) + (\Delta t)^2 - 4}{\Delta t}$$

$$= \lim_{\Delta t \to 0} (-4 + \Delta t) = -4$$

12. $g(x) = -5$

$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-5 - (-5)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$$

14. $f(x) = 3x + 2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[3(x + \Delta x) + 2] - [3x + 2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 3 = 3$$

16. $f(x) = 9 - \frac{1}{2}x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[9 - (1/2)(x + \Delta x)] - [9 - (1/2)x]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(-\frac{1}{2}\right) = -\frac{1}{2}$$

18. $f(x) = 1 - x^2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[1 - (x + \Delta x)^2] - [1 - x^2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1 - x^2 - 2x\Delta x - (\Delta x)^2 - 1 + x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} (-2x - \Delta x) = -2x$$

20. $f(x) = x^3 + x^2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^3 + (x + \Delta x)^2] - [x^3 + x^2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + x^2 + 2x\Delta x + (\Delta x)^2 - x^3 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2 + 2x + (\Delta x)) = 3x^2 + 2x$$

22. $f(x) = \dfrac{1}{x^2}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{1}{(x + \Delta x)^2} - \dfrac{1}{x^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 - (x + \Delta x)^2}{\Delta x(x + \Delta x)^2 x^2}$$

$$= \lim_{\Delta x \to 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x(x + \Delta x)^2 x^2}$$

$$= \lim_{\Delta x \to 0} \frac{-2x - \Delta x}{(x + \Delta x)^2 x^2}$$

$$= \frac{-2x}{x^4}$$

$$= -\frac{2}{x^3}$$

24. $f(x) = \dfrac{4}{\sqrt{x}}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{4}{\sqrt{x + \Delta x}} - \dfrac{4}{\sqrt{x}}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{4\sqrt{x} - 4\sqrt{x + \Delta x}}{\Delta x \sqrt{x}\sqrt{x + \Delta x}} \cdot \left(\frac{\sqrt{x} + \sqrt{x + \Delta x}}{\sqrt{x} + \sqrt{x + \Delta x}} \right)$$

$$= \lim_{\Delta x \to 0} \frac{4x - 4(x + \Delta x)}{\Delta x \sqrt{x}\sqrt{x + \Delta x}\left(\sqrt{x} + \sqrt{x + \Delta x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{-4}{\sqrt{x}\sqrt{x + \Delta x}\left(\sqrt{x} + \sqrt{x + \Delta x}\right)}$$

$$= \frac{-4}{\sqrt{x}\sqrt{x}\left(\sqrt{x} + \sqrt{x}\right)} = \frac{-2}{x\sqrt{x}}$$

26. (a) $f(x) = x^2 + 2x + 1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 + 2(x + \Delta x) + 1] - [x^2 + 2x + 1]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 + 2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x + 2) = 2x + 2$$

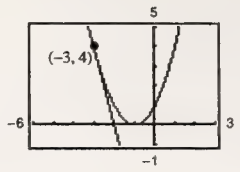
At $(-3, 4)$, the slope of the tangent line is $m = 2(-3) + 2 = -4$.

The equation of the tangent line is

$$y - 4 = -4(x + 3)$$

$$y = -4x - 8.$$

(b)



28. (a) $f(x) = x^3 + 1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^3 + 1] - (x^3 + 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 + 1 - x^3 - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} [3x^2 + 3x(\Delta x) + (\Delta x)^2] = 3x^2$$

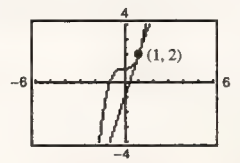
At $(1, 2)$, the slope of the tangent line is $m = 3(1)^2 = 3$.

The equation of the tangent line is

$$y - 2 = 3(x - 1)$$

$$y = 3x - 1.$$

(b)



30. (a) $f(x) = \sqrt{x - 1}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x - 1} - \sqrt{x - 1}}{\Delta x} \cdot \left(\frac{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}}{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}} \right)$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x - 1) - (x - 1)}{\Delta x\left(\sqrt{x + \Delta x - 1} + \sqrt{x - 1}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}} = \frac{1}{2\sqrt{x - 1}}$$

At $(5, 2)$, the slope of the tangent line is

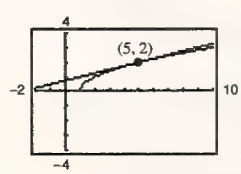
$$m = \frac{1}{2\sqrt{5 - 1}} = \frac{1}{4}$$

The equation of the tangent line is

$$y - 2 = \frac{1}{4}(x - 5)$$

$$y = \frac{1}{4}x + \frac{3}{4}$$

(b)



32. (a) $f(x) = \dfrac{1}{x+1}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{1}{x+\Delta x+1} - \dfrac{1}{x+1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+1) - (x+\Delta x+1)}{\Delta x(x+\Delta x+1)(x+1)}$$

$$= \lim_{\Delta x \to 0} -\frac{1}{(x+\Delta x+1)(x+1)}$$

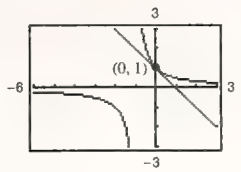
$$= -\frac{1}{(x+1)^2}$$

At (0, 1), the slope of the tangent line is

$$m = \frac{-1}{(0+1)^2} = -1.$$

The equation of the tangent line is $y = -x + 1$.

(b)



34. Using the limit definition of derivative, $f'(x) = 3x^2$. Since the slope of the given line is 3, we have

$$3x^2 = 3$$

$$x^2 = 1 \Rightarrow x = \pm 1.$$

Therefore, at the points (1, 3) and (−1, 1) the tangent lines are parallel to $3x - y - 4 = 0$. These lines have equations

$$y - 3 = 3(x - 1) \text{ and } y - 1 = 3(x + 1)$$

$$y = 3x \qquad y = 3x + 4$$

36. Using the limit definition of derivative, $f'(x) = \dfrac{-1}{2(x-1)^{3/2}}$.

Since the slope of the given line is $-\dfrac{1}{2}$, we have

$$\frac{-1}{2(x-1)^{3/2}} = -\frac{1}{2}$$

$$1 = (x-1)^{3/2}$$

$$1 = x - 1 \Rightarrow x = 2$$

At the point (2, 1), the tangent line is parallel to $x + 2y + 7 = 0$. The equation of the tangent line is

$$y - 1 = -\frac{1}{2}(x - 2)$$

$$y = -\frac{1}{2}x + 2$$

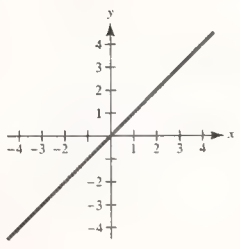
38. $h(-1) = 4$ because the tangent line passes through (−1, 4)

$$h'(-1) = \frac{6-4}{3-(-1)} = \frac{2}{4} = \frac{1}{2}$$

40. $f(x) = x^2 \Rightarrow f'(x) = 2x$ (d)

42. f' does not exist at $x = 0$. Matches (c)

44.



Answers will vary.

Sample answer: $y = x$

46. (a) Yes. $\lim_{\Delta x \to 0} \dfrac{f(x + 2\Delta x) - f(x)}{2Dx} = \lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$

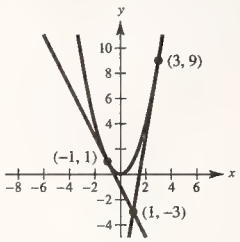
(b) No. The numerator does not approach zero.

(c) Yes.
$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x) - f(x - \Delta x) + f(x)}{2\Delta x}$$
$$= \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{2\Delta x} + \frac{f(x - \Delta x) - f(x)}{2(-\Delta x)} \right]$$
$$= \frac{1}{2} f'(x) + \frac{1}{2} f'(x) = f'(x)$$

(d) Yes. $\lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$

48. Let (x_0, y_0) be a point of tangency on the graph of f. By the limit definition for the derivative, $f'(x) = 2x$. The slope of the line through $(1, -3)$ and (x_0, y_0) equals the derivative of f at x_0:

$$\frac{-3 - y_0}{1 - x_0} = 2x_0$$
$$-3 - y_0 = (1 - x_0)2x_0$$
$$-3 - x_0^2 = 2x_0 - 2x_0^2$$
$$x_0^2 - 2x_0 - 3 = 0$$
$$(x_0 - 3)(x_0 + 1) = 0 \Rightarrow x_0 = 3, -1$$



Therefore, the points of tangency are $(3, 9)$ and $(-1, 1)$, and the corresponding slopes are 6 and -2. The equations of the tangent lines are

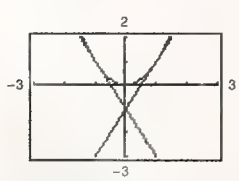
$y + 3 = 6(x - 1)$ $\quad$ $y + 3 = -2(x - 1)$

$y = 6x - 9$ $\quad$ $y = -2x - 1$

50. (a) $f(x) = x^2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - x^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

At $x = -1, f'(-1) = -2$ and the tangent line is
$y - 1 = -2(x + 1)$ or $y = -2x - 1$.
At $x = 0, f'(0) = 0$ and the tangent line is $y = 0$.
At $x = 1, f'(1) = 2$ and the tangent line is $y = 2x - 1$.

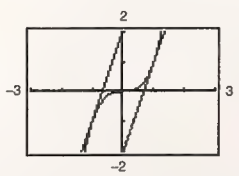


For this function, the slopes of the tangent lines are always distinct for different values of x.

(b) $g'(x) = \lim_{\Delta x \to 0} \dfrac{g(x + \Delta x) - g(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\Delta x(3x^2 + 3x(\Delta x) + (\Delta x)^2)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (3x^2 + 3x(\Delta x) + (\Delta x)^2) = 3x^2$$

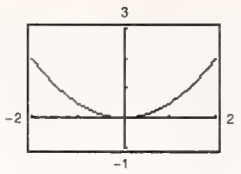
At $x = -1, g'(-1) = 3$ and the tangent line is
$y + 1 = 3(x + 1)$ or $y = 3x + 2$.
At $x = 0, g'(0) = 0$ and the tangent line is $y = 0$.
At $x = 1, g'(1) = 3$ and the tangent line is
$y - 1 = 3(x - 1)$ or $y = 3x - 2$.



For this function, the slopes of the tangent lines are sometimes the same.

52. $f(x) = \frac{1}{2}x^2$

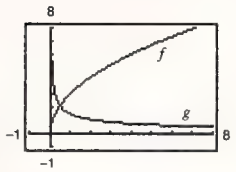
By the limit definition of the derivative we have $f'(x) = x$.



x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	2	1.125	0.5	0.125	0	0.125	0.5	1.125	2
$f'(x)$	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2

54. $g(x) = \dfrac{f(x + 0.01) - f(x)}{0.01}$

$= \left(3\sqrt{x + 0.01} - 3\sqrt{3}\right)100$

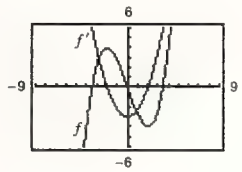


The graph of $g(x)$ is approximately the graph of $f'(x)$.

56. $f(2) = \frac{1}{4}(2^3) = 2,\ f(2.1) = 2.31525$

$f'(2) \approx \dfrac{2.31525 - 2}{2.1 - 2} = 3.1525$ [Exact: $f'(2) = 3$]

58. $f(x) = \dfrac{x^3}{4} - 3x$ and $f'(x) = \dfrac{3}{4}x^2 - 3$



60. $f(x) = x + \dfrac{1}{x}$

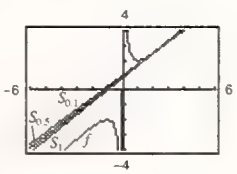
$$S_{\Delta x}(x) = \frac{f(2 + \Delta x) - f(2)}{\Delta x}(x - 2) + f(2) = \frac{(2 + \Delta x) + \dfrac{1}{2 + \Delta x} - \dfrac{5}{2}}{\Delta x}(x - 2) + \frac{5}{2}$$

$$= \frac{2(2 + \Delta x)^2 + 2 - 5(2 + \Delta x)}{2(2 + \Delta x)\,\Delta x}(x - 2) + \frac{5}{2} = \frac{(2\Delta x + 3)}{2(2 + \Delta x)}(x - 2) + \frac{5}{2}$$

(a) $\Delta x = 1$: $S_{\Delta x} = \dfrac{5}{6}(x - 2) + \dfrac{5}{2} = \dfrac{5}{6}x + \dfrac{5}{6}$

$\Delta x = 0.5$: $S_{\Delta x} = \dfrac{4}{5}(x - 2) + \dfrac{5}{2} = \dfrac{4}{5}x + \dfrac{9}{10}$

$\Delta x = 0.1$: $S_{\Delta x} = \dfrac{16}{21}(x - 2) + \dfrac{5}{2} = \dfrac{16}{21}x + \dfrac{41}{42}$



(b) As $\Delta x \to 0$, the line approaches the tangent line to f at $\left(2, \frac{5}{2}\right)$.

62. $g(x) = x(x - 1) = x^2 - x,\ c = 1$

$$g'(1) = \lim_{x \to 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - x - 0}{x - 1} = \lim_{x \to 1} \frac{x(x - 1)}{x - 1}$$

$$= \lim_{x \to 1} x = 1$$

64. $f(x) = x^3 + 2x, c = 1$

$$f'(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1} \frac{x^3 + 2x - 3}{x - 1} = \lim_{x\to 1} \frac{(x - 1)(x^2 + x + 3)}{x - 1} = \lim_{x\to 1} (x^2 + x + 3) = 5$$

66. $f(x) = \frac{1}{x}, c = 3$

$$f'(3) = \lim_{x\to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x\to 3} \frac{(1/x) - (1/3)}{x - 3} = \lim_{x\to 3} \frac{3 - x}{3x} \cdot \frac{1}{x - 3} = \lim_{x\to 3} \left(-\frac{1}{3x}\right) = -\frac{1}{9}$$

68. $g(x) = (x + 3)^{1/3}, c = -3$

$$g'(-3) = \lim_{x\to -3} \frac{g(x) - g(-3)}{x - (-3)} = \lim_{x\to -3} \frac{(x + 3)^{1/3} - 0}{x + 3} = \lim_{x\to -3} \frac{1}{(x + 3)^{2/3}}$$

Does not exist.

70. $f(x) = |x - 4|, c = 4$

$$f'(4) = \lim_{x\to 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x\to 4} \frac{|x - 4| - 0}{x - 4} = \lim_{x\to 4} \frac{|x - 4|}{x - 4}$$

Does not exist.

72. $f(x)$ is differentiable everywhere except at $x = \pm 3$. (Sharp turns in the graph.)

74. $f(x)$ is differentiable everywhere except at $x = 1$. (Discontinuity)

76. $f(x)$ is differentiable everywhere except at $x = 0$. (Sharp turn in the graph)

78. $f(x)$ is differentiable everywhere except at $x = \pm 2$. (Discontinuities)

80. $f(x)$ is differentiable everywhere except at $x = 1$. (Discontinuity)

82. $f(x) = \sqrt{1 - x^2}$

The derivative from the left does not exist because

$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{\sqrt{1 - x^2} - 0}{x - 1} = \lim_{x\to 1^-} \frac{\sqrt{1 - x^2}}{x - 1} \cdot \frac{\sqrt{1 - x^2}}{\sqrt{1 - x^2}} = \lim_{x\to 1^-} -\frac{1 + x}{\sqrt{1 - x^2}} = -\infty.$$ (Vertical tangent)

The limit from the right does not exist since f is undefined for $x > 1$. Therefore, f is not differentiable at $x = 1$.

84. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

The derivative from the left is

$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{x - 1}{x - 1} = \lim_{x\to 1^-} 1 = 1.$$

The derivative from the right is

$$\lim_{x\to 1^+} = \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1^+} (x + 1) = 2.$$

These one-sided limits are not equal. Therefore, f is not differentiable at $x = 1$.

86. Note that f is continuous at $x = 2$. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \geq 2 \end{cases}$

The derivative from the left is

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{\left(\frac{1}{2}x + 1\right) - 2}{x - 2} = \lim_{x \to 2^-} \frac{\frac{1}{2}(x - 2)}{x - 2} = \frac{1}{2}.$$

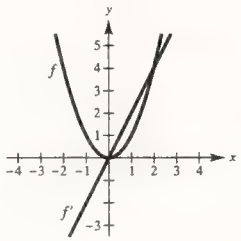
The derivative from the right is

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{\sqrt{2x} - 2}{x - 2} \cdot \frac{\sqrt{2x} + 2}{\sqrt{2x} + 2}$$

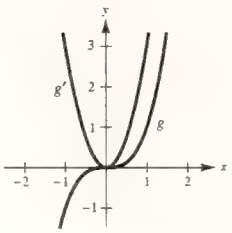
$$= \lim_{x \to 2^+} \frac{2x - 4}{(x - 2)\left(\sqrt{2x} + 2\right)} = \lim_{x \to 2^+} \frac{2(x - 2)}{(x - 2)\left(\sqrt{2x} + 2\right)} = \lim_{x \to 2^+} \frac{2}{\sqrt{2x} + 2} = \frac{1}{2}.$$

The one-sided limits are equal. Therefore, f is differentiable at $x = 2$. $\left(f'(2) = \frac{1}{2}\right)$

88. (a) $f(x) = x^2$ and $f'(x) = 2x$



(b) $g(x) = x^3$ and $g'(x) = 3x^2$



(c) The derivative is a polynomial of degree 1 less than the original function. If $h(x) = x^n$, then $h'(x) = nx^{n-1}$.

(d) If $f(x) = x^4$, then $f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^4 - x^4}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^4 + 4x^3(\Delta x) + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4 - x^4}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3)}{\Delta x}$$

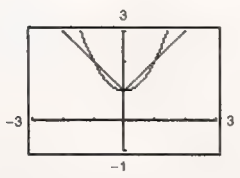
$$= \lim_{\Delta x \to 0} (4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3) = 4x^3$$

Hence, if $f(x) = x^4$, then $f'(x) = 4x^3$ which is consistent with the conjecture. However, this is not a proof, since you must verify the conjecture for all integer values of n, $n \geq 2$.

90. False. $y = |x - 2|$ is continuous at $x = 2$, but is not differentiable at $x = 2$. (Sharp turn in the graph)

92. True—see Theorem 2.1

94.



As you zoom in, the graph of $y_1 = x^2 + 1$ appears to be locally the graph of a horizontal line, whereas the graph of $y_2 = |x| + 1$ always has a sharp corner at $(0, 1)$. y_2 is not differentiable at $(0, 1)$.

Section 2.2 Basic Differentiation Rules and Rates of Change

2. (a) $y = x^{-1/2}$, $y' = -\frac{1}{2}x^{-3/2}$, $y'(1) = -\frac{1}{2}$

(b) $y = x^{-1}$, $y' = -x^{-2}$, $y'(1) = -1$

(c) $y = x^{-3/2}$, $y' = -\frac{3}{2}x^{-5/2}$, $y'(1) = -\frac{3}{2}$

(d) $y = x^{-2}$, $y' = -2x^{-3}$, $y'(1) = -2$

4. $f(x) = -2$, $f'(x) = 0$

6. $y = x^8$, $y' = 8x^7$

8. $y = \frac{1}{x^8} = x^{-8}$, $y' = 8x^{-9} = \frac{-8}{x^9}$

10. $y = \sqrt[4]{x} = x^{1/4}$, $y' = \frac{1}{4}x^{-3/4} = \frac{1}{4x^{3/4}}$

12. $g(x) = 3x - 1$, $g'(x) = 3$

14. $y = t^2 + 2t - 3$, $y' = 2t + 2$

16. $y = 8 - x^3$, $y' = -3x^2$

18. $f(x) = 2x^3 - x^2 + 3x$, $f'(x) = 6x^2 - 2x + 3$

20. $g(t) = \pi \cos t$, $g'(t) = -\pi \sin t$

22. $y = 5 + \sin x$, $y' = \cos x$

24. $y = \frac{5}{(2x)^3} + 2\cos x = \frac{5}{8}x^{-3} + 2\cos x$

$y' = \frac{5}{8}(-3)x^{-4} - 2\sin x = \frac{-15}{8x^4} - 2\sin x$

	Function	*Rewrite*	*Derivative*	*Simplify*
26.	$y = \frac{2}{3x^2}$	$y = \frac{2}{3}x^{-2}$	$y' = -\frac{4}{3}x^{-3}$	$y' = -\frac{4}{3x^3}$
28.	$y = \frac{\pi}{(3x)^2}$	$y = \frac{\pi}{9}x^{-2}$	$y' = -\frac{2\pi}{9}x^{-3}$	$y' = -\frac{2\pi}{9x^3}$
30.	$y = \frac{4}{x^{-3}}$	$y = 4x^3$	$y' = 12x^2$	$y' = 12x^2$

32. $f(t) = 3 - \frac{3}{5t}$, $\left(\frac{3}{5}, 2\right)$

$f'(t) = \frac{3}{5t^2}$

$f'\left(\frac{3}{5}\right) = \frac{5}{3}$

34. $y = 3x^3 - 6$, $(2, 18)$

$y' = 9x^2$

$y'(2) = 36$

36. $f(x) = 3(5 - x)^2$, $(5, 0)$

$= 3x^2 - 30x + 75$

$f'(x) = 6x - 30$

$f'(5) = 0$

38. $g(t) = 2 + 3\cos t$, $(\pi, -1)$

$g'(t) = -3\sin t$

$g'(\pi) = 0$

40. $f(x) = x^2 - 3x - 3x^{-2}$

$f'(x) = 2x - 3 + 6x^{-3}$

$= 2x - 3 + \frac{6}{x^3}$

42. $f(x) = x + x^{-2}$

$f'(x) = 1 - 2x^{-3}$

$= 1 - \frac{2}{x^3}$

44. $h(x) = \frac{2x^2 - 3x + 1}{x} = 2x - 3 + x^{-1}$

$h'(x) = 2 - \frac{1}{x^2} = \frac{2x^2 - 1}{x^2}$

46. $y = 3x(6x - 5x^2) = 18x^2 - 15x^3$

$y' = 36x - 45x^2$

48. $f(x) = \sqrt[3]{x} + \sqrt[5]{x} = x^{1/3} + x^{1/5}$

$f'(x) = \frac{1}{3}x^{-2/3} + \frac{1}{5}x^{-4/5} = \frac{1}{3x^{2/3}} + \frac{1}{5x^{4/5}}$

50. $f(t) = t^{2/3} - t^{1/3} + 4$

$f'(t) = \frac{2}{3}t^{-1/3} - \frac{1}{3}t^{-2/3} = \frac{2}{3t^{1/3}} - \frac{1}{3t^{2/3}}$

52. $f(x) = \dfrac{2}{\sqrt[3]{x}} + 3\cos x = 2x^{-1/3} + 3\cos x$

$f'(x) = \dfrac{-2}{3}x^{-4/3} - 3\sin x = \dfrac{-2}{3x^{4/3}} - 3\sin x$

54. (a) $y = x^3 + x$

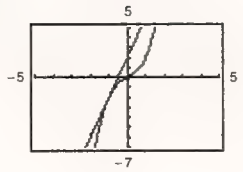
$y' = 3x^2 + 1$

At $(-1, -2)$: $y' = 3(-1)^2 + 1 = 4$.

Tangent line: $y + 2 = 4(x + 1)$

$4x - y + 2 = 0$

(b)



56. (a) $y = (x^2 + 2x)(x + 1)$

$= x^3 + 3x^2 + 2x$

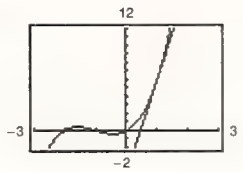
$y' = 3x^2 + 6x + 2$

At $(1, 6)$: $y' = 3(1)^2 + 6(1) + 2 = 11$.

Tangent line: $y - 6 = 11(x - 1)$

$0 = 11x - y - 5$

(b)



58. $y = x^3 + x$

$y' = 3x^2 + 1 > 0$ for all x.

Therefore, there are no horizontal tangents.

60. $y = x^2 + 1$

$y' = 2x = 0 \Rightarrow x = 0$

At $x = 0$, $y = 1$.

Horizontal tangent: $(0, 1)$

62. $y = \sqrt{3}x + 2\cos x,\ 0 \le x < 2\pi$

$y' = \sqrt{3} - 2\sin x = 0$

$\sin x = \dfrac{\sqrt{3}}{2} \Rightarrow x = \dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$

At $x = \dfrac{\pi}{3}$, $y = \dfrac{\sqrt{3}\pi + 3}{3}$.

At $x = \dfrac{2\pi}{3}$, $y = \dfrac{2\sqrt{3}\pi - 3}{3}$.

Horizontal tangents: $\left(\dfrac{\pi}{3}, \dfrac{\sqrt{3}\pi + 3}{3}\right), \left(\dfrac{2\pi}{3}, \dfrac{2\sqrt{3}\pi - 3}{3}\right)$

64. $k - x^2 = -4x + 7$ Equate functions

$-2x = -4$ Equate derivatives

Hence, $x = 2$ and $k - 4 = -8 + 7 \Rightarrow k = 3$

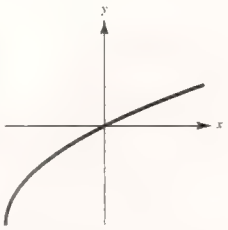
66. $k\sqrt{x} = x + 4$ Equate functions

$\dfrac{k}{2\sqrt{x}} = 1$ Equate derivatives

Hence, $k = 2\sqrt{x}$ and

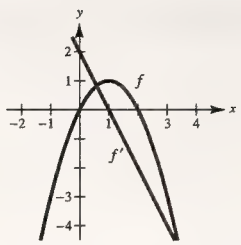
$(2\sqrt{x})\sqrt{x} = x + 4 \Rightarrow 2x = x + 4 \Rightarrow x = 4 \Rightarrow k = 4$

68. The graph of a function f such that $f' > 0$ for all x and the rate of change the function is decreasing (i.e. $f'' < 0$) would, in general, look like the graph at the right.



70. $g(x) = -5f(x) \Rightarrow g'(x) = -5f'(x)$

72.



If f is quadratic, then its derivative is a linear function.

$$f(x) = ax^2 + bx + c$$

$$f'(x) = 2ax + b$$

74. m_1 is the slope of the line tangent to $y = x$. m_2 is the slope of the line tangent to $y = 1/x$. Since

$$y = x \Rightarrow y' = 1 \Rightarrow m_1 = 1 \text{ and } y = \frac{1}{x} \Rightarrow y' = \frac{-1}{x^2} \Rightarrow m_2 = \frac{-1}{x^2}.$$

The points of intersection of $y = x$ and $y = 1/x$ are

$$x = \frac{1}{x} \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$$

At $x = \pm 1$, $m_2 = -1$. Since $m_2 = -1/m_1$, these tangent lines are perpendicular at the points intersection.

76. $f(x) = \dfrac{2}{x}$, $(5, 0)$

$$f'(x) = -\frac{2}{x^2}$$

$$-\frac{2}{x^2} = \frac{0 - y}{5 - x}$$

$$-10 + 2x = -x^2y$$

$$-10 + 2x = -x^2\left(\frac{2}{x}\right)$$

$$-10 + 2x = -2x$$

$$4x = 10$$

$$x = \frac{5}{2}, y = \frac{4}{5}$$

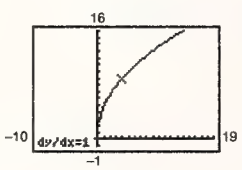
The point $\left(\frac{5}{2}, \frac{4}{5}\right)$ is on the graph of f. The slope of the tangent line is $f'\left(\frac{5}{2}\right) = -\frac{8}{25}$.

Tangent line: $$y - \frac{4}{5} = -\frac{8}{25}\left(x - \frac{5}{2}\right)$$

$$25y - 20 = -8x + 20$$

$$8x + 25y - 40 = 0$$

78. $f'(4) = 1$

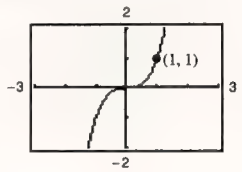


80. (a) Nearby point: (1.0073138, 1.0221024)

Secant line: $y - 1 = \dfrac{1.0221024 - 1}{1.0073138 - 1}(x - 1)$

$y = 3.022(x - 1) + 1$

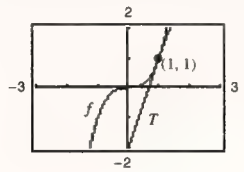
(Answers will vary.)



(b) $f'(x) = 3x^2$

$T(x) = 3(x - 1) + 1 = 3x - 2$

(c) The accuracy worsens at you move away from (1, 1).



(d)

Δx	-3	-2	-1	-0.5	-0.1	0	0.1	0.5	1	2	3
$f(x)$	-8	-1	0	0.125	0.729	1	1.331	3.375	8	27	64
$T(x)$	-8	-5	-2	-0.5	0.7	1	1.3	2.5	4	7	10

The accuracy decreases more rapidly than in Exercise 59 because $y = x^3$ is less "linear" than $y = x^{3/2}$.

82. True. If $f(x) = g(x) + c$, then $f'(x) = g'(x) + 0 = g'(x)$.

84. True. If $y = x/\pi = (1/\pi) \cdot x$, then $dy/dx = (1/\pi)(1) = 1/\pi$.

86. False. If $f(x) = \dfrac{1}{x^n} = x^{-n}$, then $f'(x) = -nx^{-n-1} = \dfrac{-n}{x^{n+1}}$

88. $f(t) = t^2 - 3, [2, 2.1]$

$f'(t) = 2t$

Instantaneous rate of change:

$(2, 1) \Rightarrow f'(2) = 2(2) = 4$

$(2.1, 1.41) \Rightarrow f'(2.1) = 4.2$

Average rate of change:

$$\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{1.41 - 1}{0.1} = 4.1$$

90. $f(x) = \sin x, \left[0, \dfrac{\pi}{6}\right]$

$f'(x) = \cos x$

Instantaneous rate of change:

$(0, 0) \Rightarrow f'(0) = 1$

$\left(\dfrac{\pi}{6}, \dfrac{1}{2}\right) \Rightarrow f'\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2} \approx 0.866$

Average rate of change:

$$\frac{f(\pi/6) - f(0)}{(\pi/6) - 0} = \frac{(1/2) - 0}{(\pi/6) - 0} = \frac{3}{\pi} \approx 0.955$$

92.

$s(t) = -16t^2 - 22t + 220$

$v(t) = -32t - 22$

$v(3) = -118$ ft/sec

$s(t) = -16t^2 - 22t + 220$

$= 112$ (height after falling 108 ft)

$-16t^2 - 22t + 108 = 0$

$-2(t - 2)(8t + 27) = 0$

$t = 2$

$v(2) = -32(2) - 22$

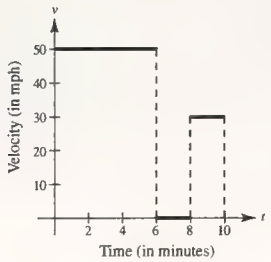
$= -86$ ft/sec

94. $s(t) = -4.9t^2 + v_0t + s_0$

$= -4.9t^2 + s_0 = 0$ when $t = 6.8$.

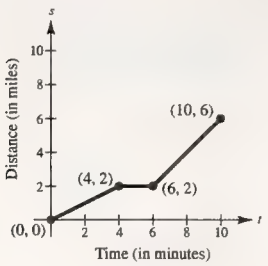
$s_0 = 4.9t^2 = 4.9(6.8)^2 = 226.6$ m

96.



(The velocity has been converted to miles per hour)

98. This graph corresponds with Exercise 75.



100. $s(t) = -\frac{1}{2}at^2 + c$ and $s'(t) = -at$.

Average velocity: $\dfrac{s(t_0 + \Delta t) - s(t_0 - \Delta t)}{(t_0 + \Delta t) - (t_0 - \Delta t)} = \dfrac{[-(1/2)a(t_0 + \Delta t)^2 + c] - [-(1/2)a(t_0 - \Delta t)^2 + c)]}{2\Delta t}$

$$= \frac{-(1/2)a({t_0}^2 + 2t_0\Delta t + (\Delta t)^2) + (1/2)a({t_0}^2 - 2t_0\Delta t + (\Delta t)^2)}{2\Delta t}$$

$$= \frac{-2at_0\Delta t}{2\Delta t}$$

$$= -at_0$$

$= s'(t_0)$ Instantaneous velocity at $t = t_0$

102. $V = s^3, \dfrac{dV}{ds} = 3s^2$

When $s = 4$ cm, $\dfrac{dV}{ds} = 48$ cm^2.

104. $C =$ (gallons of fuel used)(cost per gallon)

$$= \left(\frac{15{,}000}{x}\right)(1.25) = \frac{18{,}750}{x}$$

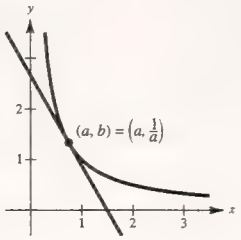
$$\frac{dC}{dx} = -\frac{18{,}750}{x^2}$$

x	10	15	20	25	30	35	40
C	1875	1250	537.5	750	625	535.71	468.75
$\frac{dC}{dx}$	−187.5	−83.333	−46.875	−30	−20.833	−15.306	−11.719

The driver who gets 15 miles per gallon would benefit more from a 1 mile per gallon increase in fuel efficiency. The rate of change is larger when $x = 15$.

106. $\dfrac{dT}{dt} = K(T - T_a)$

108. $y = \frac{1}{x}, \; x > 0$



$y' = -\frac{1}{x^2}$

At (a, b), the equation of the tangent line is

$$y - \frac{1}{a} = -\frac{1}{a^2}(x - a) \quad \text{or} \quad y = -\frac{x}{a^2} + \frac{2}{a}.$$

The x-intercept is $(2a, 0)$.

The y-intercept is $\left(0, \frac{2}{a}\right)$.

The area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}(2a)\left(\frac{2}{a}\right) = 2$.

110. $y = x^2$

$y' = 2x$

(a) Tangent lines through $(0, a)$:

$$y - a = 2x(x - 0)$$
$$x^2 - a = 2x^2$$
$$-a = x^2$$
$$\pm\sqrt{-a} = x$$

The points of tangency are $(\pm\sqrt{-a}, -a)$. At $(\sqrt{-a}, -a)$ the slope is $y'(\sqrt{-a}) = 2\sqrt{-a}$. At $(-\sqrt{-a}, -a)$ the slope is $y'(-\sqrt{-a}) = -2\sqrt{-a}$.

Tangent lines: $y + a = 2\sqrt{-a}(x - \sqrt{-a})$ and $y + a = -2\sqrt{-a}(x + \sqrt{-a})$

$$y = 2\sqrt{-a}x + a \qquad y = -2\sqrt{-a}x + a$$

Restriction: a must be negative.

(b) Tangent lines through $(a, 0)$:

$$y - 0 = 2x(x - a)$$
$$x^2 = 2x^2 - 2ax$$
$$0 = x^2 - 2ax = x(x - 2a)$$

The points of tangency are $(0, 0)$ and $(2a, 4a^2)$. At $(0, 0)$ the slope is $y'(0) = 0$. At $(2a, 4a^2)$ the slope is $y'(2a) = 4a$.

Tangent lines: $y - 0 = 0(x - 0)$ and $y - 4a^2 = 4a(x - 2a)$

$$y = 0 \qquad y = 4ax - 4a^2$$

Restriction: None, a can be any real number.

112. $f_1(x) = |\sin x|$ is differentiable for all $x \neq n\pi$, n an integer.

$f_2(x) = \sin|x|$ is differentiable for all $x \neq 0$.

You can verify this by graphing f_1 and f_2 and observing the locations of the sharp turns.

Section 2.3 The Product and Quotient Rules and Higher-Order Derivatives

2. $f(x) = (6x + 5)(x^3 - 2)$

$f'(x) = (6x + 5)(3x^2) + (x^3 - 2)(6)$

$= 18x^3 + 15x^2 + 6x^3 - 12$

$= 24x^3 + 15x^2 - 12$

4. $g(s) = \sqrt{s}(4 - s^2) = s^{1/2}(4 - s^2)$

$g'(s) = s^{1/2}(-2s) + (4 - s^2)\frac{1}{2}s^{-1/2} = -2s^{3/2} + \frac{4 - s^2}{2s^{1/2}}$

$= \frac{4 - 5s^2}{2s^{1/2}}$

6. $g(x) = \sqrt{x}\sin x$

$g'(x) = \sqrt{x}\cos x + \sin x\left(\frac{1}{2\sqrt{x}}\right) = \sqrt{x}\cos x + \frac{1}{2\sqrt{x}}\sin x$

8. $g(t) = \frac{t^2 + 2}{2t - 7}$

$g'(t) = \frac{(2t - 7)(2t) - (t^2 + 2)(2)}{(2t - 7)^2} = \frac{2t^2 - 14t - 4}{(2t - 7)^2}$

10. $h(s) = \frac{s}{\sqrt{s} - 1}$

$h'(s) = \frac{(\sqrt{s} - 1)(1) - s\left(\frac{1}{2}s^{-1/2}\right)}{(\sqrt{s} - 1)^2}$

$= \frac{\sqrt{s} - 1 - \frac{1}{2}\sqrt{s}}{(\sqrt{s} - 1)^2} = \frac{\sqrt{s} - 2}{2(\sqrt{s} - 1)^2}$

12. $f(t) = \frac{\cos t}{t^3}$

$f'(t) = \frac{t^3(-\sin t) - \cos t(3t^2)}{(t^3)^2} = -\frac{t\sin t + 3\cos t}{t^4}$

14. $f(x) = (x^2 - 2x + 1)(x^3 - 1)$

$f'(x) = (x^2 - 2x + 1)(3x^2) + (x^3 - 1)(2x - 2)$

$= 3x^2(x - 1)^2 + 2(x - 1)^2(x^2 + x + 1)$

$= (x - 1)^2(5x^2 + 2x + 2)$

$f'(1) = 0$

16. $f(x) = \frac{x + 1}{x - 1}$

$f'(x) = \frac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2}$

$= \frac{x - 1 - x - 1}{(x - 1)^2}$

$= -\frac{2}{(x - 1)^2}$

$f'(2) = -\frac{2}{(2 - 1)^2} = -2$

18. $f(x) = \frac{\sin x}{x}$

$f'(x) = \frac{(x)(\cos x) - (\sin x)(1)}{x^2}$

$= \frac{x\cos x - \sin x}{x^2}$

$f'\left(\frac{\pi}{6}\right) = \frac{(\pi/6)\left(\sqrt{3}/2\right) - (1/2)}{\pi^2/36}$

$= \frac{3\sqrt{3}\pi - 18}{\pi^2}$

$= \frac{3\left(\sqrt{3}\pi - 6\right)}{\pi^2}$

Function	Rewrite	Derivative	Simplify
20. $y = \frac{5x^2 - 3}{4}$	$y = \frac{5}{4}x^2 - \frac{3}{4}$	$y' = \frac{10}{4}x$	$y' = \frac{5x}{2}$

Function	Rewrite	Derivative	Simplify
22. $y = \dfrac{4}{5x^2}$	$y = \dfrac{4}{5}x^{-2}$	$y' = -\dfrac{8}{5}x^{-3}$	$y' = -\dfrac{8}{5x^3}$
24. $y = \dfrac{3x^2 - 5}{7}$	$y = \dfrac{3}{7}x^2 - \dfrac{5}{7}$	$y' = \dfrac{6x}{7}$	$y' = \dfrac{6}{7}x$

26. $f(x) = \dfrac{x^3 + 3x + 2}{x^2 - 1}$

$$f'(x) = \frac{(x^2 - 1)(3x^2 + 3) - (x^3 + 3x + 2)(2x)}{(x^2 - 1)^2}$$

$$= \frac{x^4 - 6x^2 - 4x - 3}{(x^2 - 1)^2}$$

28. $f(x) = x^4\left[1 - \dfrac{2}{x + 1}\right] = x^4\left[\dfrac{x - 1}{x + 1}\right]$

$$f'(x) = x^4\left[\frac{(x + 1) - (x - 1)}{(x + 1)^2}\right] + \left[\frac{x - 1}{x + 1}\right](4x^3)$$

$$= 2x^3\left[\frac{2x^2 + x - 2}{(x + 1)^2}\right]$$

30. $f(x) = \sqrt[3]{x}\left(\sqrt{x} + 3\right) = x^{1/3}(x^{1/2} + 3)$

$$f'(x) = x^{1/3}\left(\frac{1}{2}x^{-1/2}\right) + (x^{1/2} + 3)\left(\frac{1}{3}x^{-2/3}\right)$$

$$= \frac{5}{6}x^{-1/6} + x^{-2/3}$$

$$= \frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$$

Alternate solution:

$$f(x) = \sqrt[3]{x}\left(\sqrt{x} + 3\right)$$

$$= x^{5/6} + 3x^{1/3}$$

$$f'(x) = \frac{5}{6}x^{-1/6} + x^{-2/3}$$

$$= \frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$$

32. $h(x) = (x^2 - 1)^2 = x^4 - 2x^2 + 1$

$$h'(x) = 4x^3 - 4x = 4x(x^2 - 1)$$

34. $g(x) = x^2\left(\dfrac{2}{x} - \dfrac{1}{x + 1}\right) = 2x - \dfrac{x^2}{x + 1}$

$$g'(x) = 2 - \frac{(x + 1)2x - x^2(1)}{(x + 1)^2} = \frac{2(x^2 + 2x + 1) - x^2 - 2x}{(x + 1)^2} = \frac{x^2 + 2x + 2}{(x + 1)^2}$$

36. $f(x) = (x^2 - x)(x^2 + 1)(x^2 + x + 1)$

$$f'(x) = (2x - 1)(x^2 + 1)(x^2 + x + 1) + (x^2 - x)(2x)(x^2 + x + 1) + (x^2 - x)(x^2 + 1)(2x + 1)$$

$$= (2x - 1)(x^4 + x^3 + 2x^2 + x + 1) + (x^2 - x)(2x^3 + 2x^2 + 2x) + (x^2 - x)(2x^3 + x^2 + 2x + 1)$$

$$= 2x^5 + x^4 + 3x^3 + x - 1 + 2x^5 - 2x^2 + 2x^5 - x^4 + x^3 - x^2 - x$$

$$= 6x^5 + 4x^3 - 3x^2 - 1$$

38. $f(x) = \dfrac{c^2 - x^2}{c^2 + x^2}$

$$f'(x) = \frac{(c^2 + x^2)(-2x) - (c^2 - x^2)(2x)}{(c^2 + x^2)^2}$$

$$= \frac{-4xc^2}{(c^2 + x^2)^2}$$

40. $f(\theta) = (\theta + 1)\cos\theta$

$$f'(\theta) = (\theta + 1)(-\sin\theta) + (\cos\theta)(1)$$

$$= \cos\theta - (\theta + 1)\sin\theta$$

42. $f(x) = \dfrac{\sin x}{x}$

$f'(x) = \dfrac{x\cos x - \sin x}{x^2}$

44. $y = x + \cot x$

$y' = 1 - \csc^2 x = -\cot^2 x$

46. $h(s) = \dfrac{1}{s} - 10\csc s$

$h'(s) = -\dfrac{1}{s^2} + 10\csc s \cot s$

48. $y = \dfrac{\sec x}{x}$

$y' = \dfrac{x\sec x\tan x - \sec x}{x^2}$

$= \dfrac{\sec x(x\tan x - 1)}{x^2}$

50. $y = x\sin x + \cos x$

$y' = x\cos x + \sin x - \sin x = x\cos x$

52. $f(x) = \sin x\cos x$

$f'(x) = \sin x(-\sin x) + \cos x(\cos x)$

$= \cos 2x$

54. $h(\theta) = 5\theta\sec\theta + \theta\tan\theta$

$h'(\theta) = 5\theta\sec\theta\tan\theta + 5\sec\theta + \theta\sec^2\theta + \tan\theta$

56. $f(x) = \left(\dfrac{x^2 - x - 3}{x^2 + 1}\right)(x^2 + x + 1)$

$f'(x) = 2\dfrac{x^5 + 2x^3 + 2x^2 - 2}{(x^2 + 1)^2}$ (form of answer may vary)

58. $f(\theta) = \dfrac{\sin\theta}{1 - \cos\theta}$

$f'(\theta) = \dfrac{1}{\cos\theta - 1} = \dfrac{\cos\theta - 1}{(1 - \cos\theta)^2}$

(form of answer may vary)

60. $f(x) = \tan x\cot x = 1$

$f'(x) = 0$

$f'(1) = 0$

62. $f(x) = \sin x(\sin x + \cos x)$

$f'(x) = \sin x(\cos x - \sin x) + (\sin x + \cos x)\cos x$

$= \sin x\cos x - \sin^2 x + \sin x\cos x + \cos^2 x$

$= \sin 2x + \cos 2x$

$f'\left(\dfrac{\pi}{4}\right) = \sin\dfrac{\pi}{2} + \cos\dfrac{\pi}{2} = 1$

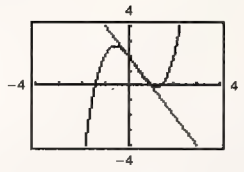
64. (a) $f(x) = (x - 1)(x^2 - 2)$, $(0, 2)$

$f'(x) = (x - 1)(2x) + (x^2 - 2)(1) = 3x^2 - 2x - 2$

$f'(0) = -2$ = slope at $(0, 2)$.

Tangent line: $y - 2 = -2x \Rightarrow y = -2x + 2$

(b)



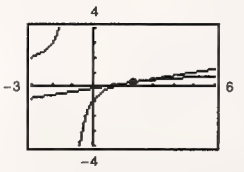
66. (a) $f(x) = \dfrac{x - 1}{x + 1}$, $\left(2, \dfrac{1}{3}\right)$

$f'(x) = \dfrac{(x + 1)(1) - (x - 1)(1)}{(x + 1)^2} = \dfrac{2}{(x + 1)^2}$

$f'(2) = \dfrac{2}{9}$ = slope at $\left(2, \dfrac{1}{3}\right)$.

Tangent line: $y - \dfrac{1}{3} = \dfrac{2}{9}(x - 2) \Rightarrow y = \dfrac{2}{9}x - \dfrac{1}{9}$

(b)



68. (a) $f(x) = \sec x, \quad \left(\frac{\pi}{3}, 2\right)$

$f'(x) = \sec x \tan x$

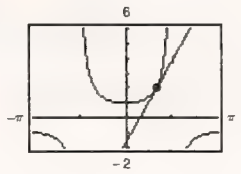
$f'\left(\frac{\pi}{3}\right) = 2\sqrt{3} =$ slope at $\left(\frac{\pi}{3}, 2\right)$.

Tangent line:

$y - 2 = 2\sqrt{3}\left(x - \frac{\pi}{3}\right)$

$6\sqrt{3}x - 3y + 6 - 2\sqrt{3}\pi = 0$

(b)



70. $f(x) = \frac{x^2}{x^2 + 1}$

$f'(x) = \frac{(x^2 + 1)(2x) - (x^2)(2x)}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}$

$f'(x) = 0$ when $x = 0$.

Horizontal tangent is at $(0, 0)$.

72. $f'(x) = \frac{x(\cos x - 3) - (\sin x - 3x)(1)}{x^2} = \frac{x\cos x - \sin x}{x^2}$

$g'(x) = \frac{x(\cos x + 2) - (\sin x + 2x)(1)}{x^2} = \frac{x\cos x - \sin x}{x^2}$

$g(x) = \frac{\sin x + 2x}{x} = \frac{\sin x - 3x + 5x}{x} = f(x) + 5$

f and g differ by a constant.

74. $f(x) = \frac{\cos x}{x^n} = x^{-n}\cos x$

$f'(x) = -x^{-n}\sin x - nx^{-n-1}\cos x$

$= -x^{-n-1}(x\sin x + n\cos x)$

$= -\frac{x\sin x + n\cos x}{x^{n+1}}$

When $n = 1$: $f'(x) = -\frac{x\sin x + \cos x}{x^2}$.

When $n = 2$: $f'(x) = -\frac{x\sin x + 2\cos x}{x^3}$.

When $n = 3$: $f'(x) = -\frac{x\sin x + 3\cos x}{x^4}$.

When $n = 4$: $f'(x) = -\frac{x\sin x + 4\cos x}{x^5}$.

For general n, $f'(x) = -\frac{x\sin x + n\cos x}{x^{n+1}}$.

76. $V = \pi r^2 h = \pi(t + 2)\left(\frac{1}{2}\sqrt{t}\right)$

$= \frac{1}{2}(t^{3/2} + 2t^{1/2})\pi$

$V'(t) = \frac{1}{2}\left(\frac{3}{2}t^{1/2} + t^{-1/2}\right)\pi = \frac{3t + 2}{4t^{1/2}}\pi$ cubic inches/sec

78. $P = \frac{k}{V}$

$\frac{dP}{dV} = -\frac{k}{V^2}$

80. $f(x) = \sec x$

$g(x) = \csc x, \quad [0, 2\pi)$

$f'(x) = g'(x)$

$$\sec x \tan x = -\csc x \cot x \Rightarrow \frac{\sec x \tan x}{\csc x \cot x} = -1 \Rightarrow \frac{\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}}{\frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}} = -1 \Rightarrow$$

$$\frac{\sin^3 x}{\cos^3 x} = -1 \Rightarrow \tan^3 x = -1 \Rightarrow \tan x = -1$$

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$

82. (a) $n(t) = -9.6643t^2 + 90.7414t + 77.5029$

$v(t) = -276.4643t^2 + 2987.6929t + 1809.9714$

(b) $A = \dfrac{v(t)}{n(t)} \approx \dfrac{-276.46t^2 + 2987.69t + 1809.97}{-9.66t^2 + 90.74t + 77.50}$

A represents the average retail value (in millions of dollars) per 1000 motor homes.

(c) $A'(t) \approx \dfrac{40.46(x^2 - 2.09x + 17.83)}{(x^2 - 9.39x - 8.02)^2}$

84. $f(x) = x + \dfrac{32}{x^2}$

$f'(x) = 1 - \dfrac{64}{x^3}$

$f''(x) = \dfrac{192}{x^4}$

86. $f(x) = \dfrac{x^2 + 2x - 1}{x} = x + 2 - \dfrac{1}{x}$

$f'(x) = 1 + \dfrac{1}{x^2}$

$f''(x) = -\dfrac{2}{x^3}$

88. $f(x) = \sec x$

$f'(x) = \sec x \tan x$

$f''(x) = \sec x(\sec^2 x) + \tan x(\sec x \tan x)$

$= \sec x(\sec^2 x + \tan^2 x)$

90. $f''(x) = 2 - 2x^{-1}$

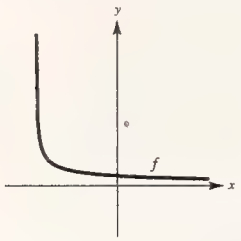
$f'''(x) = 2x^{-2} = \dfrac{2}{x^2}$

92. $f^{(4)}(x) = 2x + 1$

$f^{(5)}(x) = 2$

$f^{(6)}(x) = 0$

94. The graph of a differentiable function f such that $f > 0$ and $f' < 0$ for all real numbers x would in general look like the graph below.



96. $f(x) = 4 - h(x)$

$f'(x) = -h'(x)$

$f'(2) = -h'(2) = -4$

98. $f(x) = g(x)h(x)$

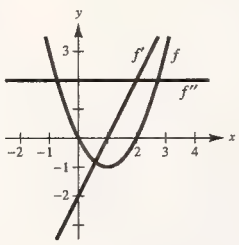
$f'(x) = g(x)h'(x) + h(x)g'(x)$

$f'(2) = g(2)h'(2) + h(2)g'(2)$

$= (3)(4) + (-1)(-2)$

$= 14$

100.



It appears that f is quadratic; so f' would be linear and f'' would be constant.

102. $s(t) = -8.25t^2 + 66t$

$v(t) = -16.50t + 66$

$a(t) = -16.50$

t(sec)	0	1	2	3	4
$s(t)$ (ft)	0	57.75	99	123.75	132
$v(t) = s'(t)$ (ft/sec)	66	49.5	33	16.5	0
$a(t) = v'(t)$ (ft/sec^2)	−16.5	−16.5	−16.5	−16.5	−16.5

Average velocity on:

$[0, 1]$ is $\dfrac{57.75 - 0}{1 - 0} = 57.75.$

$[1, 2]$ is $\dfrac{99 - 57.75}{2 - 1} = 41.25.$

$[2, 3]$ is $\dfrac{123.75 - 99}{3 - 2} = 24.75.$

$[3, 4]$ is $\dfrac{132 - 123.75}{4 - 3} = 8.25.$

104. (a) $f(x) = x^n$

$f^n(x) = n(n-1)(n-2)\cdots(2)(1) = n!$

(b) $f(x) = \dfrac{1}{x}$

$f^{(n)}(x) = \dfrac{(-1)^n(n)(n-1)(n-2)\cdots(2)(1)}{x^{n+1}}$

$= \dfrac{(-1)^n n!}{x^{n+1}}$

Note: $n! = n(n-1)\cdots 3 \cdot 2 \cdot 1$ (read "n factorial.")

106. $[xf(x)]' = xf'(x) + f(x)$

$[xf(x)]'' = xf''(x) + f'(x) + f'(x) = xf''(x) + 2f'(x)$

$[xf(x)]''' = xf'''(x) + f''(x) + 2f''(x) = xf'''(x) + 3f''(x)$

In general, $[xf(x)]^{(n)} = xf^{(n)}(x) + nf^{(n-1)}(x)$.

108. $f(x) = \sin x \qquad f\left(\dfrac{\pi}{2}\right) = 1$

$f'(x) = \cos x \qquad f'\left(\dfrac{\pi}{2}\right) = 0$

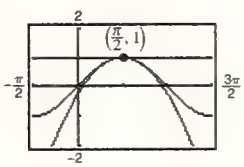
$f''(x) = -\sin x \qquad f''\left(\dfrac{\pi}{2}\right) = -1$

(a) $P_1(x) = f'(a)(x-a) + f(a) = 0\left(x - \dfrac{\pi}{2}\right) + 1 = 1$

$P_2(x) = \dfrac{1}{2}f''(a)(x-a)^2 + f'(a)(x-a) + f(a) = \dfrac{1}{2}(-1)\left(x - \dfrac{\pi}{2}\right)^2 + 1$

$= 1 - \dfrac{1}{2}\left(x - \dfrac{\pi}{2}\right)^2$

(b)



(c) P_2 is a better approximation than P_1.

(d) The accuracy worsens as you move farther away from $x = a = \dfrac{\pi}{2}$.

110. True. y is a fourth-degree polynomial.

$\dfrac{d^n y}{dx^n} = 0$ when $n > 4$.

112. True

114. True. If $v(t) = c$ then

$a(t) = v'(t) = 0$.

116. (a) $(fg' - f'g)' = fg'' + f'g' - f'g' - f''g$

$= fg'' - f''g$ True

(b) $(fg)'' = (fg' + f'g)'$

$= fg'' + f'g' + f'g' + f''g$

$= fg'' + 2f'g' + f''g$

$\neq fg'' + f''g$ False

Section 2.4 The Chain Rule

$y = f(g(x))$	$u = g(x)$	$y = f(u)$

2. $y = \dfrac{1}{\sqrt{x+1}}$ $\quad u = x + 1$ $\quad y = u^{-1/2}$

4. $y = 3\tan(\pi x^2)$ $\quad u = \pi x^2$ $\quad y = 3\tan u$

6. $y = \cos\dfrac{3x}{2}$ $\quad u = \dfrac{3x}{2}$ $\quad y = \cos u$

8. $y = (2x^3 + 1)^2$

$y' = 2(2x^3 + 1)(6x^2) = 12x^2(2x^3 + 1)$

10. $y = 3(4 - x^2)^5$

$y' = 15(4 - x^2)^4(-2x) = -30x(4 - x^2)^4$

12. $f(t) = (9t + 2)^{2/3}$

$f'(t) = \dfrac{2}{3}(9t + 2)^{-1/3}(9) = \dfrac{6}{\sqrt[3]{9t + 2}}$

14. $g(x) = \sqrt{5 - 3x} = (5 - 3x)^{1/2}$

$g'(x) = \dfrac{1}{2}(5 - 3x)^{-1/2}(-3) = \dfrac{-3}{2\sqrt{5 - 3x}}$

16. $g(x) = \sqrt{x^2 - 2x + 1} = \sqrt{(x-1)^2} = |x - 1|$

$g'(x) = \begin{cases} 1, & x > 1 \\ -1, & x < 1 \end{cases}$

18. $f(x) = -3^4\sqrt{2 - 9x}$

$f(x) = -3(2 - 9x)^{1/4}$

$f'(x) = -\dfrac{3}{4}(2 - 9x)^{-3/4}(-9) = \dfrac{27}{4(2 - 9x)^{3/4}}$

20. $s(t) = \dfrac{1}{t^2 + 3t - 1}$

$s(t) = (t^2 + 3t - 1)^{-1}$

$s'(t) = -1(t^2 + 3t - 1)^{-2}(2t + 3)$

$= \dfrac{-(2t + 3)}{(t^2 + 3t - 1)^2}$

22. $y = -\dfrac{5}{(t + 3)^3}$

$y = -5(t + 3)^{-3}$

$y' = 15(t + 3)^{-4} = \dfrac{15}{(t + 3)^4}$

24. $g(t) = \sqrt{\dfrac{1}{t^2 - 2}}$

$g(t) = (t^2 - 2)^{-1/2}$

$g'(t) = -\dfrac{1}{2}(t^2 - 2)^{-3/2}(2t) = -\dfrac{t}{(t^2 - 2)^{3/2}}$

26. $f(x) = x(3x - 9)^3$

$f'(x) = x[3(3x - 9)^2(3)] + (3x - 9)^3(1)$

$= (3x - 9)^2[9x + 3x - 9]$

$= 27(x - 3)^2(4x - 3)$

28. $y = \dfrac{1}{2}x^2\sqrt{16 - x^2}$

$y' = \dfrac{1}{2}x^2\left(\dfrac{1}{2}(16 - x^2)^{-1/2}(-2x)\right) + x(16 - x^2)^{1/2}$

$= \dfrac{-x^3}{2\sqrt{16 - x^2}} + x\sqrt{16 - x^2} = \dfrac{-x(3x^2 - 32)}{2\sqrt{16 - x^2}}$

30. $y = \dfrac{x}{\sqrt{x^4 + 4}}$

$y' = \dfrac{(x^4 + 4)^{1/2}(1) - x\dfrac{1}{2}(x^4 + 4)^{-1/2}(4x^3)}{x^4 + 4}$

$= \dfrac{x^4 + 4 - 2x^4}{(x^4 + 4)^{3/2}} = \dfrac{4 - x^4}{(x^4 + 4)^{3/2}}$

32. $h(t) = \left(\dfrac{t^2}{t^3 + 2}\right)^2$

$h'(t) = 2\left(\dfrac{t^2}{t^3 + 2}\right)\left(\dfrac{(t^3 + 2)(2t) - t^2(3t^2)}{(t^3 + 2)^2}\right) = \dfrac{2t^2(4t - t^4)}{(t^3 + 2)^3} = \dfrac{2t^3(4 - t^3)}{(t^3 + 2)^3}$

34. $g(x) = \left(\dfrac{3x^2 - 2}{2x + 3}\right)^3$

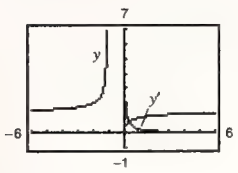
$$g'(x) = 3\left(\frac{3x^2 - 2}{2x + 3}\right)^2\left(\frac{(2x + 3)(6x) - (3x^2 - 2)(2)}{(2x + 3)^2}\right)$$

$$= \frac{3(3x^2 - 2)^2(6x^2 + 18x + 4)}{(2x + 3)^4} = \frac{6(3x^2 - 2)^2(3x^2 + 9x + 2)}{(2x + 3)^4}$$

36. $y = \sqrt{\dfrac{2x}{x + 1}}$

$$y' = \frac{1}{\sqrt{2x}(x + 1)^{3/2}}$$

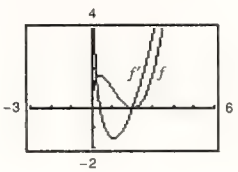
y' has no zeros.



38. $f(x) = \sqrt{x}(2 - x)^2$

$$f'(x) = \frac{(x - 2)(5x - 2)}{2\sqrt{x}}$$

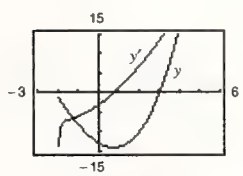
The zeros of f' correspond to the points on the graph of f where the tangent lines are horizontal.



40. $y = (t^2 - 9)\sqrt{t + 2}$

$$y' = \frac{5t^2 + 8t - 9}{2\sqrt{t + 2}}$$

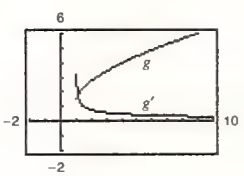
The zero of y' corresponds to the point on the graph of y where the tangent line is horizontal.



42. $g(x) = \sqrt{x - 1} + \sqrt{x + 1}$

$$g'(x) = \frac{1}{2\sqrt{x - 1}} + \frac{1}{2\sqrt{x + 1}}$$

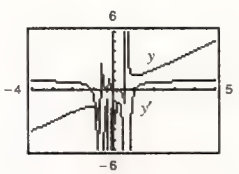
g' has no zeros.



44. $y = x^2 \tan\dfrac{1}{x}$

$$\frac{dy}{dx} = 2x \tan\frac{1}{x} - \sec^2\frac{1}{x}$$

The zeros of y' correspond to the points on the graph of y where the tangent lines are horizontal.



46. (a) $y = \sin 3x$

$$y' = 3\cos 3x$$

$$y'(0) = 3$$

3 cycles in $[0, 2\pi]$

(b) $y = \sin\left(\dfrac{x}{2}\right)$

$$y' = \left(\frac{1}{2}\right)\cos\left(\frac{x}{2}\right)$$

$$y'(0) = \frac{1}{2}$$

Half cycle in $[0, 2\pi]$

The slope of $\sin ax$ at the origin is a.

48. $y = \sin \pi x$

$$\frac{dy}{dx} = \pi \cos \pi x$$

50. $h(x) = \sec(x^2)$

$$h'(x) = 2x \sec(x^2) \tan(x^2)$$

52. $y = \cos(1 - 2x)^2 = \cos((1 - 2x)^2)$

$$y' = -\sin(1 - 2x)^2(2(1 - 2x)(-2)) = 4(1 - 2x)\sin(1 - 2x)^2$$

54. $g(\theta) = \sec\left(\frac{1}{2}\theta\right)\tan\left(\frac{1}{2}\theta\right)$

$$g'(\theta) = \sec\left(\frac{1}{2}\theta\right)\sec^2\left(\frac{1}{2}\theta\right)\frac{1}{2} + \tan\left(\frac{1}{2}\theta\right)\sec\left(\frac{1}{2}\theta\right)\tan\left(\frac{1}{2}\theta\right)\frac{1}{2}$$

$$= \frac{1}{2}\sec\left(\frac{1}{2}\theta\right)\left[\sec^2\left(\frac{1}{2}\theta\right) + \tan^2\left(\frac{1}{2}\theta\right)\right]$$

56. $g(v) = \dfrac{\cos v}{\csc v} = \cos v \cdot \sin v$

$$g'(v) = \cos v(\cos v) + \sin v(-\sin v) = \cos^2 v - \sin^2 v = \cos 2v$$

58. $y = 2\tan^3 x$

$$y' = 6\tan^2 x \cdot \sec^2 x$$

60. $g(t) = 5\cos^2 \pi t = 5(\cos \pi t)^2$

$$g'(t) = 10\cos \pi t(-\sin \pi t)(\pi)$$

$$= -10\pi(\sin \pi t)(\cos \pi t) = -5\pi \sin 2\pi t$$

62. $h(t) = 2\cot^2(\pi t + 2)$

$$h'(t) = 4\cot(\pi t + 2)(-\csc^2(\pi t + 2)(\pi))$$

$$= -4\pi \cot(\pi t + 2)\csc^2(\pi t + 2)$$

64. $y = 3x - 5\cos(\pi x)^2$

$$= 3x - 5\cos(\pi^2 x^2)$$

$$\frac{dy}{dx} = 3 + 5\sin(\pi^2 x^2)(2\pi^2 x)$$

$$= 3 + 10\pi^2 x \sin(\pi x)^2$$

66. $y = \sin x^{1/3} + (\sin x)^{1/3}$

$$y' = \cos x^{1/3}\left(\frac{1}{3}x^{-2/3}\right) + \frac{1}{3}(\sin x)^{-2/3}\cos x$$

$$= \frac{1}{3}\left[\frac{\cos x^{1/3}}{x^{2/3}} + \frac{\cos x}{(\sin x)^{2/3}}\right]$$

68. $y = (3x^3 + 4x)^{1/5}, \quad (2, 2)$

$$y' = \frac{1}{5}(3x^3 + 4x)^{-4/5}(9x^2 + 4)$$

$$= \frac{9x^2 + 4}{5(3x^3 + 4x)^{4/5}}$$

$$y'(2) = \frac{1}{2}$$

70. $f(x) = \dfrac{1}{(x^2 - 3x)^2} = (x^2 - 3x)^{-2}, \quad \left(4, \frac{1}{16}\right)$

$$f'(x) = -2(x^2 - 3x)^{-3}(2x - 3) = \frac{-2(2x - 3)}{(x^2 - 3x)^3}$$

$$f'(4) = -\frac{5}{32}$$

72. $f(x) = \dfrac{x + 1}{2x - 3}, \quad (2, 3)$

$$f'(x) = \frac{(2x - 3)(1) - (x + 1)(2)}{(2x - 3)^2} = \frac{-5}{(2x - 3)^2}$$

$$f'(2) = -5$$

74. $y = \dfrac{1}{x} + \sqrt{\cos x}, \quad \left(\frac{\pi}{2}, \frac{2}{\pi}\right)$

$$y' = -\frac{1}{x^2} - \frac{\sin x}{2\sqrt{\cos x}}$$

$y'(\pi/2)$ is undefined.

76. (a) $f(x) = \frac{1}{3}x\sqrt{x^2+5}$, $(2, 2)$

$$f'(x) = \frac{1}{3}x\left[\frac{1}{2}(x^2+5)^{-1/2}(2x)\right] + \frac{1}{3}(x^2+5)^{1/2}$$

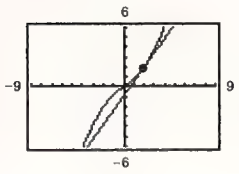
$$= \frac{x^2}{3\sqrt{x^2+5}} + \frac{1}{3}\sqrt{x^2+5}$$

$$f'(2) = \frac{4}{3(3)} + \frac{1}{3}(3) = \frac{13}{9}$$

Tangent line:

$$y - 2 = \frac{13}{9}(x-2) \Rightarrow 13x - 9y - 8 = 0$$

(b)



78. (a) $f(x) = \tan^2 x$, $\left(\frac{\pi}{4}, 1\right)$

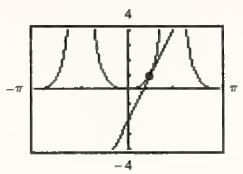
$$f'(x) = 2\tan x \sec^2 x$$

$$f'\left(\frac{\pi}{4}\right) = 2(1)(2) = 4$$

Tangent line:

$$y - 1 = 4\left(x - \frac{\pi}{4}\right) \Rightarrow 4x - y + (1 - \pi) = 0$$

(b)



80. $f(x) = (x-2)^{-1}$

$$f'(x) = -(x-2)^{-2} = \frac{-1}{(x-2)^2}$$

$$f''(x) = 2(x-2)^{-3} = \frac{2}{(x-2)^3}$$

82. $f(x) = \sec^2 \pi x$

$$f'(x) = 2\sec \pi x(\pi \sec \pi x \tan \pi x)$$

$$= 2\pi \sec^2 \pi x \tan \pi x$$

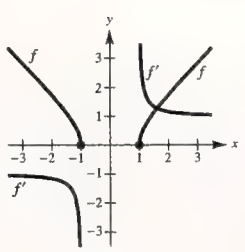
$$f''(x) = 2\pi \sec^2 \pi x(\sec^2 \pi x)(\pi) + 2\pi \tan \pi x(2\pi \sec^2 \pi x \tan \pi x)$$

$$= 2\pi^2 \sec^4 \pi x + 4\pi^2 \sec^2 \pi x \tan^2 \pi x$$

$$= 2\pi^2 \sec^2 \pi x(\sec^2 \pi x + 2\tan^2 \pi x)$$

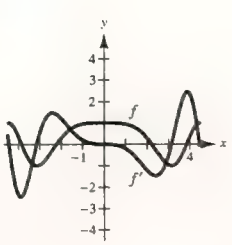
$$= 2\pi^2 \sec^2 \pi x(3\sec^2 \pi x - 2)$$

84.



f is decreasing on $(-\infty, -1)$ so f' must be negative there. f is increasing on $(1, \infty)$ so f' must be postive there.

86.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

88. $g(x) = f(x^2)$

$$g'(x) = f'(x^2)(2x) \Rightarrow g'(x) = 2xf'(x^2)$$

90. (a) $g(x) = \sin^2 x + \cos^2 x = 1 \Rightarrow g'(x) = 0$

$g'(x) = 2 \sin x \cos x + 2 \cos x(-\sin x) = 0$

(b) $\tan^2 x + 1 = \sec^2 x$

$g(x) + 1 = f(x)$

Taking derivatives of both sides,

$g'(x) = f'(x)$.

Equivalently, $f'(x) = 2 \sec x \cdot \sec x \cdot \tan x$ and $g'(x) = 2 \tan x \cdot \sec^2 x$, which are the same.

92. $y = \frac{1}{3} \cos 12t - \frac{1}{4} \sin 12t$

$v = y' = \frac{1}{3}[-12 \sin 12t] - \frac{1}{4}[12 \cos 12t]$

$= -4 \sin 12t - 3 \cos 12t$

When $t = \pi/8$, $y = 0.25$ feet and $v = 4$ feet per second.

94. $y = A \cos \omega t$

(a) Amplitude: $A = \dfrac{3.5}{2} = 1.75$

$y = 1.75 \cos \omega t$

Period: $10 \Rightarrow \omega = \dfrac{2\pi}{10} = \dfrac{\pi}{5}$

$y = 1.75 \cos \dfrac{\pi t}{5}$

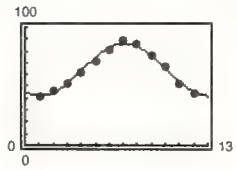
(b) $v = y' = 1.75\left[-\dfrac{\pi}{5} \sin \dfrac{\pi t}{5}\right]$

$= -0.35\pi \sin \dfrac{\pi t}{5}$

96. (a) Using a graphing utility, or by trial and error, you obtain a model of the form

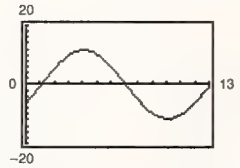
$T(t) = 64.18 - 22.15 \sin\left(\dfrac{\pi t}{6} + 1\right)$

(b)



(c) $T'(t) = -22.15 \cos\left(\dfrac{\pi t}{6} + 1\right)\left(\dfrac{\pi}{6}\right)$

$= -11.60 \cos\left(\dfrac{\pi t}{6} + 1\right)$



(d) The temperature changes most rapidly when $t \approx 4.1$ (April) and $t \approx 10.1$ (October). The temperature changes most slowly ($T'(t) = 0$) when $t \approx 1.1$ (January) and $t \approx 7.1$ (July).

98. (a) $g(x) = f(x) - 2 \Rightarrow g'(x) = f'(x)$

(b) $h(x) = 2f(x) \Rightarrow h'(x) = 2f'(x)$

(c) $r(x) = f(-3x) \Rightarrow r'(x) = f'(-3x)(-3) = -3f'(-3x)$

Hence, you need to know $f'(-3x)$.

$r'(0) = -3f'(0) = (-3)\left(-\frac{1}{3}\right) = 1$

$r'(-1) = -3f'(3) = (-3)(-4) = 12$

(d) $s(x) = f(x + 2) \Rightarrow s'(x) = f'(x + 2)$

Hence, you need to know $f'(x + 2)$.

$s'(-2) = f'(0) = -\frac{1}{3}$, etc.

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$h'(x)$	8	$\frac{4}{3}$	$-\frac{2}{3}$	-2	-4	-8
$r'(x)$		12	1			
$s'(x)$	$-\frac{1}{3}$	-1	-2	-4		

100. $f(x + p) = f(x)$ for all x.

(a) Yes, $f'(x + p) = f'(x)$, which shows that f' is periodic as well.

(b) Yes, let $g(x) = f(2x)$, so $g'(x) = 2f'(2x)$. Since f' is periodic, so is g'.

102. If $f(-x) = -f(x)$, then

$$\frac{d}{dx}[f(-x)] = \frac{d}{x}[-f(x)]$$

$$f'(-x)(-1) = -f'(x)$$

$$f'(-x) = f'(x).$$

Thus, $f'(x)$ is even.

104. $|u| = \sqrt{u^2}$

$$\frac{d}{dx}[|u|] = \frac{d}{dx}\left[\sqrt{u^2}\right] = \frac{1}{2}(u^2)^{-1/2}(2uu') = \frac{uu'}{\sqrt{u^2}} = u'\frac{u}{|u|}, u \neq 0$$

106. $f(x) = |x^2 - 4|$

$$f'(x) = 2x\left(\frac{x^2 - 4}{|x^2 - 4|}\right), x \neq \pm 2$$

108. $f(x) = |\sin x|$

$$f'(x) = \cos x\left(\frac{\sin x}{|\sin x|}\right), x \neq k\pi$$

110. (a) $f(x) = \sec(2x)$

$$f'(x) = 2(\sec 2x)(\tan 2x)$$

$$f''(x) = 2[2(\sec 2x)(\tan 2x)]\tan 2x + 2(\sec 2x)(\sec^2 2x)(2)$$

$$= 4[(\sec 2x)(\tan^2 2x) + \sec^3 2x]$$

$$f\left(\frac{\pi}{6}\right) = \sec\left(\frac{\pi}{3}\right) = 2$$

$$f'\left(\frac{\pi}{6}\right) = 2\sec\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right) = 4\sqrt{3}$$

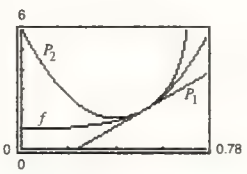
$$f''\left(\frac{\pi}{6}\right) = 4[2(3) + 2^3] = 56$$

$$P_1(x) = 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$

$$P_2(x) = \frac{1}{2}(56)\left(x - \frac{\pi}{6}\right)^2 + 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$

$$= 28\left(x - \frac{\pi}{6}\right)^2 + 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$

(b)



(c) P_2 is a better approximation than P_1.

(d) The accuracy worsens as you move away from $x = \pi/6$.

112. False. If $f(x) = \sin^2 2x$, then $f'(x) = 2(\sin 2x)(2\cos 2x)$.

114. False. First apply the Product Rule.

Section 2.5 Implicit Differentiations

2. $x^2 - y^2 = 16$

$2x - 2yy' = 0$

$$y' = \frac{x}{y}$$

4. $x^3 + y^3 = 8$

$3x^2 + 3y^2y' = 0$

$$y' = -\frac{x^2}{y^2}$$

6. $x^2y + y^2x = -2$

$x^2y' + 2xy + y^2 + 2yxy' = 0$

$(x^2 + 2xy)y' = -(y^2 + 2xy)$

$$y' = \frac{-y(y + 2x)}{x(x + 2y)}$$

8. $(xy)^{1/2} - x + 2y = 0$

$\frac{1}{2}(xy)^{-1/2}(xy' + y) - 1 + 2y' = 0$

$\frac{x}{2\sqrt{xy}}y' + \frac{y}{2\sqrt{xy}} - 1 + 2y' = 0$

$xy' + y - 2\sqrt{xy} + 4\sqrt{xy}\,y' = 0$

$$y' = \frac{2\sqrt{xy} - y}{4\sqrt{xy} + x}$$

10. $2 \sin x \cos y = 1$

$2[\sin x(-\sin y)y' + \cos y(\cos x)] = 0$

$$y' = \frac{\cos x \cos y}{\sin x \sin y}$$

$$= \cot x \cot y$$

12. $(\sin \pi x + \cos \pi y)^2 = 2$

$2(\sin \pi x + \cos \pi y)[\pi \cos \pi x - \pi(\sin \pi y)y'] = 0$

$\pi \cos \pi x - \pi(\sin \pi y)y' = 0$

$$y' = \frac{\cos \pi x}{\sin \pi y}$$

14. $\cot y = x - y$

$(-\csc^2 y)y' = 1 - y'$

$$y' = \frac{1}{1 - \csc^2 y}$$

$$= \frac{1}{-\cot^2 y} = -\tan^2 y$$

16. $x = \sec\frac{1}{y}$

$$1 = -\frac{y'}{y^2}\sec\frac{1}{y}\tan\frac{1}{y}$$

$$y' = \frac{-y^2}{\sec(1/y)\tan(1/y)} = -y^2\cos\left(\frac{1}{y}\right)\cot\left(\frac{1}{y}\right)$$

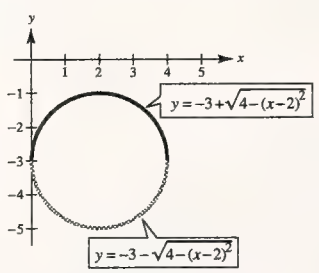
18. (a) $(x^2 - 4x + 4) + (y^2 + 6y + 9) = -9 + 4 + 9$

$(x - 2)^2 + (y + 3)^2 = 4$ (Circle)

$(y + 3)^2 = 4 - (x - 2)^2$

$$y = -3 \pm \sqrt{4 - (x - 2)^2}$$

(b)



(c) Explicitly:

$$\frac{dy}{dx} = \pm\frac{1}{2}[4 - (x - 2)^2]^{-1/2}(-2)(x - 2)$$

$$= \frac{\mp(x - 2)}{(\sqrt{4 - (x - 2)^2}}$$

$$= \frac{-(x - 2)}{\pm\sqrt{4 - (x - 2)^2}}$$

$$= \frac{-(x - 2)}{-3 \pm \sqrt{4 - (x - 2)^2} + 3}$$

$$= \frac{-(x - 2)}{y + 3}$$

(d) Implicitly:

$2x + 2yy' - 4 + 6y' = 0$

$(2y + 6)y' = -2(x - 2)$

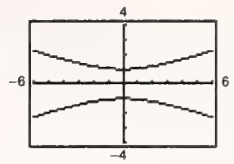
$$y' = \frac{-(x - 2)}{y + 3}$$

20. (a) $9y^2 = x^2 + 9$

$$y^2 = \frac{x^2}{9} + 1 = \frac{x^2 + 9}{9}$$

$$y = \frac{\pm\sqrt{x^2 + 9}}{3}$$

(b)



(c) Explicitly: $\dfrac{dy}{dx} = \dfrac{\pm\frac{1}{2}(x^2 + 9)^{-1/2}(2x)}{3} = \dfrac{\pm x}{3\sqrt{x^2 + 9}} = \dfrac{\pm x}{3(\pm 3y)} = \dfrac{x}{9y}$

(d) Implicitly: $9y^2 - x^2 = 9$

$$18yy' - 2x = 0$$

$$18yy' = 2x$$

$$y' = \frac{2x}{18y} = \frac{x}{9y}$$

22. $x^2 - y^3 = 0$

$$2x - 3y^2y' = 0$$

$$y' = \frac{2x}{3y^2}$$

At $(1, 1)$: $y' = \dfrac{2}{3}$.

24. $(x + y)^3 = x^3 + y^3$

$$x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3$$

$$3x^2y + 3xy^2 = 0$$

$$x^2y + xy^2 = 0$$

$$x^2y' + 2xy + 2xyy' + y^2 = 0$$

$$(x^2 + 2xy)y' = -(y^2 + 2xy)$$

$$y' = -\frac{y(y + 2x)}{x(x + 2y)}$$

At $(-1, 1)$: $y' = -1$.

26. $x^3 + y^3 = 4xy + 1$

$$3x^2 + 3y^2y' = 4xy' + 4y$$

$$(3y^2 - 4x)y' = 4y - 3x^2$$

$$y' = \frac{4y - 3x^2}{(3y^2 - 4x)}$$

At $(2, 1)$, $y' = \dfrac{4 - 12}{3 - 8} = \dfrac{8}{5}$

28. $x \cos y = 1$

$$x[-y' \sin y] + \cos y = 0$$

$$y' = \frac{\cos y}{x \sin y}$$

$$= \frac{1}{x}\cot y = \frac{\cot y}{x}$$

At $\left(2, \dfrac{\pi}{3}\right)$: $y' = \dfrac{1}{2\sqrt{3}}$.

30. $(4 - x)y^2 = x^3$

$$(4 - x)(2yy') + y^2(-1) = 3x^2$$

$$y' = \frac{3x^2 + y^2}{2y(4 - x)}$$

At $(2, 2)$: $y' = 2$.

32. $x^3 + y^3 - 6xy = 0$

$$3x^2 + 3y^2y' - 6xy' - 6y = 0$$

$$y'(3y^2 - 6x) = 6y - 3x^2$$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}$$

At $\left(\dfrac{4}{3}, \dfrac{8}{3}\right)$: $y' = \dfrac{(16/3) - (16/9)}{(64/9) - (8/3)} = \dfrac{32}{40} = \dfrac{4}{5}$.

34. $\cos y = x$

$$-\sin y \cdot y' = 1$$

$$y' = \frac{-1}{\sin y},\ 0 < y < \pi$$

$$\sin^2 y + \cos^2 y = 1$$

$$\sin^2 y = 1 - \cos^2 y$$

$$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$$

$$y' = \frac{-1}{\sqrt{1 - x^2}},\ -1 < x < 1$$

36.

$$x^2y^2 - 2x = 3$$

$$2x^2yy' + 2xy^2 - 2 = 0$$

$$x^2yy' + xy^2 - 1 = 0$$

$$y' = \frac{1 - xy^2}{x^2y}$$

$$2xyy' + x^2(y')^2 + x^2yy'' + 2xyy' + y^2 = 0$$

$$4xyy' + x^2(y')^2 + x^2yy'' + y^2 = 0$$

$$\frac{4 - 4xy^2}{x} + \frac{(1 - xy^2)^2}{x^2y^2} + x^2yy'' + y^2 = 0$$

$$4xy^2 - 4x^2y^4 + 1 - 2xy^2 + x^2y^4 + x^4y^3y'' + x^2y^4 = 0$$

$$x^4y^3y'' = 2x^2y^4 - 2xy^2 - 1$$

$$y'' = \frac{2x^2y^4 - 2xy^2 - 1}{x^4y^3}$$

38. $1 - xy = x - y$

$$y - xy = x - 1$$

$$y = \frac{x - 1}{1 - x} = -1$$

$$y' = 0$$

$$y'' = 0$$

40. $y^2 = 4x$

$$2yy' = 4$$

$$y' = \frac{2}{y}$$

$$y'' = -2y^{-2}y' = \left[\frac{-2}{y^2}\right] \cdot \frac{2}{y} = \frac{-4}{y^3}$$

42. $y^2 = \dfrac{x - 1}{x^2 + 1}$

$$2yy' = \frac{(x^2 + 1)(1) - (x - 1)(2x)}{(x^2 + 1)^2}$$

$$= \frac{x^2 + 1 - 2x^2 + 2x}{(x^2 + 1)^2}$$

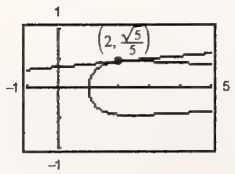
$$y' = \frac{1 + 2x - x^2}{2y(x^2 + 1)^2}$$

At $\left(2, \dfrac{\sqrt{5}}{5}\right)$: $y' = \dfrac{1 + 4 - 4}{[(2\sqrt{5})/5](4 + 1)^2} = \dfrac{1}{10\sqrt{5}}$.

Tangent line: $y - \dfrac{\sqrt{5}}{5} = \dfrac{1}{10\sqrt{5}}(x - 2)$

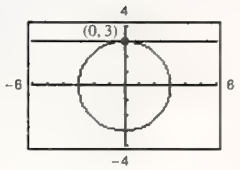
$$10\sqrt{5}y - 10 = x - 2$$

$$x - 10\sqrt{5}y + 8 = 0$$



44. $x^2 + y^2 = 9$

$$y' = \frac{-x}{y}$$

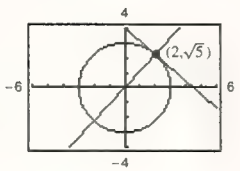


At $(0, 3)$:

Tangent line: $y = 3$

Normal line: $x = 0$.

At $(2, \sqrt{5})$:



Tangent line: $y - \sqrt{5} = \frac{-2}{\sqrt{5}}(x - 2) \Rightarrow 2x + \sqrt{5}y - 9 = 0$

Normal line: $y - \sqrt{5} = \frac{\sqrt{5}}{2}(x - 2) \Rightarrow \sqrt{5}x - 2y = 0$.

46. $y^2 = 4x$

$2yy' = 4$

$y' = \frac{2}{y} = 1$ at $(1, 2)$

Equation of normal at $(1, 2)$ is $y - 2 = -1(x - 1)$, $y = 3 - x$. The centers of the circles must be on the normal and at a distance of 4 units from $(1, 2)$. Therefore,

$$(x - 1)^2 + [(3 - x) - 2]^2 = 16$$

$$2(x - 1)^2 = 16$$

$$x = 1 \pm 2\sqrt{2}.$$

Centers of the circles: $(1 + 2\sqrt{2}, 2 - 2\sqrt{2})$ and $(1 - 2\sqrt{2}, 2 + 2\sqrt{2})$

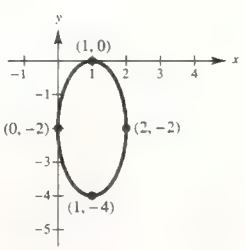
Equations: $(x - 1 - 2\sqrt{2})^2 + (y - 2 + 2\sqrt{2})^2 = 16$

$(x - 1 + 2\sqrt{2})^2 + (y - 2 - 2\sqrt{2})^2 = 16$

48. $4x^2 + y^2 - 8x + 4y + 4 = 0$

$8x + 2yy' - 8 + 4y' = 0$

$$y' = \frac{8 - 8x}{2y + 4} = \frac{4 - 4x}{y + 2}$$



Horizontal tangents occur when $x = 1$:

$$4(1)^2 + y^2 - 8(1) + 4y + 4 = 0$$

$$y^2 + 4y = y(y + 4) = 0 \Rightarrow y = 0, -4$$

Horizontal tangents: $(1, 0)$, $(1, -4)$.

Vertical tangents occur when $y = -2$:

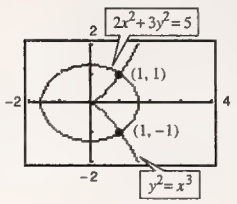
$$4x^2 + (-2)^2 - 8x + 4(-2) + 4 = 0$$

$$4x^2 - 8x = 4x(x - 2) = 0 \Rightarrow x = 0, 2$$

Vertical tangents: $(0, -2)$, $(2, -2)$.

50. Find the points of intersection by letting $y^2 = x^3$ in the equation $2x^2 + 3y^2 = 5$.

$2x^2 + 3x^3 = 5$ and $3x^3 + 2x^2 - 5 = 0$



Intersect when $x = 1$.

Points of intersection: $(1, \pm 1)$

$y^2 = x^3$:	$2x^2 + 3y^2 = 5$:
$2yy' = 3x^2$	$4x + 6yy' = 0$
$y' = \dfrac{3x^2}{2y}$	$y' = -\dfrac{2x}{3y}$

At $(1, 1)$, the slopes are:

$y' = \dfrac{3}{2}$ $\qquad$ $y' = -\dfrac{2}{3}$.

At $(1, -1)$, the slopes are:

$y' = -\dfrac{3}{2}$ $\qquad$ $y' = \dfrac{2}{3}$.

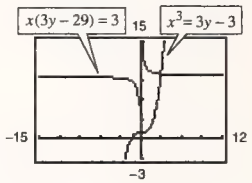
Tangents are perpendicular.

52. Rewriting each equation and differentiating,

$x^3 = 3(y - 1)$ $\qquad$ $x(3y - 29) = 3$

$y = \dfrac{x^3}{3} + 1$ $\qquad$ $y = \dfrac{1}{3}\left(\dfrac{3}{x} + 29\right)$

$y' = x^2$ $\qquad$ $y' = -\dfrac{1}{x^2}$.

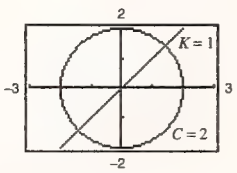


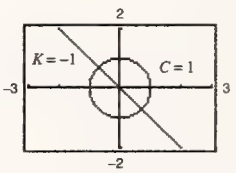
For each value of x, the derivatives are negative reciprocals of each other. Thus, the tangent lines are orthogonal at both points of intersection.

54. $x^2 + y^2 = C^2$ $\qquad$ $y = Kx$

$2x + 2yy' = 0$ $\qquad$ $y' = K$

$y' = -\dfrac{x}{y}$





At the point of intersection (x, y) the product of the slopes is $(-x/y)(K) = (-x/Kx)(K) = -1$. The curves are orthogonal.

56. $x^2 - 3xy^2 + y^3 = 10$

(a) $2x - 3y^2 - 6xyy' + 3y^2y' = 0$

$(-6xy + 3y^2)y' = 3y^2 - 2x$

$y' = \dfrac{3y^2 - 2x}{3y^2 - 6xy}$

(b) $2x\dfrac{dx}{dt} - 3y^2\dfrac{dx}{dt} - 6xy\dfrac{dy}{dt} + 3y^2\dfrac{dy}{dt} = 0$

$(2x - 3y^2)\dfrac{dx}{dt} = (6xy - 3y^2)\dfrac{dy}{dt}$

58. (a) $4 \sin x \cos y = 1$

$4 \sin x(-\sin y)y' + 4 \cos x \cos y = 0$

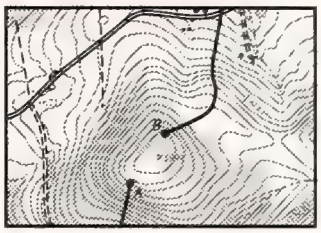
$y' = \dfrac{\cos x \cos y}{\sin x \sin y}$

(b) $4 \sin x(-\sin y)\dfrac{dy}{dt} + 4 \cos x\dfrac{dx}{dt}\cos y = 0$

$\cos x \cos y\dfrac{dx}{dt} = \sin x \sin y\dfrac{dy}{dt}$

60. Given an implicit equation, first differentiate both sides with respect to x. Collect all terms involving y' on the left, and all other terms to the right. Factor out y' on the left side. Finally, divide both sides by the left-hand factor that does not contain y'.

62.



Use starting point B.

64. $\sqrt{x} + \sqrt{y} = \sqrt{c}$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Tangent line at (x_0, y_0):

$$y - y_0 = -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$$

x-intercept: $\left(x_0 + \sqrt{x_0}\sqrt{y_0}, 0\right)$

y-intercept: $\left(0, y_0 + \sqrt{x_0}\sqrt{y_0}\right)$

Sum of intercepts:

$$\left(x_0 + \sqrt{x_0}\sqrt{y_0}\right) + \left(y_0 + \sqrt{x_0}\sqrt{y_0}\right) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = \left(\sqrt{x_0} + \sqrt{y_0}\right)^2 = \left(\sqrt{c}\right)^2 = c.$$

Section 2.6 Related Rates

2. $y = 2(x^2 - 3x)$

$$\frac{dy}{dt} = (4x - 6)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{1}{4x - 6}\frac{dy}{dt}$$

(a) When $x = 3$ and $\frac{dx}{dt} = 2$, $\frac{dy}{dt} = [4(3) - 6](2) = 12$

(b) When $x = 1$ and $\frac{dy}{dt} = 5$, $\frac{dx}{dt} = \frac{1}{4(1) - 6}(5) = -\frac{5}{2}$

4. $x^2 + y^2 = 25$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \left(-\frac{x}{y}\right)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(-\frac{y}{x}\right)\frac{dy}{dt}$$

(a) When $x = 3$, $y = 4$, and $dx/dt = 8$,

$$\frac{dy}{dt} = -\frac{3}{4}(8) = -6$$

(b) When $x = 4$, $y = 3$, and $dy/dt = -2$,

$$\frac{dx}{dt} = -\frac{3}{4}(-2) = \frac{3}{2}.$$

6. $y = \dfrac{1}{1 + x^2}$

$\dfrac{dx}{dt} = 2$

$\dfrac{dy}{dt} = \left[\dfrac{-2x}{(1 + x^2)^2}\right]\dfrac{dx}{dt}$

(a) When $x = -2$,

$$\frac{dy}{dt} = \frac{-2(-2)(2)}{25} = \frac{8}{25}\text{ cm/sec.}$$

(b) When $x = 0$,

$$\frac{dy}{dt} = 0\text{ cm/sec.}$$

(c) When $x = 2$,

$$\frac{dy}{dt} = \frac{-2(2)(2)}{25} = \frac{-8}{25}\text{ cm/sec.}$$

8. $y = \sin x$

$\dfrac{dx}{dt} = 2$

$\dfrac{dy}{dt} = \cos x \dfrac{dx}{dt}$

(a) When $x = \pi/6$,

$$\frac{dy}{dt} = \left(\cos\frac{\pi}{6}\right)(2) = \sqrt{3}\text{ cm/sec.}$$

(b) When $x = \pi/4$,

$$\frac{dy}{dt} = \left(\cos\frac{\pi}{4}\right)(2) = \sqrt{2}\text{ cm/sec.}$$

(c) When $x = \pi/3$,

$$\frac{dy}{dt} = \left(\cos\frac{\pi}{3}\right)(2) = 1\text{ cm/sec.}$$

10. (a) $\dfrac{dx}{dt}$ negative $\Rightarrow \dfrac{dy}{dt}$ negative

(b) $\dfrac{dy}{dt}$ positive $\Rightarrow \dfrac{dx}{dt}$ positive

12. Answers will vary. See page 145.

14. $D = \sqrt{x^2 + y^2} = \sqrt{x^2 + \sin^2 x}$

$\dfrac{dx}{dt} = 2$

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + \sin^2 x)^{-1/2}(2x + 2\sin x\cos x)\frac{dx}{dt} = \frac{x + \sin x\cos x}{\sqrt{x^2 + \sin^2 x}}\frac{dx}{dt} = \frac{2 + 2\sin x\cos x}{\sqrt{x^2 + \sin^2 x}}$$

16. $A = \pi r^2$

$\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$

If dr/dt is constant, dA/dt is not constant.

$\dfrac{dA}{dt}$ depends on r and $\dfrac{dr}{dt}$.

18. $V = \dfrac{4}{3}\pi r^3$

$\dfrac{dr}{dt} = 2$

$\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$

(a) When $r = 6$, $\dfrac{dV}{dt} = 4\pi(6)^2(2) = 288\pi\text{ in}^3/\text{min.}$

When $r = 24$, $\dfrac{dV}{dt} = 4\pi(24)^2(2) = 4608\pi\text{ in}^3/\text{min.}$

(b) If dr/dt is constant, dV/dt is proportional to r^2.

20. $V = x^3$

$\dfrac{dx}{dt} = 3$

$\dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt}$

(a) When $x = 1$,

$$\frac{dV}{dt} = 3(1)^2(3) = 9\text{ cm}^3/\text{sec.}$$

(b) When $x = 10$,

$$\frac{dV}{dt} = 3(10)^2(3) = 900\text{ cm}^3/\text{sec.}$$

22. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(3r) = \pi r^3$

$\frac{dr}{dt} = 2$

$\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}$

(a) When $r = 6$,

$$\frac{dV}{dt} = 3\pi(6)^2(2) = 216\pi \text{ in}^3/\text{min}.$$

(b) When $r = 24$,

$$\frac{dV}{dt} = 3\pi(24)^2(2) = 3456\pi \text{ in}^3/\text{min}.$$

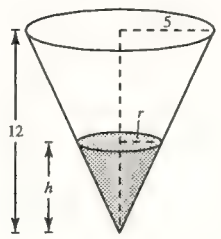
24. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \frac{25}{144}h^3 = \frac{25\pi}{3(144)}h^3$

$\left(\text{By similar triangles, } \frac{r}{5} = \frac{h}{12} \Rightarrow r = \frac{5}{12}h.\right)$

$\frac{dV}{dt} = 10$

$\frac{dV}{dt} = \frac{25\pi}{144}h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \left(\frac{144}{25\pi h^2}\right)\frac{dV}{dt}$

When $h = 8$, $\frac{dh}{dt} = \frac{144}{25\pi(64)}(10) = \frac{9}{10\pi}$ ft/min.



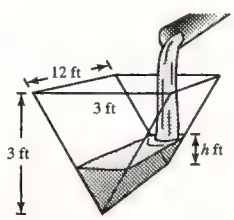
26. $V = \frac{1}{2}bh(12) = 6bh = 6h^2$ (since $b = h$)

(a) $\frac{dV}{dt} = 12h\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{12h}\frac{dV}{dt}$

When $h = 1$ and $\frac{dV}{dt} = 2$, $\frac{dh}{dt} = \frac{1}{12(1)}(2) = \frac{1}{6}$ ft/min

(b) If $\frac{dh}{dt} = \frac{3}{8}$ in/min $= \frac{1}{32}$ ft/min and $h = 2$ feet, then

$\frac{dV}{dt} = (12)(2)\left(\frac{1}{32}\right) = \frac{3}{4}$ ft^3/min.



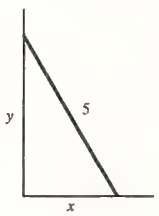
28. $x^2 + y^2 = 25$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\frac{dx}{dt} = -\frac{y}{x} \cdot \frac{dy}{dt} = -\frac{0.15y}{x}$ since $\frac{dy}{dt} = 0.15$.

When $x = 2.5$,

$y = \sqrt{18.75}$, $\frac{dx}{dt} = -\frac{\sqrt{18.75}}{2.5}0.15 \approx -0.26$ m/sec



30. Let L be the length of the rope.

(a) $L^2 = 144 + x^2$

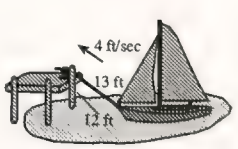
$2L\frac{dL}{dt} = 2x\frac{dx}{dt}$

$\frac{dx}{dt} = \frac{L}{x} \cdot \frac{dL}{dt} = -\frac{4L}{x}$ since $\frac{dL}{dt} = -4$ ft/sec.

When $L = 13$,

$x = \sqrt{L^2 - 144} = \sqrt{169 - 144} = 5$

$\frac{dx}{dt} = -\frac{4(13)}{5} = -\frac{52}{5} = -10.4$ ft/sec.



Speed of the boat increases as it approaches the dock.

(b) If $\frac{dx}{dt} = -4$, and $L = 13$,

$\frac{dL}{dt} = \frac{x}{L}\frac{dx}{dt}$

$= \frac{5}{13}(-4)$

$= \frac{-20}{13}$ ft/sec

As $L \to 12^+$, $\left|\frac{dL}{dt}\right|$ decreases, so the speed decreases.

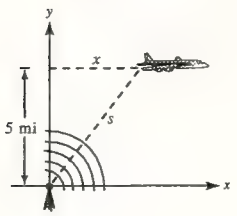
32. $x^2 + y^2 = s^2$

$$2x\frac{dx}{dt} + 0 = 2s\frac{ds}{dt} \qquad \left(\text{since } \frac{dy}{dt} = 0\right)$$

$$\frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt}$$

When $s = 10, x = \sqrt{100 - 25} = \sqrt{75} = 5\sqrt{3}$

$$\frac{dx}{dt} = \frac{10}{5\sqrt{3}}(240) = \frac{480}{\sqrt{3}} = 160\sqrt{3} \approx 277.13 \text{ mph.}$$

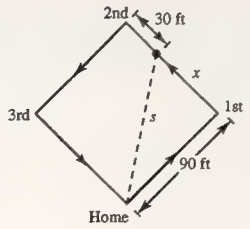


34. $s^2 = 90^2 + x^2$

$x = 60$

$\frac{dx}{dt} = 28$

$\frac{ds}{dt} = \frac{x}{s} \cdot \frac{dx}{dt}$



When $x = 60$,

$$s = \sqrt{90^2 + 60^2} = 30\sqrt{13}$$

$$\frac{ds}{dt} = \frac{60}{30\sqrt{13}}(28) = \frac{56}{\sqrt{13}} \approx 15.53 \text{ ft/sec.}$$

36. (a) $\frac{20}{6} = \frac{y}{y - x}$

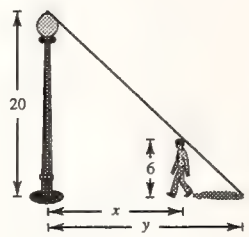
$20y - 20x = 6y$

$14y = 20x$

$y = \frac{10}{7}x$

$\frac{dx}{dt} = -5$

$$\frac{dy}{dt} = \frac{10}{7}\frac{dx}{dt} = \frac{10}{7}(-5) = \frac{-50}{7} \text{ ft/sec}$$



(b) $$\frac{d(y - x)}{dt} = \frac{dy}{dt} - \frac{dx}{dt} = \frac{-50}{7} - (-5) = \frac{-50}{7} + \frac{35}{7} = \frac{-15}{7} \text{ ft/sec}$$

38. $x(t) = \frac{3}{5}\sin \pi t, x^2 + y^2 = 1$

(a) Period: $\frac{2\pi}{\pi} = 2$ seconds

(b) When $x = \frac{3}{5}, y = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$ m.

Lowest point: $\left(0, \frac{4}{5}\right)$

(c) When $x = \frac{3}{10}, y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$ and

$$\frac{3}{10} = \frac{3}{5}\sin \pi t \Rightarrow \sin \pi t = \frac{1}{2} \Rightarrow t = \frac{1}{6}$$

$$\frac{dx}{dt} = \frac{3}{5}\pi \cos \pi t$$

$$x^2 + y^2 = 1$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}.$$

Thus, $\frac{dy}{dt} = \frac{-3/10}{\sqrt{15}/4} \cdot \frac{3}{5}\pi \cos\left(\frac{\pi}{6}\right)$

$$= \frac{-9\pi}{25\sqrt{5}} = \frac{-9\sqrt{5}\pi}{125}.$$

Speed $= \left|\frac{-9\sqrt{5}\pi}{125}\right| \approx 0.5058$ m/sec

40. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$

$\dfrac{dR_1}{dt} = 1$

$\dfrac{dR_2}{dt} = 1.5$

$\dfrac{1}{R^2} \cdot \dfrac{dR}{dt} = \dfrac{1}{R_1^2} \cdot \dfrac{dR_1}{dt} + \dfrac{1}{R_2^2} \cdot \dfrac{dR_2}{dt}$

When $R_1 = 50$ and $R_2 = 75$,

$R = 30$

$\dfrac{dR}{dt} = (30)^2\left[\dfrac{1}{(50)^2}(1) + \dfrac{1}{(75)^2}(1.5)\right]$

$= 0.6$ ohms/sec.

42. $rg \tan\theta = v^2$

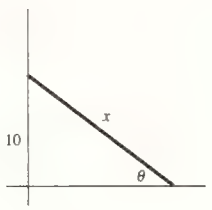
$32r \tan\theta = v^2$, r is a constant.

$32r \sec^2\theta \dfrac{d\theta}{dt} = 2v\dfrac{dv}{dt}$

$\dfrac{dv}{dt} = \dfrac{16r}{v}\sec^2\theta\,\dfrac{d\theta}{dt}$

Likewise, $\dfrac{d\theta}{dt} = \dfrac{v}{16r}\cos^2\theta\,\dfrac{dv}{dt}$.

44. $\sin\theta = \dfrac{10}{x}$



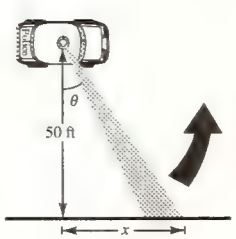
$\dfrac{dx}{dt} = (-1)$ft/sec

$\cos\theta\left(\dfrac{d\theta}{dt}\right) = \dfrac{-10}{x^2} \cdot \dfrac{dx}{dt}$

$\dfrac{d\theta}{dt} = \dfrac{-10}{x^2}\dfrac{dx}{dt}(\sec\theta)$

$= \dfrac{-10}{25^2}(-1)\dfrac{25}{\sqrt{25^2 - 10^2}} = \dfrac{10}{25}\dfrac{1}{5\sqrt{21}} = \dfrac{2}{25\sqrt{21}} = \dfrac{2\sqrt{21}}{525} \approx 0.017$ rad/sec

46. $\tan\theta = \dfrac{x}{50}$



$\dfrac{d\theta}{dt} = 30(2\pi) = 60\pi$ rad/min $= \pi$ rad/sec

$\sec^2\theta\left(\dfrac{d\theta}{dt}\right) = \dfrac{1}{50}\left(\dfrac{dx}{dt}\right)$

$\dfrac{dx}{dt} = 50\sec^2\theta\left(\dfrac{d\theta}{dt}\right)$

(a) When $\theta = 30°$, $\dfrac{dx}{dt} = \dfrac{200\pi}{3}$ ft/sec.

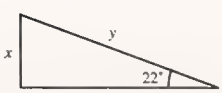
(b) When $\theta = 60°$, $\dfrac{dx}{dt} = 200\pi$ ft/sec.

(c) When $\theta = 70°$, $\dfrac{dx}{dt} \approx 427.43\pi$ ft/sec.

48. $\sin 22° = \dfrac{x}{y}$

$0 = -\dfrac{x}{y^2} \cdot \dfrac{dy}{dt} + \dfrac{1}{y} \cdot \dfrac{dx}{dt}$

$\dfrac{dx}{dt} = \dfrac{x}{y} \cdot \dfrac{dy}{dt} = (\sin 22°)(240) \approx 89.9056$ mi/hr



50. (a) $dy/dt = 3(dx/dt)$ means that y changes three times as fast as x changes.

(b) y changes slowly when $x \approx 0$ or $x \approx L$. y changes more rapidly when x is near the middle of the interval.

52. $L^2 = 144 + x^2$; acceleration of the boat $= \dfrac{d^2x}{dt^2}$.

First derivative: $2L\dfrac{dL}{dt} = 2x\dfrac{dx}{dt}$

$$L\frac{dL}{dt} = x\frac{dx}{dt}$$

Second derivative: $L\dfrac{d^2L}{dt^2} + \dfrac{dL}{dt}\cdot\dfrac{dL}{dt} = x\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt}\cdot\dfrac{dx}{dt}$

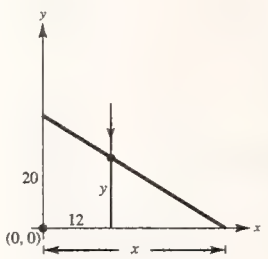
$$\frac{d^2x}{dt^2} = \left(\frac{1}{x}\right)\left[L\frac{d^2L}{dt^2} + \left(\frac{dL}{dt}\right)^2 - \left(\frac{dx}{dt}\right)^2\right]$$

When $L = 13$, $x = 5$, $\dfrac{dx}{dt} = -10.4$, and $\dfrac{dL}{dt} = -4$ (see Exercise 30). Since $\dfrac{dL}{dt}$ is constant, $\dfrac{d^2L}{dt^2} = 0$.

$$\frac{d^2x}{dt^2} = \frac{1}{5}[13(0) + (-4)^2 - (-10.4)^2]$$

$$= \frac{1}{5}[16 - 108.16] = \frac{1}{5}[-92.16] = -18.432 \text{ ft/sec}^2$$

54. $y(t) = -4.9t^2 + 20$



$$\frac{dy}{dt} = -9.8t$$

$y(1) = -4.9 + 20 = 15.1$

$y'(1) = -9.8$

By similar triangles, $\dfrac{20}{x} = \dfrac{y}{x - 12}$

$$20x - 240 = xy.$$

When $y = 15.1$, $20x - 240 = x(15.1)$

$$(20 - 15.1)x = 240$$

$$x = \frac{240}{4.9}.$$

$$20x - 240 = xy$$

$$20\frac{dx}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x}{20 - y}\frac{dy}{dt}$$

At $t = 1$, $\dfrac{dx}{dt} = \dfrac{240/4.9}{20 - 15.1}(-9.8) \approx -97.96$ m/sec.

Review Exercises for Chapter 2

2. $f(x) = \dfrac{x+1}{x-1}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{x+\Delta x+1}{x+\Delta x-1} - \dfrac{x+1}{x-1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x+1)(x-1) - (x+\Delta x-1)(x+1)}{\Delta x(x+\Delta x-1)(x-1)}$$

$$= \lim_{\Delta x \to 0} \frac{(x^2+x\Delta x+x-x-\Delta x-1) - (x^2+x\Delta x-x+x+\Delta x-1)}{\Delta x(x+\Delta x-1)(x-1)}$$

$$= \lim_{\Delta x \to 0} \frac{-2\Delta x}{\Delta x(x+\Delta x-1)(x-1)} = \lim_{\Delta x \to 0} \frac{-2}{(x+\Delta x-1)(x-1)} = \frac{-2}{(x-1)^2}$$

4. $f(x) = \dfrac{2}{x}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{2}{x+\Delta x} - \dfrac{2}{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x - (2x+2\Delta x)}{\Delta x(x+\Delta x)x}$$

$$= \lim_{\Delta x \to 0} \frac{-2\Delta x}{\Delta x(x+\Delta x)x}$$

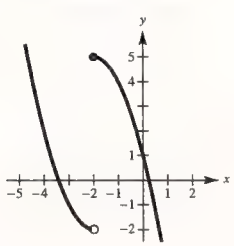
$$= \lim_{\Delta x \to 0} \frac{-2}{(x+\Delta x)x} = \frac{-2}{x^2}$$

6. f is differentiable for all $x \neq -3$.

8. $f(x) = \begin{cases} x^2+4x+2, & \text{if } x < -2 \\ 1-4x-x^2, & \text{if } x \geq -2 \end{cases}$

(a) Nonremovable discontinuity at $x = -2$.

(b) Not differentiable at $x = -2$ because the function is discontinuous there.



10. Using the limit defintion, you obtain $h'(x) = \dfrac{3}{8} - 4x$.

At $x = -2$, $h'(-2) = \dfrac{3}{8} - 4(-2) = \dfrac{67}{8}$.

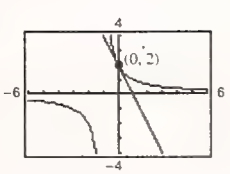
12. (a) Using the limit definition, $f'(x) = \dfrac{-2}{(x+1)^2}$.

At $x = 0, f'(0) = -2$. The tangent line is

$$y - 2 = -2(x - 0)$$

$$y = -2x + 2$$

(b)



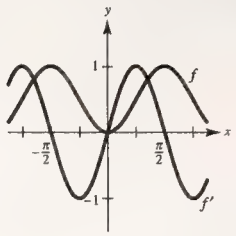
14. $f'(2) = \lim_{x\to 2} \dfrac{f(x) - f(2)}{x - 2}$

$$= \lim_{x\to 2} \frac{\dfrac{1}{x+1} - \dfrac{1}{3}}{x - 2}$$

$$= \lim_{x\to 2} \frac{3 - x - 1}{(x - 2)(x + 1)3}$$

$$= \lim_{x\to 2} \frac{-1}{(x + 1)3} = \frac{-1}{9}$$

16.



18. $y = -12$

$y' = 0$

20. $g(x) = x^{12}$

$g'(x) = 12x^{11}$

22. $f(t) = -8t^5$

$f'(t) = -40t^4$

24. $g(s) = 4s^4 - 5s^2$

$g'(s) = 16s^3 - 10s$

26. $f(x) = x^{1/2} - x^{-1/2}$

$$f'(x) = \frac{1}{2}x^{-1/2} + \frac{1}{2}x^{-3/2} = \frac{x + 1}{2x^{3/2}}$$

28. $h(x) = \dfrac{2}{9}x^{-2}$

$$h'(x) = \frac{-4}{9}x^{-3} = \frac{-4}{9x^3}$$

30. $g(\alpha) = 4\cos\alpha + 6$

$g'(\alpha) = -4\sin\alpha$

32. $g(\alpha) = \dfrac{5\sin\alpha}{3} - 2\alpha$

$$g'(\alpha) = \frac{5\cos\alpha}{3} - 2$$

34. $s = -16t^2 + s_0$

First ball:

$$-16t^2 + 100 = 0$$

$$t = \sqrt{\frac{100}{16}} = \frac{10}{4} = 2.5 \text{ seconds to hit ground}$$

Second ball:

$$-16t^2 + 75 = 0$$

$$t^2 = \sqrt{\frac{75}{16}} = \frac{5\sqrt{3}}{4} \approx 2.165 \text{ seconds to hit ground}$$

Since the second ball was released one second after the first ball, the first ball will hit the ground first. The second ball will hit the ground $3.165 - 2.5 = 0.665$ second later.

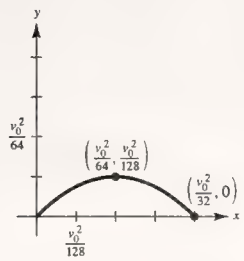
36. $s(t) = -16t^2 + 14{,}400 = 0$

$16t^2 = 14{,}400$

$t = 30$ sec

Since 600 mph $= \frac{1}{6}$ mi/sec, in 30 seconds the bomb will move horizontally $(\frac{1}{6})(30) = 5$ miles.

38.



(a) $y = x - \dfrac{32}{{v_0}^2}x^2 = x\left(1 - \dfrac{32}{{v_0}^2}x\right)$

$= 0$ if $x = 0$ or $x = \dfrac{{v_0}^2}{32}$.

Projectile strikes the ground when $x = {v_0}^2/32$.

Projectile reaches its maximum height at $x = {v_0}^2/64$. (one-half the distance)

(b) $y' = 1 - \dfrac{64}{{v_0}^2}x$

When $x = \dfrac{{v_0}^2}{64}$, $y' = 1 - \dfrac{64}{{v_0}^2}\left(\dfrac{{v_0}^2}{64}\right) = 0$.

(c) $y = x - \dfrac{32}{{v_0}^2}x^2 = x\left(1 - \dfrac{32}{{v_0}^2}x\right) = 0$

when $x = 0$ and $x = {x_0}^2/32$. Therefore, the range is $x = {v_0}^2/32$. When the initial velocity is doubled the range is

$$x = \frac{(2v_0)^2}{32} = \frac{4{v_0}^2}{32}$$

or four times the initial range. From part (a), the maximum height occurs when $x = {v_0}^2/64$. The maximum height is

$$y\left(\frac{{v_0}^2}{64}\right) = \frac{{v_0}^2}{64} - \frac{32}{{v_0}^2}\left(\frac{{v_0}^2}{64}\right)^2 = \frac{{v_0}^2}{64} - \frac{{v_0}^2}{128} = \frac{{v_0}^2}{128}.$$

If the initial velocity is doubled, the maximum height is

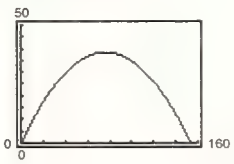
$$y\left[\frac{(2v_0)^2}{64}\right] = \frac{(2v_0)^2}{128} = 4\left(\frac{{v_0}^2}{128}\right)$$

or four times the original maximum height.

(d) $v_0 = 70$ ft/sec

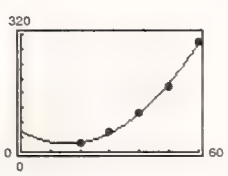
Range: $x = \dfrac{{v_0}^2}{32} = \dfrac{(70)^2}{32} = 153.125$ ft

Maximum height: $y = \dfrac{{v_0}^2}{128} = \dfrac{(70)^2}{128} \approx 38.28$ ft

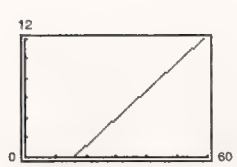


40. (a) $y = 0.14x^2 - 4.43x + 58.4$

(b)



(c)



(d) If $x = 65$, $y \approx 362$ feet.

(e) As the speed increases, the stopping distance increases at an increasing rate.

42. $g(x) = (x^3 - 3x)(x + 2)$

$g'(x) = (x^3 - 3x)(1) + (x + 2)(3x^2 - 3)$

$= x^3 - 3x + 3x^3 + 6x^2 - 3x - 6$

$= 4x^3 + 6x^2 - 6x - 6$

44. $f(t) = t^3 \cos t$

$f'(t) = t^3(-\sin t) + \cos t(3t^2)$

$= -t^3 \sin t + 3t^2 \cos t$

46. $f(x) = \dfrac{x + 1}{x - 1}$

$f'(x) = \dfrac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2}$

$= \dfrac{-2}{(x - 1)^2}$

48. $f(x) = \dfrac{6x - 5}{x^2 + 1}$

$f'(x) = \dfrac{(x^2 + 1)(6) - (6x - 5)(2x)}{(x^2 + 1)^2}$

$= \dfrac{2(3 + 5x - 3x^2)}{(x^2 + 1)^2}$

50. $f(x) = 9(3x^2 - 2x)^{-1}$

$f'(x) = -9(3x^2 - 2x)^{-2}(6x - 2) = \dfrac{18(1 - 3x)}{(3x^2 - 2x)^2}$

52. $y = \dfrac{\sin x}{x^2}$

$y' = \dfrac{(x^2)\cos x - (\sin x)(2x)}{x^4} = \dfrac{x \cos x - 2 \sin x}{x^3}$

54. $y = 2x - x^2 \tan x$

$y' = 2 - x^2 \sec^2 x - 2x \tan x$

56. $y = \dfrac{1 + \sin x}{1 - \sin x}$

$y' = \dfrac{(1 - \sin x)\cos x - (1 + \sin x)(-\cos x)}{(1 - \sin x)^2}$

$= \dfrac{2 \cos x}{(1 - \sin x)^2}$

58. $v(t) = 36 - t^2, 0 \le t \le 6$

$a(t) = v'(t) = -2t$

$v(4) = 36 - 16 = 20$ m/sec

$a(4) = -8$ m/sec

60. $f(x) = 12x^{1/4}$

$f'(x) = 3x^{-3/4}$

$f''(x) = \dfrac{-9}{4}x^{-7/4} = \dfrac{-9}{4x^{7/4}}$

62. $h(t) = 4 \sin t - 5 \cos t$

$h'(t) = 4 \cos t + 5 \sin t$

$h''(t) = -4 \sin t + 5 \cos t$

64. $y = \dfrac{(10 - \cos x)}{x}$

$xy + \cos x = 10$

$xy' + y - \sin x = 0$

$xy' = \sin x - y$

$xy' + y = (\sin x - y) + y = \sin x$

66. $f(x) = (x^2 - 1)^{1/3}$

$f'(x) = \dfrac{1}{3}(x^2 - 1)^{-2/3}(2x)$

$= \dfrac{2x}{3(x^2 - 1)^{2/3}}$

68. $f(x) = \left(x^2 + \dfrac{1}{x}\right)^5$

$f'(x) = 5\left(x^2 + \dfrac{1}{x}\right)^4\left(2x - \dfrac{1}{x^2}\right)$

70. $h(\theta) = \dfrac{\theta}{(1-\theta)^3}$

$$h'(\theta) = \frac{(1-\theta)^3 - \theta[3(1-\theta)^2(-1)]}{(1-\theta)^6}$$

$$= \frac{(1-\theta)^2(1-\theta+3\theta)}{(1-\theta)^6} = \frac{2\theta+1}{(1-\theta)^4}$$

72. $y = 1 - \cos 2x + 2\cos^2 x$

$$y' = 2\sin 2x - 4\cos x \sin x$$

$$= 2[2\sin x \cos x] - 4\sin x \cos x$$

$$= 0$$

74. $y = \csc 3x + \cot 3x$

$$y' = -3\csc 3x \cot 3x - 3\csc^2 3x$$

$$= -3\csc 3x(\cot 3x + \csc 3x)$$

76. $y = \dfrac{\sec^7 x}{7} - \dfrac{\sec^5 x}{5}$

$$y' = \sec^6 x(\sec x \tan x) - \sec^4 x(\sec x \tan x)$$

$$= \sec^5 x \tan x(\sec^2 x - 1)$$

$$= \sec^5 x \tan^3 x$$

78. $f(x) = \dfrac{3x}{\sqrt{x^2+1}}$

$$f'(x) = \frac{3(x^2+1)^{1/2} - 3x\frac{1}{2}(x^2+1)^{-1/2}(2x)}{x^2+1}$$

$$= \frac{3(x^2+1) - 3x^2}{(x^2+1)^{3/2}}$$

$$= \frac{3}{(x^2+1)^{3/2}}$$

80. $y = \dfrac{\cos(x-1)}{x-1}$

$$y' = \frac{-(x-1)\sin(x-1) - \cos(x-1)(1)}{(x-1)^2}$$

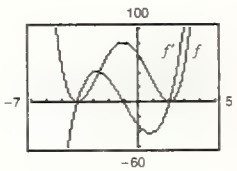
$$= -\frac{1}{(x-1)^2}[(x-1)\sin(x-1) + \cos(x-1)]$$

82. $f(x) = [(x-2)(x+4)]^2 = (x^2+2x-8)^2$

$$f'(x) = 4(x^3+3x^2-6x-8)$$

$$= 4(x-2)(x+1)(x+4)$$

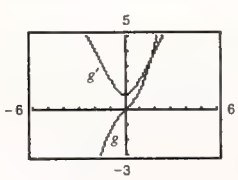
The zeros of f' correspond to the points on the graph of f where the tangent line is horizontal.



84. $g(x) = x(x^2+1)^{1/2}$

$$g'(x) = \frac{2x^2+1}{\sqrt{x^2+1}}$$

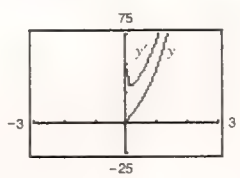
g' does not equal zero for any value of x. The graph of g has no horizontal tangent lines.



86. $y = \sqrt{3x}(x+2)^3$

$$y' = \frac{3(x+2)^2(7x+2)}{2\sqrt{3x}}$$

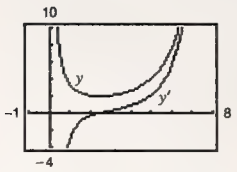
y' does not equal zero for any x in the domain. The graph has no horizontal tangent lines.



88. $y = 2\csc^3(\sqrt{x})$

$$y' = -\frac{3}{\sqrt{x}}\csc^3\sqrt{x}\cot\sqrt{x}$$

The zero of y' corresponds to the point on the graph of y where the tangent line is horizontal.



90. $y = x^{-1} + \tan x$

$$y' = -x^{-2} + \sec^2 x$$

$$y'' = 2x^{-3} + 2\sec x(\sec x \tan x)$$

$$= \frac{2}{x^3} + 2\sec^2 x \tan x$$

92. $y = \sin^2 x$

$$y' = 2\sin x \cos x = \sin 2x$$

$$y'' = 2\cos 2x$$

94. $g(x) = \dfrac{6x - 5}{x^2 + 1}$

$$g'(x) = \frac{2(-3x^2 + 5x + 3)}{(x^2 + 1)^2}$$

$$g''(x) = \frac{2(6x^3 - 15x^2 - 18x + 5)}{(x^2 + 1)^3}$$

96. $h(x) = x\sqrt{x^2 - 1}$

$$h'(x) = \frac{2x^2 - 1}{\sqrt{x^2 - 1}}$$

$$h''(x) = \frac{x(2x^2 - 3)}{(x^2 - 1)^{3/2}}$$

98. $v = \sqrt{2gh} = \sqrt{2(32)h} = 8\sqrt{h}$

$$\frac{dv}{dh} = \frac{4}{\sqrt{h}}$$

(a) When $h = 9$, $\dfrac{dv}{dh} = \dfrac{4}{3}$ ft/sec.

(b) When $h = 4$, $\dfrac{dv}{dh} = 2$ ft/sec.

100. $x^2 + 9y^2 - 4x + 3y = 0$

$$2x + 18yy' - 4 + 3y' = 0$$

$$3(6y + 1)y' = 4 - 2x$$

$$y' = \frac{4 - 2x}{3(6y + 1)}$$

102.

$$y^2 = x^3 - x^2y + xy - y^2$$

$$0 = x^3 - x^2y + xy - 2y^2$$

$$0 = 3x^2 - x^2y' - 2xy + xy' + y - 4yy'$$

$$(x^2 - x + 4y)y' = 3x^2 - 2xy + y$$

$$y' = \frac{3x^2 - 2xy + y}{x^2 - x + 4y}$$

104.

$$\cos(x + y) = x$$

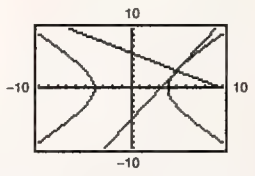
$$-(1 + y')\sin(x + y) = 1$$

$$-y'\sin(x + y) = 1 + \sin(x + y)$$

$$y' = -\frac{1 + \sin(x + y)}{\sin(x + y)}$$

$$= -\csc(x + 1) - 1$$

106. $x^2 - y^2 = 16$



$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}$$

At (5, 3): $y' = \dfrac{5}{3}$

Tangent line: $y - 3 = \dfrac{5}{3}(x - 5)$

$$5x - 3y - 16 = 0$$

Normal line: $y - 3 = -\dfrac{3}{5}(x - 5)$

$$3x + 5y - 30 = 0$$

108. Surface area $= A = 6x^2$, x length of edge.

$$\frac{dx}{dt} = 5$$

$$\frac{da}{dt} = 12x\frac{dx}{dt} = 12(4.5)(5) = 270 \text{ cm}^2/\text{sec}$$

110. $\tan\theta = x$

$$\frac{d\theta}{dt} = 3(2\pi) \text{ rad/min}$$

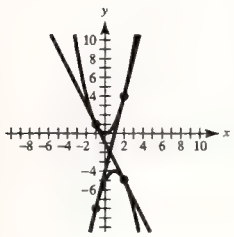
$$\sec^2\theta\left(\frac{d\theta}{dt}\right) = \frac{dx}{dt}$$

$$\frac{dx}{dt} = (\tan^2\theta + 1)(6\pi) = 6\pi(x^2 + 1)$$

When $x = \frac{1}{2}$, $\frac{dx}{dt} = 6\pi\left(\frac{1}{4} + 1\right) = \frac{15\pi}{2}$ km/min $= 450\pi$ km/hr.

Problem Solving for Chapter 2

2.



Let (a, a^2) and $(b, -b^2 + 2b - 5)$ be the points of tangency.

For $y = x^2$, $y' = 2x$ and for $y = -x^2 + 2x - 5$, $y' = -2x + 2$.

Thus, $2a = -2b + 2 \Rightarrow a + b = 1$, or $a = 1 - b$. Furthermore, the slope of the common tangent line is

$$\frac{a^2 - (-b^2 + 2b - 5)}{a - b} = \frac{(1 - b)^2 + b^2 - 2b + 5}{(1 - b) - b} = -2b + 2$$

$$\Rightarrow \frac{1 - 2b + b^2 + b^2 - 2b + 5}{1 - 2b} = -2b + 2$$

$$\Rightarrow 2b^2 - 4b + 6 = 4b^2 - 6b + 2$$

$$\Rightarrow 2b^2 - 2b - 4 = 0$$

$$\Rightarrow b^2 - b - 2 = 0$$

$$\Rightarrow (b - 2)(b + 1) = 0$$

$$b = 2, -1$$

For $b = 2$, $a = 1 - b = -1$ and the points of tangency are $(-1, 1)$, $(2, -5)$. The tangent line has slope -2:

$y - 1 = -2(x = 1) \Rightarrow y = -2x - 1$

For $b = -1$, $a = 1 - b = 2$ and the points of tangency are $(2, 4)$ and $(-1, -8)$. The tangent line has slope 4:

$y - 4 = 4(x - 2) \Rightarrow y = 4x - 4$.

4. (a) $y = x^2$, $y' = 2x$. Slope $= 4$ at $(2, 4)$.

Tangent line: $y - 4 = 4(x - 2)$

$$y = 4x - 4$$

(b) Slope of normal line: $-\frac{1}{4}$.

Normal line: $y - 4 = -\frac{1}{4}(x - 2)$

$$y = -\frac{1}{4}x + \frac{9}{2}$$

$y = -\frac{1}{4}x + \frac{9}{2} = x^2 \Rightarrow 4x^2 + x - 18 = 0 \Rightarrow (4x + 9)(x - 2) = 0$

$x = 2, -\frac{9}{4}$. Second intersection point: $\left(-\frac{9}{4}, \frac{81}{16}\right)$

(c) Tangent line: $y = 0$

Normal line: $x = 0$

—CONTINUED—

4. —CONTINUED—

(d) Let (a, a^2), $a \neq 0$, be a point on the parabola $y = x^2$. Tangent line at (a, a^2) is $y = 2a(x - a) + a^2$. Normal line at (a, a^2) is $y = -\frac{1}{2a}(x - a) + a^2$. To find points of intersection, solve

$$x^2 = -\frac{1}{2a}(x - a) + a^2$$

$$x^2 + \frac{1}{2a}x = a^2 + \frac{1}{2}$$

$$x^2 + \frac{1}{2a}x + \frac{1}{16a^2} = a^2 + \frac{1}{2} + \frac{1}{16a^2}$$

$$\left(x + \frac{1}{4a}\right)^2 = \left(a + \frac{1}{4a}\right)^2$$

$$x + \frac{1}{4a} = \pm\left(a + \frac{1}{4a}\right)$$

$x + \frac{1}{4a} = a + \frac{1}{4a} \Rightarrow x = a$ (Point of tangency)

$x + \frac{1}{4a} = -\left(a + \frac{1}{4a}\right) \Rightarrow x = -a - \frac{1}{2a} = -\frac{2a^2 + 1}{2a}$

The normal line intersects a second time at $x = -\frac{2a^2 + 1}{2a}$.

6. $f(x) = a + b\cos cx$

$f'(x) = -bc\sin cx$

At $(0, 1)$: $a + b = 1$ Equation 1

At $\left(\frac{\pi}{4}, \frac{3}{2}\right)$: $a + b\cos\left(\frac{c\pi}{4}\right) = \frac{3}{2}$ Equation 2

$-bc\sin\left(\frac{c\pi}{4}\right) = 1$ Equation 3

From Equation 1, $a = 1 - b$. Equation 2 becomes $(1 - b) + b\cos\left(\frac{c\pi}{4}\right) = \frac{3}{2} \Rightarrow -b + b\cos\frac{c\pi}{4} = \frac{1}{2}$

From Equation 3, $b = \frac{-1}{c\sin\left(\frac{c\pi}{4}\right)}$. Thus $\frac{1}{c\sin\left(\frac{c\pi}{4}\right)} + \frac{-1}{c\sin\left(\frac{c\pi}{4}\right)}\cos\left(\frac{c\pi}{4}\right) = \frac{1}{2}$

$1 - \cos\left(\frac{c\pi}{4}\right) = \frac{1}{2}c\sin\left(\frac{c\pi}{4}\right)$

Graphing the equation $g(c) = \frac{1}{2}c\sin\left(\frac{c\pi}{4}\right) + \cos\left(\frac{c\pi}{4}\right) - 1$, you see that many values of c will work.

One answer: $c = 2, b = -\frac{1}{2}, a = \frac{3}{2} \Rightarrow f(x) = \frac{3}{2} - \frac{1}{2}\cos 2x$

8. (a) $b^2y^2 = x^3(a - x);\ a, b > 0$

$$y^2 = \frac{x^3(a-x)}{b^2}$$

Graph $y_1 = \dfrac{\sqrt{x^3(a-x)}}{b}$ and $y_2 = -\dfrac{\sqrt{x^3(a-x)}}{b}$

(b) a determines the x-intercept on the right: $(a, 0)$.

b affects the height.

(c) Differentiating implicitly.

$$2b^2yy' = 3x^2(a-x) - x^3 = 3ax^2 - 4x^3$$

$$y' = \frac{(3ax^2 - 4x^3)}{2b^2y} = 0$$

$$\Rightarrow 3ax^2 = 4x^3$$

$$3a = 4x$$

$$x = \frac{3a}{4}.$$

$$b^2y^2 = \left(\frac{3a}{4}\right)^3\left(a - \frac{3a}{4}\right) = \frac{27a^3}{64}\left(\frac{1}{4}a\right)$$

$$y^2 = \frac{27a^4}{256b^2} \Rightarrow y = \pm\frac{3\sqrt{3}a^2}{16b}$$

Two points: $\left(\dfrac{3a}{4}, \dfrac{3\sqrt{3}a^2}{16b}\right), \left(\dfrac{3a}{4}, \dfrac{-3\sqrt{3}a^2}{16b}\right)$

10. (a) $y = x^{1/3} \Rightarrow \dfrac{dy}{dt} = \dfrac{1}{3}x^{-2/3}\dfrac{dx}{dt}$

$$1 = \frac{1}{3}(8)^{-2/3}\frac{dx}{dt}$$

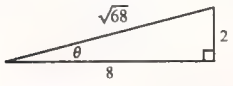
$$\frac{dx}{dt} = 12 \text{ cm/sec}$$

(b) $D = \sqrt{x^2 + y^2} \Rightarrow \dfrac{dD}{dt} = \dfrac{1}{2}(x^2 + y^2)\left(2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}\right)$

$$= \frac{x\dfrac{dx}{dt} + y\dfrac{dy}{dt}}{\sqrt{x^2 + y^2}}$$

$$= \frac{8(12) + 2(1)}{\sqrt{64 + 4}} = \frac{98}{\sqrt{68}} = \frac{49}{\sqrt{17}} \text{ cm/sec.}$$

(c) $\tan\theta = \dfrac{y}{x} \Rightarrow \sec^2\theta \cdot \dfrac{d\theta}{dt} = \dfrac{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}}{x^2}$



From the triangle, $\sec\theta = \dfrac{\sqrt{68}}{8}$. Hence $\dfrac{d\theta}{dt} = \dfrac{8(1) - 2(12)}{64\left(\dfrac{68}{64}\right)} = \dfrac{-16}{68} = \dfrac{-4}{17}$ rad/sec

12. $E'(x) = \lim_{\Delta x \to 0} \dfrac{E(x + \Delta x) - E(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{E(x)E(\Delta x) - E(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} E(x)\left(\frac{E(\Delta x) - 1}{\Delta x}\right)$$

$$= E(x) \lim_{\Delta x \to 0} \frac{E(\Delta x) - 1}{\Delta x}$$

But, $E'(0) = \lim_{\Delta x \to 0} \dfrac{E(\Delta x) - E(0)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{E(\Delta x) - 1}{\Delta x} = 1.$

Thus, $E'(x) = E(x)E'(0) = E(x)$ exists for all x.

For example: $E(x) = e^x$.

14. (a) $v(t) = -\dfrac{27}{5}t + 27$ ft/sec

$a(t) = -\dfrac{27}{5}$ ft/sec^2

(b) $v(t) = -\dfrac{27}{5}t + 27 = 0 \Rightarrow \dfrac{27}{5}t = 27 \Rightarrow t = 5$ seconds

$S(5) = -\dfrac{27}{10}(5)^2 + 27(5) + 6 = 73.5$ feet

(c) The acceleration due to gravity on Earth is greater in magnitude than that on the moon.

CHAPTER 3
Applications of Differentiation

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CHAPTER 3
Applications of Differentiation

Section 3.1 Extrema on an Interval

Solutions to Even-Numbered Exercises

2. $f(x) = \cos\frac{\pi x}{2}$

$f'(x) = -\frac{\pi}{2}\sin\frac{\pi x}{2}$

$f'(0) = 0$

$f'(2) = 0$

4. $f(x) = -3x\sqrt{x+1}$

$f'(x) = -3x\left[\frac{1}{2}(x+1)^{-1/2}\right] + \sqrt{x+1}(-3)$

$= -\frac{3}{2}(x+1)^{-1/2}[x + 2(x+1)]$

$= -\frac{3}{2}(x+1)^{-1/2}(3x+2)$

$f'\left(-\frac{2}{3}\right) = 0$

6. Using the limit definition of the derivative,

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{(4-|x|)-4}{x} = 1$$

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{(4-|x|)-4}{x-0} = -1$$

$f'(0)$ does not exist, since the one-sided derivatives are not equal.

8. Critical number: $x = 0$.

$x = 0$: neither

10. Critical numbers: $x = 2, 5$

$x = 2$: neither

$x = 5$: absolute maximum

12. $g(x) = x^2(x^2-4) = x^4 - 4x^2$

$g'(x) = 4x^3 - 8x = 4x(x^2-2)$

Critical numbers: $x = 0, x = \pm\sqrt{2}$

14. $f(x) = \frac{4x}{x^2+1}$

$f'(x) = \frac{(x^2+1)(4) - (4x)(2x)}{(x^2+1)^2} = \frac{4(1-x^2)}{(x^2+1)^2}$

Critical numbers: $x = \pm 1$

16. $f(\theta) = 2\sec\theta + \tan\theta, 0 < \theta < 2\pi$

$f'(\theta) = 2\sec\theta\tan\theta + \sec^2\theta$

$= \sec\theta(2\tan\theta + \sec\theta)$

$= \sec\theta\left[2\left(\frac{\sin\theta}{\cos\theta}\right) + \frac{1}{\cos\theta}\right]$

$= \sec^2\theta(2\sin\theta + 1)$

On $(0, 2\pi)$, critical numbers: $\theta = \frac{7\pi}{6}, \theta = \frac{11\pi}{6}$

18. $f(x) = \dfrac{2x + 5}{3}, [0, 5]$

$f'(x) = \dfrac{2}{3} \Longrightarrow$ No critical numbers

Left endpoint: $\left(0, \dfrac{5}{3}\right)$ Minimum

Right endpoint: (5, 5) Maximum

20. $f(x) = x^2 + 2x - 4, [-1, 1]$

$f'(x) = 2x + 2 = 2(x + 1)$

Left endpoint: $(-1, -5)$ Minimum

Right endpoint: $(1, -1)$ Maximum

22. $f(x) = x^3 - 12x, [0, 4]$

$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$

Left endpoint: (0, 0)

Critical number: $(2, -16)$ Minimum

Right endpoint: (4, 16) Maximum

Note: $x = -2$ is not in the interval.

24. $g(x) = \sqrt[3]{x}, [-1, 1]$

$g'(x) = \dfrac{1}{3x^{2/3}}$

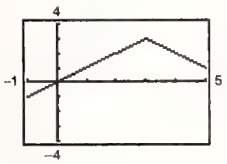
Left endpoint: $(-1, -1)$ Minimum

Critical number: (0, 0)

Right endpoint: (1, 1) Maximum

26. $y = 3 - |t - 3|, [-1, 5]$

From the graph, you see that $t = 3$ is a critical number.



Left endpoint: $(-1, -1)$ Minimum

Right endpoint: (5, 1)

Critical number: (3, 3) Maximum

28. $h(t) = \dfrac{t}{t - 2}, [3, 5]$

$h'(t) = \dfrac{-2}{(t - 2)^2}$

Left endpoint: (3, 3) Maximum

Right endpoint: $\left(5, \dfrac{5}{3}\right)$ Minimum

30. $g(x) = \sec x, \left[-\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$

$g'(x) = \sec x \tan x$

Left endpoint: $\left(-\dfrac{\pi}{6}, \dfrac{2}{\sqrt{3}}\right) \approx \left(-\dfrac{\pi}{6}, 1.1547\right)$

Right endpoint: $\left(\dfrac{\pi}{3}, 2\right)$ Maximum

Critical number: (0, 1) Minimum

32. $y = x^2 - 2 - \cos x, [-1, 3]$

$y' = 2x - \sin x$

Left endpoint: $(-1, -1.5403)$

Right endpoint: (3, 7.99) Maximum

Critical number: $(0, -3)$ Minimum

34. (a) Minimum: (4, 1)

Maximum: (1, 4)

(b) Maximum: (1, 4)

(c) Minimum: (4, 1)

(d) No extrema

36. (a) Minima: $(-2, 0)$ and (2, 0)

Maximum: (0, 2)

(b) Minimum: $(-2, 0)$

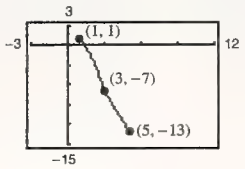
(c) Maximum: (0, 2)

(d) Maximum: $(1, \sqrt{3})$

38. $f(x) = \begin{cases} 2 - x^2, & 1 \le x < 3 \\ 2 - 3x, & 3 \le x \le 5 \end{cases}$

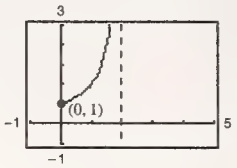
Left endpoint: (1, 1) Maximum

Right endpoint: (5, −13) Minimum

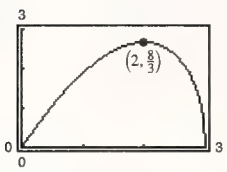


40. $f(x) = \dfrac{2}{2 - x}, [0, 2)$

Left endpoint: (0, 1) Minimum



42. (a)



Maximum: $\left(2, \dfrac{8}{3}\right)$

Minimum: (0, 0), (3, 0)

(b) $f(x) = \dfrac{4}{3}x\sqrt{3 - x}, [0, 3]$

$$f'(x) = \frac{4}{3}\left[x\left(\frac{1}{2}\right)(3 - x)^{-1/2}(-1) + (3 - x)^{1/2}(1)\right]$$

$$= \frac{4}{3}(3 - x)^{-1/2}\left(\frac{1}{2}\right)[-x + 2(3 - x)]$$

$$= \frac{2(6 - 3x)}{3\sqrt{3 - x}} = \frac{6(2 - x)}{3\sqrt{3 - x}} = \frac{2(2 - x)}{\sqrt{3 - x}}$$

Critical number: $x = 2$

$f(0) = 0$ Minimum

$f(3) = 0$ Minimum

$f(2) = \dfrac{8}{3}$

Maximum: $\left(2, \dfrac{8}{3}\right)$

44. $f(x) = \dfrac{1}{x^2 + 1}, \left[\dfrac{1}{2}, 3\right]$

$$f'(x) = \frac{-2x}{(x^2 + 1)^2}$$

$$f''(x) = \frac{-2(1 - 3x^2)}{(x^2 + 1)^3}$$

$$f'''(x) = \frac{24x - 24x^3}{(x^2 + 1)^4}$$

Setting $f''' = 0$, we have $x = 0, \pm 1$.

$|f''(1)| = \dfrac{1}{2}$ is the maximum value.

46. $f(x) = \dfrac{1}{x^2 + 1}, [-1, 1]$

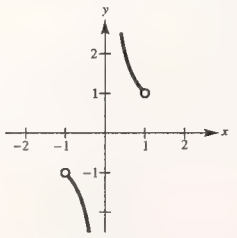
$$f'''(x) = \frac{24x - 24x^3}{(x^2 + 1)^4}$$ (See Exercise 44.)

$$f^{(4)}(x) = \frac{24(5x^4 - 10x^2 + 1)}{(x^2 + 1)^5}$$

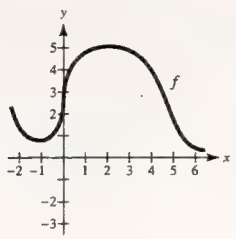
$$f^{(5)}(x) = \frac{-240x(3x^4 - 10x^2 + 3)}{(x^2 + 1)^6}$$

$|f^{(4)}(0)| = 24$ is the maximum value.

48. Let $f(x) = 1/x$. f is continuous on (0, 1) but does not have a maximum. f is also continuous on $(-1, 0)$ but does not have a minimum. This can occur if one of the endpoints is an infinite discontinuity.



50.



52. (a) No

(b) Yes

54. (a) No

(b) Yes

56. $x = \dfrac{v^2 \sin 2\theta}{32}, \dfrac{\pi}{4} \le \theta \le \dfrac{3\pi}{4}$

$\dfrac{d\theta}{dt}$ is constant.

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} \text{ (by the Chain Rule)}$$

$$= \frac{v^2 \cos 2\theta}{16}\frac{d\theta}{dt}$$

In the interval $[\pi/4, 3\pi/4]$, $\theta = \pi/4, 3\pi/4$ indicate minimums for dx/dt and $\theta = \pi/2$ indicates a maximum for dx/dt. This implies that the sprinkler waters longest when $\theta = \pi/4$ and $3\pi/4$. Thus, the lawn farthest from the spinkler gets the most water.

58. $C = 2x + \dfrac{300{,}000}{x}, 1 \le x \le 300$

$C(1) = 300{,}002$

$C(300) = 1600$

$$C' = 2 - \frac{300{,}000}{x^2} = 0$$

$2x^2 = 300{,}000$

$x^2 = 150{,}000$

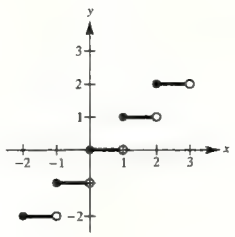
$x = 100\sqrt{15} \approx 387 > 300$ (outside of interval)

C is minimized when $x = 300$ units.

Yes, if $1 \le x \le 400$, then $x = 387$ would minimize C.

60. $f(x) = [\![x]\!]$

The derivative of f is undefined at every integer and is zero at any noninteger real number. All real numbers are critical numbers.



62. True. This is stated in the Extreme Value Theorem.

64. False. Let $f(x) = x^2$. $x = 0$ is a critical number of f.

$g(x) = f(x - k)$

$= (x - k)^2$

$x = k$ is a critical number of g.

Section 3.2 Rolle's Theorem and the Mean Value Theorem

2. Rolle's Theorem does not apply to $f(x) = \cot(x/2)$ over $[\pi, 3\pi]$ since f is not continuous at $x = 2\pi$.

4. $f(x) = x(x - 3)$

x-intercepts: $(0, 0), (3, 0)$

$f'(x) - 2x - 3 = 0$ at $x = \dfrac{3}{2}$.

6. $f(x) = -3x\sqrt{x + 1}$

x-intercepts: $(-1, 0), (0, 0)$

$$f'(x) = -3x\frac{1}{2}(x + 1)^{-1/2} - 3(x + 1)^{1/2} = -3(x + 1)^{-1/2}\left(\frac{x}{2} + (x + 1)\right)$$

$$f'(x) = -3(x + 1)^{-1/2}\left(\frac{3}{2}x + 1\right) = 0 \text{ at } x = -\frac{2}{3}.$$

8. $f(x) = x^2 - 5x + 4,\ [1, 4]$

$f(1) = f(4) = 0$

f is continuous on $[1, 4]$. f is differentiable on $(1, 4)$.

Rolle's Theorem applies.

$$f'(x) = 2x - 5$$

$$2x - 5 = 0 \Rightarrow x = \frac{5}{2}$$

c value: $\frac{5}{2}$

10. $f(x) = (x - 3)(x + 1)^2, [-1, 3]$

$f(-1) = f(3) = 0$

f is continuous on $[-1, 3]$. f is differentiable on $(-1, 3)$. Rolle's Theorem applies.

$$f'(x) = (x - 3)(2)(x + 1) + (x + 1)^2$$
$$= (x + 1)[2x - 6 + x + 1]$$
$$= (x + 1)(3x - 5)$$

c value: $\frac{5}{3}$

12. $f(x) = 3 - |x - 3|, [0, 6]$

$f(0) = f(6) = 0$

f is continuous on $[0, 6]$. f is not differentiable on $(0, 6)$ since $f'(3)$ does not exist. Rolle's Theorem does not apply.

14. $f(x) = \dfrac{x^2 - 1}{x},\ [-1, 1]$

$f(-1) = f(1) = 0$

f is not continuous on $[-1, 1]$ since $f(0)$ does not exist. Rolle's Theorem does not apply.

16. $f(x) = \cos x,\ [0, 2\pi]$

$f(0) = f(2\pi) = 1$

f is continuous on $[0, 2\pi]$. f is differentiable on $(0, 2\pi)$. Rolle's Theorem applies.

$$f'(x) = -\sin x$$

c value: π

18. $f(x) = \cos 2x,\ \left[-\frac{\pi}{12}, \frac{\pi}{6}\right]$

$$f\left(-\frac{\pi}{12}\right) = \frac{\sqrt{3}}{2}$$

$$f\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$f\left(-\frac{\pi}{12}\right) \neq f\left(\frac{\pi}{0}\right)$$

Rolle's Theorem does not apply.

20. $f(x) = \sec x,\ \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

$$f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) = \sqrt{2}$$

f is continuous on $[-\pi/4, \pi/4]$. f is differentiable on $(-\pi/4, \pi/4)$. Rolle's Theorem applies.

$$f'(x) = \sec x \tan x$$

$$\sec x \tan x = 0$$

$$x = 0$$

c value: 0

22. $f(x) = x - x^{1/3},\ [0, 1]$

$f(0) = f(1) = 0$

f is continuous on $[0, 1]$. f is differentiable on $(0, 1)$. (**Note:** f is not differentiable at $x = 0$.) Rolle's Theorem applies.

$$f'(x) = 1 - \frac{1}{3\sqrt[3]{x^2}} = 0$$

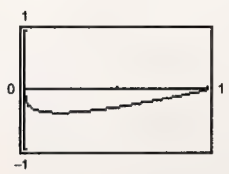
$$1 = \frac{1}{3\sqrt[3]{x^2}}$$

$$\sqrt[3]{x^2} = \frac{1}{3}$$

$$x^2 = \frac{1}{27}$$

$$x = \sqrt{\frac{1}{27}} = \frac{\sqrt{3}}{9}$$

c value: $\frac{\sqrt{3}}{9} \approx 0.1925$



24. $f(x) = \frac{x}{2} - \sin\frac{\pi x}{6}$, $[-1, 0]$

$f(-1) = f(0) = 0$

f is continuous on $[-1, 0]$. f is differentiable on $(-1, 0)$. Rolle's Theorem applies.

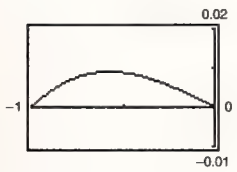
$$f'(x) = \frac{1}{2} - \frac{\pi}{6}\cos\frac{\pi x}{6} = 0$$

$$\cos\frac{\pi x}{6} = \frac{3}{\pi}$$

$$x = -\frac{6}{\pi}\arccos\frac{3}{\pi} \quad \text{[Value needed in } (-1, 0).\text{]}$$

$$\approx -0.5756 \text{ radian}$$

c value: -0.5756



26. $C(x) = 10\left(\frac{1}{x} + \frac{x}{x+3}\right)$

(a) $C(3) = C(6) = \frac{25}{3}$

(b) $$C'(x) = 10\left(-\frac{1}{x^2} + \frac{3}{(x+3)^2}\right) = 0$$

$$\frac{3}{x^2 + 6x + 9} = \frac{1}{x^2}$$

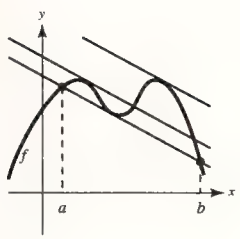
$$2x^2 - 6x - 9 = 0$$

$$x = \frac{6 \pm \sqrt{108}}{4}$$

$$= \frac{6 \pm 6\sqrt{3}}{4} = \frac{3 \pm 3\sqrt{3}}{2}$$

In the interval $(3, 6)$: $c = \frac{3 + 3\sqrt{3}}{2} \approx 4.098$.

28.



30. $f(x) = |x - 3|$, $[0, 6]$

f is not differentiable at $x = 3$.

32. $f(x) = x(x^2 - x - 2)$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$.

$$\frac{f(1) - f(-1)}{1 - (-1)} = -1$$

$$f'(x) = 3x^2 - 2x - 2 = -1$$

$$(3x + 1)(x - 1) = 0$$

$$c = -\frac{1}{3}$$

34. $f(x) = (x + 1)/x$ is continuous on $[1/2, 2]$ and differentiable on $(1/2, 2)$.

$$\frac{f(2) - f(1/2)}{2 - (1/2)} = \frac{(3/2) - 3}{3/2} = -1$$

$$f'(x) = \frac{-1}{x^2} = -1$$

$$x^2 = 1$$

$$c = 1$$

36. $f(x) = x^3$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

$$\frac{f(1) - f(0)}{1 - 0} = \frac{1 - 0}{1} = 1$$

$$f'(x) = 3x^2 = 1$$

$$x = \pm\frac{\sqrt{3}}{3}$$

In the interval $(0, 1)$: $c = \frac{\sqrt{3}}{3}$.

38. $f(x) = 2\sin x + \sin 2x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

$$f'(x) = 2\cos x + 2\cos 2x = 0$$

$$2[\cos x + 2\cos^2 x - 1] = 0$$

$$2(2\cos x - 1)(\cos x + 1) = 0$$

$$\cos x = \frac{1}{2}$$

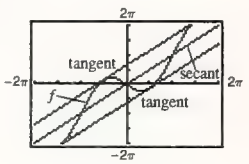
$$\cos x = -1$$

$$x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

In the interval $(0, \pi)$: $c = \frac{\pi}{3}$.

40. $f(x) = x - 2\sin x$ on $[-\pi, \pi]$

(a)



(b) Secant line:

$$\text{slope} = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)} = \frac{\pi - (-\pi)}{2\pi} = 1$$

$$y - \pi = 1(x - \pi)$$

$$y = x$$

(c) $f'(x) = 1 - 2\cos x = 1$

$$\cos x = 0$$

$$c = \pm\frac{\pi}{2}, \qquad f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 2$$

$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} + 2$$

Tangent lines: $y - \left(\frac{\pi}{2} - 2\right) = 1\left(x - \frac{\pi}{2}\right)$

$$y = x - 2$$

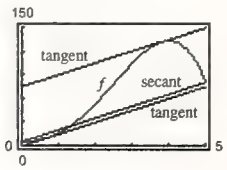
$$y - \left(-\frac{\pi}{2} + 2\right) = 1\left(x + \frac{\pi}{2}\right)$$

$$y = x + 2$$

42. $f(x) = -x^4 + 4x^3 + 8x^2 + 5$, $(0, 5)$, $(5, 80)$

$$m = \frac{80 - 5}{5 - 0} = 15$$

(a)



(b) Secant line: $y - 5 = 15(x - 0)$

$$0 = 15x - y + 5$$

$$f'(x) = -4x^3 + 12x^2 + 16x$$

$$\frac{f(5) - f(1)}{5 - 1} = 15$$

$$-4c^3 + 12c^2 + 16c = 15$$

$$0 = 4c^3 - 12c^2 - 16c + 15$$

$$c \approx 0.67 \text{ or } c \approx 3.79$$

(c) First tangent line: $y - f(c) = m(x - c)$

$$y - 9.59 = 15(x - 0.67)$$

$$0 = 15x - y - 0.46$$

Second tangent line: $y - f(c) = m(x - c)$

$$y - 131.35 = 15(x - 3.79)$$

$$0 = 15x - y + 74.5$$

44. $S(t) = 200\left(5 - \frac{9}{2+t}\right)$

(a) $\frac{S(12) - S(0)}{12 - 0} = \frac{200[5 - (9/14)] - 200[5 - (9/2)]}{12} = \frac{450}{7}$

(b) $S'(t) = 200\left(\frac{9}{(2+t)^2}\right) = \frac{450}{7}$

$$\frac{1}{(2+t)^2} = \frac{1}{28}$$

$$2 + t = 2\sqrt{7}$$

$$t = 2\sqrt{7} - 2 \approx 3.2915 \text{ months}$$

$S'(t)$ is equal to the average value in April.

46. $f(a) = f(b)$ and $f'(c) = 0$ where c is in the interval (a, b).

(a) $g(x) = f(x) + k$

$g(a) = g(b) = f(a) + k$

$g'(x) = f'(x) \Rightarrow g'(c) = 0$

Interval: $[a, b]$

Critical number of g: c

(b) $g(x) = f(x - k)$

$g(a + k) = g(b + k) = f(a)$

$g'(x) = f'(x - k)$

$g'(c + k) = f'(c) = 0$

Interval: $[a + k, b + k]$

Critical number of g: $c + k$

(c) $g(x) = f(kx)$

$g\left(\frac{a}{k}\right) = g\left(\frac{b}{k}\right) = f(a)$

$g'(x) = kf'(kx)$

$g'\left(\frac{c}{k}\right) = kf'(c) = 0$

Interval: $\left[\frac{a}{k}, \frac{b}{k}\right]$

Critical number of g: $\frac{c}{k}$

48. Let $T(t)$ be the temperature of the object. Then $T(0) = 1500°$ and $T(5) = 390°$. The average temperature over the interval $[0, 5]$ is

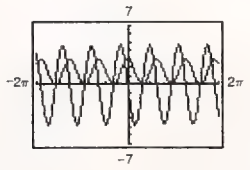
$$\frac{390 - 1500}{5 - 0} = -222° \text{ F/hr.}$$

By the Mean Value Theorem, there exists a time to, $0 < t_0 < 5$, such that $T'(t_0) = -222$.

50. $f(x) = 3\cos^2\left(\frac{\pi x}{2}\right)$, $f'(x) = 6\cos\left(\frac{\pi x}{2}\right)\left(-\sin\left(\frac{\pi x}{2}\right)\right)\left(\frac{\pi}{2}\right)$

$$= -3\pi\cos\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi x}{2}\right)$$

(a)



(b) f and f' are both continuous on the entire real line.

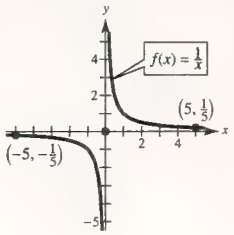
(c) Since $f(-1) = f(1) = 0$, Rolle's Theorem applies on $[-1, 1]$. Since $f(1) = 0$ and $f(2) = 3$, Rolle's Theorem does not apply on $[1, 2]$.

(d) $\lim_{x \to 3^-} f'(x) = 0$

$\lim_{x \to 3^+} f'(x) = 0$

52. f is not continuous on $[-5, 5]$.

Example: $f(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$



54. False. f must also be continuous *and* differentiable on each interval. Let

$$f(x) = \frac{x^3 - 4x}{x^2 - 1}.$$

56. True

58. Suppose $f(x)$ is not constant on (a, b). Then there exists x_1 and x_2 in (a, b) such that $f(x_1) \neq f(x_2)$. Then by the Mean Value Theorem, there exists c in (a, b) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0.$$

This contradicts the fact that $f'(x) = 0$ for all x in (a, b).

60. Suppose $f(x)$ has two fixed points c_1 and c_2. Then, by the Mean Value Theorem, there exists c such that

$$f'(c) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = \frac{c_2 - c_1}{c_2 - c_1} = 1.$$

This contradicts the fact that $f'(x) < 1$ for all x.

62. Let $f(x) = \cos x$. f is continuous and differentiable for all real numbers. By the Mean Value Theorem, for any interval $[a, b]$, there exists c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\frac{\cos b - \cos a}{b - a} = -\sin c$$

$$\cos b - \cos a = (-\sin c)(b - a)$$

$$|\cos b - \cos a| = |-\sin c||b - a|$$

$$|\cos b - \cos a| \leq |b - a| \text{ since } |-\sin c| \leq 1.$$

Section 3.3 Increasing and Decreasing Functions and the First Derivative Test

2. $y = -(x + 1)^2$

Increasing on: $(-\infty, -1)$

Decreasing on: $(-1, \infty)$

4. $f(x) = x^4 - 2x^2$

Increasing on: $(-1, 0), (1, \infty)$

Decreasing on: $(-\infty, -1), (0, 1)$

6. $y = \dfrac{x^2}{x + 1}$

$y' = \dfrac{x(x + 2)}{(x + 1)^2}$

Critical numbers: $x = 0, -2$ Discontinuity: $x = -1$

Test intervals:	$-\infty < x < -2$	$-2 < x < -1$	$-1 < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$y' > 0$	$y' < 0$	$y' < 0$	$y' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on $(-\infty, -2), (0, \infty)$

Decreasing on $(-2, -1), (-1, 0)$

8. $h(x) = 27x - x^3$

$h'(x) = 27 - 3x^2 = 3(3 - x)(3 + x)$

$h'(x) = 0$

Critical numbers: $x = \pm 3$

Test intervals:	$-\infty < x < -3$	$-3 < x < 3$	$3 < x < \infty$
Sign of $h'(x)$:	$h' < 0$	$h' > 0$	$h' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on $(-3, 3)$

Decreasing on $(-\infty, -3), (3, \infty)$

10. $y = x + \dfrac{4}{x}$

$y' = \dfrac{(x - 2)(x + 2)}{x^2}$

Critical numbers: $x = \pm 2$ Discontinuity: 0

Test intervals:	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of y':	$y' > 0$	$y' < 0$	$y' < 0$	$y' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing: $(-\infty, -2), (2, \infty)$

Decreasing: $(-2, 0), (0, 2)$

12. $f(x) = x^2 + 8x + 10$

$f'(x) = 2x + 8 = 0$

Critical number: $x = -4$

Test intervals:	$-\infty < x < -4$	$-4 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(-4, \infty)$

Decreasing on: $(-\infty, -4)$

Relative minimum: $(-4, -6)$

14. $f(x) = -(x^2 + 8x + 12)$

$f'(x) = -2x - 8 = 0$

Critical number: $x = -4$

Test intervals:	$-\infty < x < -4$	$-4 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, -4)$

Decreasing on: $(-4, \infty)$

Relative maximum: $(-4, 4)$

16. $f(x) = x^3 - 6x^2 + 15$

$f'(x) = 3x^2 - 12x = 3x(x - 4)$

Critical numbers: $x = 0, 4$

Test intervals:	$-\infty < x < 0$	$0 < x < 4$	$4 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on $(-\infty, 0)$, $(4, \infty)$

Decreasing on $(0, 4)$

Relative maximum: $(0, 15)$

Relative minimum: $(4, -17)$

18. $f(x) = (x + 2)^2(x - 1)$

$f'(x) = 3x(x + 2)$

Critical numbers: $x = -2, 0$

Test intervals:	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -2)$, $(0, \infty)$

Decreasing on: $(-2, 0)$

Relative maximum: $(-2, 0)$

Relative minimum: $(0, -4)$

20. $f(x) = x^4 - 32x + 4$

$f'(x) = 4x^3 - 32 = 4(x^3 - 8)$

Critical number: $x = 2$

Test intervals:	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(2, \infty)$

Decreasing on: $(-\infty, 2)$

Relative minimum: $(2, -44)$

22. $f(x) = x^{2/3} - 4$

$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$

Critical number: $x = 0$

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(0, \infty)$

Decreasing on: $(-\infty, 0)$

Relative minimum: $(0, -4)$

24. $f(x) = (x - 1)^{1/3}$

$f'(x) = \frac{1}{3(x - 1)^{2/3}}$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, \infty)$

No relative extrema

26. $f(x) = |x + 3| - 1$

$f'(x) = \frac{x + 3}{|x + 3|} = \begin{cases} 1, & x > -3 \\ -1, & x < -3 \end{cases}$

Critical number: $x = -3$

Test intervals:	$-\infty < x < -3$	$-3 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(-3, \infty)$

Decreasing on: $(-\infty, -3)$

Relative minimum: $(-3, -1)$

28. $f(x) = \frac{x}{x + 1}$

$f'(x) = \frac{(x + 1)(1) - (x)(1)}{(x + 1)^2} = \frac{1}{(x + 1)^2}$

Discontinuity: $x = -1$

Test intervals:	$-\infty < x < -1$	$-1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, -1), (-1, \infty)$

No relative extrema

30. $f(x) = \dfrac{x+3}{x^2} = \dfrac{1}{x} + \dfrac{3}{x^2}$

$f'(x) = -\dfrac{1}{x^2} - \dfrac{6}{x^3} = \dfrac{-(x+6)}{x^3}$

Critical number: $x = -6$

Discontinuity: $x = 0$

Test intervals:	$-\infty < x < -6$	$-6 < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(-6, 0)$

Decreasing on: $(-\infty, -6), (0, \infty)$

Relative minimum: $\left(-6, -\dfrac{1}{12}\right)$

32. $f(x) = \dfrac{x^2 - 3x - 4}{x - 2}$

$f'(x) = \dfrac{(x-2)(2x-3) - (x^2 - 3x - 4)(1)}{(x-2)^2} = \dfrac{x^2 - 4x + 10}{(x-2)^2}$

Discontinuity: $x = 2$

Test intervals:	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, 2), (2, \infty)$

No relative extrema

34. $f(x) = \sin x \cos x = \dfrac{1}{2} \sin 2x, \ 0 < x < 2\pi$

$f'(x) = \cos 2x = 0$

Critical numbers: $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

Test intervals:	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < \frac{7\pi}{4}$	$\frac{7\pi}{4} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \dfrac{\pi}{4}\right), \left(\dfrac{3\pi}{4}, \dfrac{5\pi}{4}\right), \left(\dfrac{7\pi}{4}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right), \left(\dfrac{5\pi}{4}, \dfrac{7\pi}{4}\right)$

Relative maxima: $\left(\dfrac{\pi}{4}, \dfrac{1}{2}\right), \left(\dfrac{5\pi}{4}, \dfrac{1}{2}\right)$

Relative minima: $\left(\dfrac{3\pi}{4}, -\dfrac{1}{2}\right), \left(\dfrac{7\pi}{4}, -\dfrac{1}{2}\right)$

36. $f(x) = \dfrac{\sin x}{1 + \cos^2 x}$, $0 < x < 2\pi$

$f'(x) = \dfrac{\cos x(2 + \sin^2 x)}{(1 + \cos^2 x)^2} = 0$

Critical numbers: $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

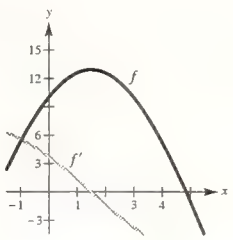
Relative maximum: $\left(\dfrac{\pi}{2}, 1\right)$

Relative minimum: $\left(\dfrac{3\pi}{2}, -1\right)$

38. $f(x) = 10\left(5 - \sqrt{x^2 - 3x + 16}\right)$, $[0, 5]$

(a) $f'(x) = -\dfrac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}}$

(b)



(c) $-\dfrac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}} = 0$

Critical number: $x = \dfrac{3}{2}$

(d) Intervals:

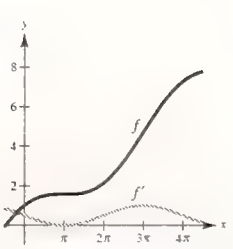
$\left(0, \frac{3}{2}\right)$	$\left(\frac{3}{2}, 5\right)$
$f'(x) > 0$	$f'(x) < 0$
Increasing	Decreasing

f is increasing when f' is positive and decreasing when f' is negative.

40. $f(x) = \dfrac{x}{2} + \cos\dfrac{x}{2}$, $[0, 4\pi]$

(a) $f'(x) = \dfrac{1}{2} - \dfrac{1}{2}\sin\dfrac{x}{2}$

(b)



(c) $\dfrac{1}{2} - \dfrac{1}{2}\sin\dfrac{x}{2} = 0$

$\sin\dfrac{x}{2} = 1$

$\dfrac{x}{2} = \dfrac{\pi}{2}$

Critical number: $x = \pi$

(d) Intervals:

$(0, \pi)$	$(\pi, 4\pi)$
$f'(x) > 0$	$f'(x) > 0$
Increasing	Increasing

f is increasing when f' is positive.

42. $f(t) = \cos^2 t - \sin^2 t = -2\sin^2 t = g(t), -2 < t < 2$

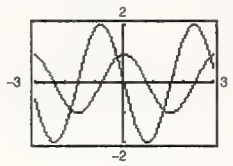
$f'(t) = -4\sin t \cos t = -2\sin 2t$

f symmetric with respect to y-axis

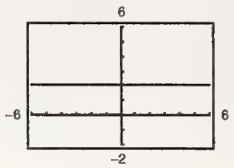
zeros of f: $\pm\frac{\pi}{4}$

Relative maximum: $(0, 1)$

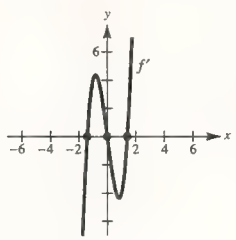
Relative minimum: $\left(-\frac{\pi}{2}, -1\right), \left(\frac{\pi}{2}, -1\right)$



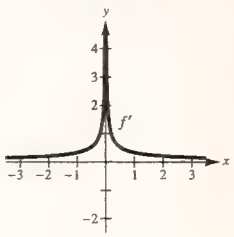
44. $f(x)$ is a line of slope $\approx 2 \Rightarrow f'(x) = 2$.



46. f is a 4th degree polynomial $\Rightarrow f'$ is a cubic polynomial.



48. f has positive slope



In Exercises 50–54, $f'(x) > 0$ on $(-\infty, -4)$, $f'(x) < 0$ on $(-4, 6)$ and $f'(x) > 0$ on $(6, \infty)$.

50. $g(x) = 3f(x) - 3$

$g'(x) = 3f'(x)$

$g'(-5) = 3f'(-5) > 0$

52. $g(x) = -f(x)$

$g'(x) = -f'(x)$

$g'(0) = -f'(0) > 0$

54. $g(x) = f(x - 10)$

$g'(x) = f'(x - 10)$

$g'(8) = f'(-2) < 0$

56. Critical number: $x = 5$

$f'(4) = -2.5 \Rightarrow f$ is decreasing at $x = 4$.

$f'(6) = 3 \Rightarrow f$ is increasing at $x = 6$.

$(5, f(5))$ is a relative minimum.

58. $s(t) = 4.9(\sin\theta)t^2$

(a) $v(t) = 9.8(\sin\theta)t$ speed $= |9.8(\sin\theta)t|$

(b) If $\theta = \pi/2$, the speed is maximum,

$v(t) = 9.8t$.

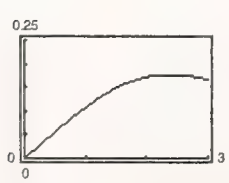
60. $C = \dfrac{3t}{27 + t^3}, t \geq 0$

(a)

t	0	0.5	1	1.5	2	2.5	3
$C(t)$	0	0.055	0.107	0.148	0.171	0.176	0.167

The concentration seems greater near $t = 2.5$ hours.

(b)



The concentration is greatest when $t \approx 2.38$ hours.

(c) $C' = \dfrac{(27 + t^3)(3) - (3t)(3t^2)}{(27 + t^3)^2}$

$= \dfrac{3(27 - 2t^3)}{(27 + t^3)^2}$

$C' = 0$ when $t = 3/\sqrt[3]{2} \approx 2.38$ hours.

By the First Derivative Test, this is a maximum.

62. $P = 2.44x - \frac{x^2}{20{,}000} - 5000, 0 \le x \le 35{,}000$

$P' = 2.44 - \frac{x}{10{,}000} = 0$

$x = 24{,}400$

Increasing when $0 < x < 24{,}400$ hamburgers.

Decreasing when $24{,}400 < x < 35{,}000$ hamburgers.

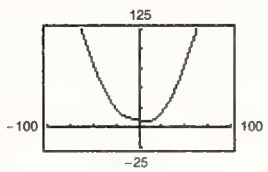
Test intervals:	$0 < x < 24{,}400$	$24{,}400 < x < 35{,}000$
Sign of P':	$P' > 0$	$P' < 0$

64. $R = \sqrt{0.001T^4 - 4T + 100}$

(a) $R' = \frac{0.004T^3 - 4}{2\sqrt{0.001T^4 - 4T + 100}} = 0$

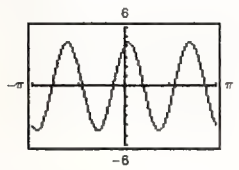
$T = 10°, R \approx 8.3666\Omega$

(b)



The minimum resistance is approximately $R \approx 8.37\Omega$ at $T = 10°$.

66. $f(x) = 2\sin(3x) + 4\cos(3x)$



The maximum value is approximately 4.472. You could use calculus by finding $f'(x)$ and then observing that the maximum value of f occurs at a point where $f'(x) = 0$. For instance, $f'(0.154) \approx 0$, and $f(0.154) = 4.472$.

68. (a) Use a cubic polynomial $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.

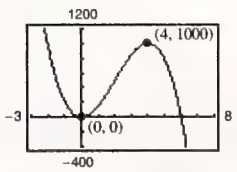
(b) $f'(x) = 3a_3x^2 + 2a_2x + a_1$

(0, 0):	$0 = a_0$	$(f(0) = 0)$
	$0 = a_1$	$(f'(0) = 0)$
(4, 1000):	$1000 = 64a_3 + 16a_2$	$(f(4) = 1000)$
	$0 = 48a_3 + 8a_2$	$(f'(4) = 0)$

(c) The solution is $a_0 = a_1 = 0, a_2 = \frac{375}{2}, a_3 = \frac{-125}{4}$

$f(x) = \frac{-125}{4}x^3 + \frac{375}{2}x^2.$

(d)



70. (a) Use a fourth degree polynomial $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

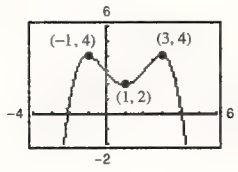
(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

$(1, 2)$:	$2 = a_4 + a_3 + a_2 + a_1 + a_0$	$(f(1) = 2)$
	$0 = 4a_4 + 3a_3 + 2a_2 + a_1$	$(f'(1) = 0)$
$(-1, 4)$:	$4 = a_4 - a_3 + a_2 - a_1 + a_0$	$(f(-1) = 4)$
	$0 = -4a_4 + 3a_3 - 2a_2 + a_1$	$(f'(-1) = 0)$
$(3, 4)$:	$4 = 81a_4 + 27a_3 + 9a_2 + 3a_1 + a_0$	$(f(3) = 4)$
	$0 = 108a_4 + 27a_3 + 6a_2 + a_1$	$(f'(3) = 0)$

(c) The solution is $a_0 = \frac{23}{8}, a_1 = -\frac{3}{2}, a_2 = \frac{1}{4}, a_3 = \frac{1}{2}, a_4 = -\frac{1}{8}$

$f(x) = -\frac{1}{8}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2 - \frac{3}{2}x + \frac{23}{8}$.

(d)



72. False

Let $h(x) = f(x)g(x)$ where $f(x) = g(x) = x$. Then $h(x) = x^2$ is decreasing on $(-\infty, 0)$.

74. True

If $f(x)$ is an nth-degree polynomial, then the degree of $f'(x)$ is $n - 1$.

76. False.

The function might not be continuous.

78. Suppose $f'(x)$ changes from positive to negative at c. Then there exists a and b in I such that $f'(x) > 0$ for all x in (a, c) and $f'(x) < 0$ for all x in (c, b). By Theorem 3.5, f is increasing on (a, c) and decreasing on (c, b). Therefore, $f(c)$ is a maximum of f on (a, b) and thus, a relative maximum of f.

Section 3.4 Concavity and the Second Derivative Test

2. $y = -x^3 + 3x^2 - 2, y'' = -6x + 6$

Concave upward: $(-\infty, 1)$

Concave downward: $(1, \infty)$

4. $f(x) = \frac{x^2 - 1}{2x + 1}, y'' = \frac{-6}{(2x + 1)^3}$

Concave upward: $\left(-\infty, -\frac{1}{2}\right)$

Concave downward: $\left(-\frac{1}{2}, \infty\right)$

6. $y = \frac{1}{270}(-3x^5 + 40x^3 + 135x), y'' = \frac{-2}{9}x(x - 2)(x + 2)$

Concave upward: $(-\infty, -2), (0, 2)$

Concave downward: $(-2, 0), (2, \infty)$

8. $h(x) = x^5 - 5x + 2$

$h'(x) = 5x^4 - 5$

$h''(x) = 20x^3$

Concave upward: $(0, \infty)$

Concave downward: $(-\infty, 0)$

10. $y = x + 2\csc x, \quad (-\pi, \pi)$

$y' = 1 - 2\csc x \cot x$

$y'' = -2\csc x(-\csc^2 x) - 2\cot x(-\csc x \cot x)$

$= 2(\csc^3 x + \csc x \cot^2 x)$

Concave upward: $(0, \pi)$

Concave downward: $(-\pi, 0)$

12. $f(x) = 2x^3 - 3x^2 - 12x + 5$

$f'(x) = 6x^2 - 6x - 12$

$f''(x) = 12x - 6$

$f''(x) = 12x - 6 = 0$ when $x = \frac{1}{2}$.

Test interval	$-\infty < x < \frac{1}{2}$	$\frac{1}{2} < x < \infty$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave downward	Concave upward

Point of inflection: $\left(\frac{1}{2}, -\frac{13}{2}\right)$

14. $f(x) = 2x^4 - 8x + 3$

$f'(x) = 8x^3 - 8$

$f''(x) = 24x^2 = 0$ when $x = 0$.

However, $(0, 3)$ is not a point of inflection since $f''(x) \geq 0$ for all x.

Concave upward on $(-\infty, \infty)$

16. $f(x) = x^3(x - 4)$

$f'(x) = x^3 + 3x^2(x - 4)$

$= x^2[x + 3(x - 4)] = 4x^2(x - 3)$

$f''(x) = 4x^2 + 8x(x - 3) = 4x[x + 2(x - 3)] = 12x(x - 2) = 0$

$f''(x) = 12x(x - 2) = 0$ when $x = 0, 2$.

Test interval	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of $f''(x)$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Points of inflection: $(0, 0), (2, -16)$

18. $f(x) = x\sqrt{x + 1}$, Domain: $[-1, \infty)$

$$f'(x) = (x)\frac{1}{2}(x + 1)^{-1/2} + \sqrt{x + 1} = \frac{3x + 2}{2\sqrt{x + 1}}$$

$$f''(x) = \frac{6\sqrt{x + 1} - (3x + 2)(x + 1)^{-1/2}}{4(x + 1)} = \frac{3x + 4}{4(x + 1)^{3/2}}$$

$f''(x) > 0$ on the entire domain of f (except for $x = -1$, for which $f''(x)$ is undefined).

There are no points of inflection.

Concave upward on $(-1, \infty)$

20. $f(x) = \dfrac{x + 1}{\sqrt{x}}$ Domain: $x > 0$

$$f'(x) = \frac{x - 1}{2x^{3/2}}$$

$$f''(x) = \frac{3 - x}{4x^{5/2}}$$

Point of inflection: $\left(3, \dfrac{4}{\sqrt{3}}\right) = \left(3, \dfrac{4\sqrt{3}}{3}\right)$

Test intervals	$0 < x < 3$	$3 < x < \infty$
Sign of $f''(x)$	$f'' > 0$	$f'' < 0$
Conclusion	Concave upward	Concave downward

22. $f(x) = 2\csc\frac{3x}{2}, 0 < x < 2\pi$

$f'(x) = -3\csc\frac{3x}{2}\cot\frac{3x}{2}$

$f''(x) = \frac{9}{2}\left(\csc^3\frac{3x}{2} + \csc\frac{3x}{2}\cot^2\frac{3x}{2}\right) \neq 0$ for any x in the domain of f.

Concave upward: $\left(0, \frac{2\pi}{3}\right), \left(\frac{4\pi}{3}, 2\pi\right)$

Concave downward: $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$

No points of inflection

24. $f(x) = \sin x + \cos x, 0 \le x \le 2\pi$

$f'(x) = \cos x - \sin x$

$f''(x) = -\sin x - \cos x$

$f''(x) = 0$ when $x = \frac{3\pi}{4}, \frac{7\pi}{4}$.

Test interval:	$0 < x < \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \frac{7\pi}{4}$	$\frac{7\pi}{4} < x < 2\pi$
Sign of $f''(x)$:	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

Points of inflection: $\left(\frac{3\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$

26. $f(x) = x + 2\cos x, [0, 2\pi]$

$f'(x) = 1 - 2\sin x$

$f''(x) = -2\cos x$

$f''(x) = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < \frac{3\pi}{2}$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$	$f'' < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

Points of inflection: $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

28. $f(x) = x^2 + 3x - 8$

$f'(x) = 2x + 3$

$f''(x) = 2$

Critical number: $x = -\frac{3}{2}$

$f''\left(-\frac{3}{2}\right) > 0$

Therefore, $\left(-\frac{3}{2}, -\frac{41}{4}\right)$ is a relative minimum.

30. $f(x) = -(x - 5)^2$

$f'(x) = -2(x - 5)$

$f''(x) = -2$

Critical number: $x = 5$

$f''(5) < 0$

Therefore, (5, 0) is a relative maximum.

32. $f(x) = x^3 - 9x^2 + 27x$

$f'(x) = 3x^2 - 18x + 27 = 3(x - 3)^2$

$f''(x) = 6(x - 3)$

Critical number: $x = 3$

However, $f''(3) = 0$, so we must use the First Derivative Test. $f'(x) \geq 0$ for all x and, therefore, there are no relative extrema.

34. $g(x) = -\frac{1}{8}(x + 2)^2(x - 4)^2$

$g'(x) = \frac{-(x - 4)(x - 1)(x + 2)}{2}$

$g''(x) = 3 + 3x - \frac{3}{2}x^2$

Critical numbers: $x = -2, 1, 4$

$g''(-2) = -9 < 0$

$(-2, 0)$ is a relative maximum.

$g''(1) = 9/2 > 0$

$(1, -10.125)$ is a relative minimum.

$g''(4) = -9 < 0$

$(4, 0)$ is a relative maximum.

36. $f(x) = \sqrt{x^2 + 1}$

$f'(x) = \frac{x}{\sqrt{x^2 + 1}}$

Critical number: $x = 0$

$f''(x) = \frac{1}{(x^2 + 1)^{3/2}}$

$f''(0) = 1 > 0$

Therefore, $(0, 1)$ is a relative minimum.

38. $f(x) = \frac{x}{x - 1}$

$f'(x) = \frac{-1}{(x - 1)^2}$

There are no critical numbers and $x = 1$ is not in the domain. There are no relative extrema.

40. $f(x) = 2 \sin x + \cos 2x, 0 \leq x \leq 2\pi$

$f'(x) = 2 \cos x - 2 \sin 2x = 2 \cos x - 4 \sin x \cos x = 2 \cos x(1 - 2 \sin x) = 0$ when $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

$f''(x) = -2 \sin x - 4 \cos 2x$

$f''\left(\frac{\pi}{6}\right) < 0$

$f''\left(\frac{\pi}{2}\right) > 0$

$f''\left(\frac{5\pi}{6}\right) < 0$

$f''\left(\frac{3\pi}{2}\right) > 0$

Relative maxima: $\left(\frac{\pi}{6}, \frac{3}{2}\right), \left(\frac{5\pi}{6}, \frac{3}{2}\right)$

Relative minima: $\left(\frac{\pi}{2}, 1\right), \left(\frac{3\pi}{2}, -3\right)$

42. $f(x) = x^2\sqrt{6 - x^2}, [-\sqrt{6}, \sqrt{6}]$

(a) $f'(x) = \dfrac{3x(4 - x^2)}{\sqrt{6 - x^2}}$

$f'(x) = 0$ when $x = 0, x = \pm 2$.

$f''(x) = \dfrac{6(x^4 - 9x^2 + 12)}{(6 - x^2)^{3/2}}$

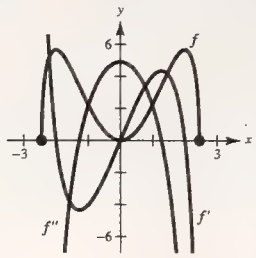
$f''(x) = 0$ when $x = \pm\sqrt{\dfrac{9 - \sqrt{33}}{2}}$.

(b) $f''(0) > 0 \Rightarrow (0, 0)$ is a relative minimum.

$f''(\pm 2) < 0 \Rightarrow \left(\pm 2, 4\sqrt{2}\right)$ are relative maxima.

Points of inflection: $(\pm 1.2758, 3.4035)$

(c)



The graph of f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

44. $f(x) = \sqrt{2x}\sin x, [0, 2\pi]$

(a) $f'(x) = \sqrt{2x}\cos x + \dfrac{\sin x}{\sqrt{2x}}$

Critical numbers: $x \approx 1.84, 4.82$

$f''(x) = -\sqrt{2x}\sin x + \dfrac{\cos x}{\sqrt{2x}} + \dfrac{\cos x}{\sqrt{2x}} - \dfrac{\sin x}{2x\sqrt{2x}}$

$= \dfrac{2\cos x}{\sqrt{2x}} - \dfrac{(4x^2 + 1)\sin x}{2x\sqrt{2x}}$

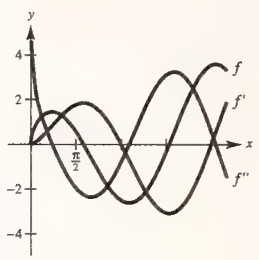
$= \dfrac{4x\cos x - (4x^2 + 1)\sin x}{2x\sqrt{2x}}$

(b) Relative maximum: $(1.84, 1.85)$

Relative minimum: $(4.82, -3.09)$

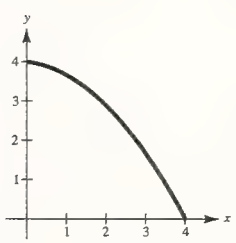
Points of inflection: $(0.75, 0.83), (3.42, -0.72)$

(c)



f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

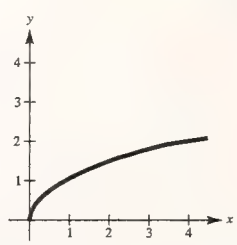
46. (a)



$f' < 0$ means f decreasing

f' decreasing means concave downward

(b)



$f' > 0$ means f increasing

f' decreasing means concave downward

48. (a) The rate of change of sales is increasing.
$S'' > 0$

(b) The rate of change of sales is decreasing.
$S' > 0, S'' < 0$

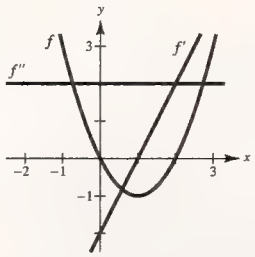
(c) The rate of change of sales is constant.
$S' = C, S'' = 0$

(d) Sales are steady.
$S = C, S' = 0, S'' = 0$

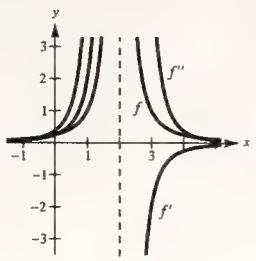
(e) Sales are declining, but at a lower rate.
$S' < 0, S'' > 0$

(f) Sales have bottomed out and have started to rise.
$S' > 0$

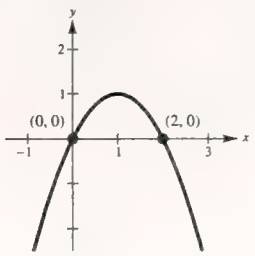
50.



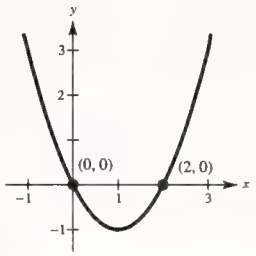
52.



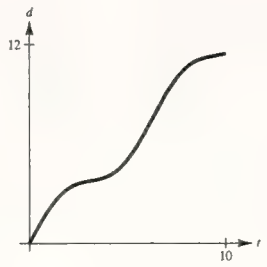
54.



56.



58. (a)



(b) Since the depth d is always increasing, there are no relative extrema. $f'(x) > 0$

(c) The rate of change of d is decreasing until you reach the widest point of the jug, then the rate increases until you reach the narrowest part of the jug's neck, then the rate decreases until you reach the top of the jug.

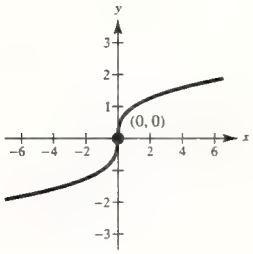
60. (a) $f(x) = \sqrt[3]{x}$

$f'(x) = \frac{1}{3}x^{-2/3}$

$f''(x) = -\frac{2}{9}x^{-5/3}$

Inflection point: $(0, 0)$

(b) $f''(x)$ does not exist at $x = 0$.



62. $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: $(2, 4)$

Relative minimum: $(4, 2)$

Point of inflection: $(3, 3)$

$f'(x) = 3ax^2 + 2bx + c, f''(x) = 6ax + 2b$

$$\left.\begin{array}{l} f(2) = 8a + 4b + 2c + d = 4 \\ f(4) = 64a + 16b + 4c + d = 2 \end{array}\right\} 56a + 12b + 2c = -2 \Rightarrow 28a + 6b + c = -1$$

$f'(2) = 12a + 4b + c = 0,\ f'(4) = 48a + 8b + c = 0,\ f''(3) = 18a + 2b = 0$

$$\begin{array}{rlrl} 28a + 6b + c &= -1 & 18a + 2b &= 0 \\ 12a + 4b + c &= 0 & 16a + 2b &= -1 \\ 16a + 2b &= -1 & 2a &= 1 \end{array}$$

$a = \frac{1}{2}, b = -\frac{9}{2}, c = 12, d = -6$

$f(x) = \frac{1}{2}x^3 - \frac{9}{2}x^2 + 12x - 6$

64. (a) line OA: $y = -0.06x$ slope: -0.06

line CB: $y = 0.04x + 50$ slope: 0.04

$f(x) = ax^3 + bx^2 + cx + d$

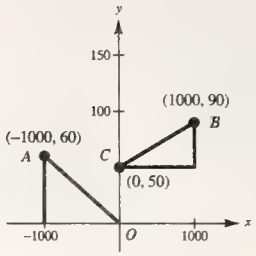
$f'(x) = 3ax^2 + 2bx + c$

$(-1000, 60)$: $60 = (-1000)^3a + (1000)^2b - 1000c + d$

$-0.06 = (1000)^2 3a - 2000b + c$

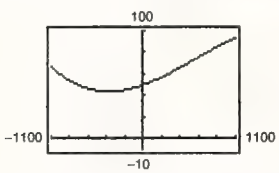
$(1000, 90)$: $90 = (1000)^3a + (1000)^2b + 1000c + d$

$0.04 = (1000)^2 3a + 2000b + c$

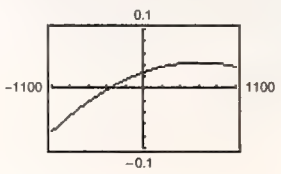


The solution to this system of 4 equations is $a = -1.25 \times 10^{-8}$, $b = 0.000025$, $c = 0.0275$, and $d = 50$.

(b) $y = -1.25 \times 10^{-8}x^3 + 0.000025x^2 + 0.0275x + 50$



(c)



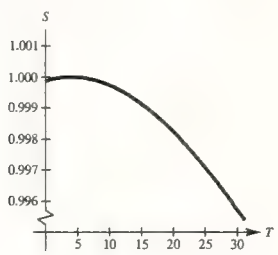
(d) The steepest part of the road is 6% at the point A.

66. $S = \dfrac{5.755T^3}{10^8} - \dfrac{8.521T^2}{10^6} + \dfrac{0.654T}{10^4} + 0.99987, \quad 0 < T < 25$

(a) The maximum occurs when $T \approx 4°$ and $S \approx 0.999999$.

(c) $S(20°) \approx 0.9982$

(b)



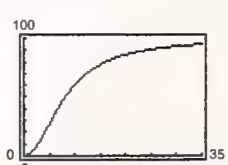
68. $C = 2x + \dfrac{300{,}000}{x}$

$C' = 2 - \dfrac{300{,}000}{x^2} = 0$ when $x = 100\sqrt{15} \approx 387$

By the First Derivative Test, C is minimized when $x \approx 387$ units.

70. $S = \dfrac{100t^2}{65 + t^2}, \quad t > 0$

(a)



(b) $S'(t) = \dfrac{13{,}000t}{(65 + t^2)^2}$

$S''(t) = \dfrac{13{,}000(65 - 3t^2)}{(65 + t^2)^3} = 0 \implies t = 4.65$

S is concave upwards on $(0, 4.65)$, concave downwards on $(4.65, 30)$.

(c) $S'(t) > 0$ for $t > 0$.

As t increases, the speed increases, but at a slower rate.

72. $f(x) = 2(\sin x + \cos x), \qquad f(0) = 2$

$f'(x) = 2(\cos x - \sin x), \qquad f'(0) = 2$

$f''(x) = 2(-\sin x - \cos x), \qquad f''(0) = -2$

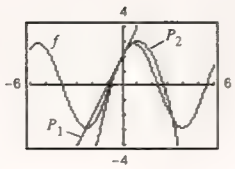
$P_1(x) = 2 + 2(x - 0) = 2(1 + x)$

$P_1'(x) = 2$

$P_2(x) = 2 + 2(x - 0) + \frac{1}{2}(-2)(x - 0)^2 = 2 + 2x - x^2$

$P_2'(x) = 2 - 2x$

$P_2''(x) = -2$



The values of f, P_1, P_2, and their first derivatives are equal at $x = 0$. The values of the second derivatives of f and P_2 are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

74. $f(x) = \dfrac{\sqrt{x}}{x - 1}, \qquad f(2) = \sqrt{2}$

$f'(x) = \dfrac{-(x + 1)}{2\sqrt{x}(x - 1)^2}, \qquad f'(2) = -\dfrac{3}{2\sqrt{2}} = -\dfrac{3\sqrt{2}}{4}$

$f''(x) = \dfrac{3x^2 + 6x - 1}{4x^{3/2}(x - 1)^3}, \qquad f''(2) = \dfrac{23}{8\sqrt{2}} = \dfrac{23\sqrt{2}}{16}$

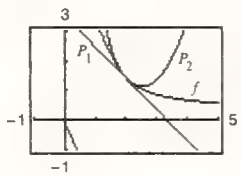
$P_1(x) = \sqrt{2} + \left(-\dfrac{3\sqrt{2}}{4}\right)(x - 2) = -\dfrac{3\sqrt{2}}{4}x + \dfrac{5\sqrt{2}}{2}$

$P_1'(x) = -\dfrac{3\sqrt{2}}{4}$

$P_2(x) = \sqrt{2} + \left(-\dfrac{3\sqrt{2}}{4}\right)(x - 2) + \dfrac{1}{2}\left(\dfrac{23\sqrt{2}}{16}\right)(x - 2)^2 = \sqrt{2} - \dfrac{3\sqrt{2}}{4}(x - 2) + \dfrac{23\sqrt{2}}{32}(x - 2)^2$

$P_2'(x) = -\dfrac{3\sqrt{2}}{4} + \dfrac{23\sqrt{2}}{16}(x - 2)$

$P_2''(x) = \dfrac{23\sqrt{2}}{16}$



The values of f, P_1, P_2 and their first derivatives are equal at $x = 2$. The values of the second derivatives of f and P_2 are equal at $x = 2$. The approximations worsen as you move away from $x = 2$.

76. $f(x) = x(x - 6)^2 = x^3 - 12x^2 + 36x$

$f'(x) = 3x^2 - 24x + 36 = 3(x - 2)(x - 6) = 0$

$f''(x) = 6x - 24 = 6(x - 4) = 0$

Relative extrema: (2, 32) and (6, 0)

Point of inflection (4, 16) is midway between the relative extrema of f.

78. $p(x) = ax^3 + bx^2 + cx + d$

$p'(x) = 3ax^2 + 2bx + c$

$p''(x) = 6ax + 2b$

$6ax + 2b = 0$

$$x = -\frac{b}{3a}$$

The sign of $p''(x)$ changes at $x = -b/3a$. Therefore, $(-b/3a, p(-b/3a))$ is a point of inflection.

$$p\left(-\frac{b}{3a}\right) = a\left(-\frac{b^3}{27a^3}\right) + b\left(\frac{b^2}{9a^2}\right) + c\left(-\frac{b}{3a}\right) + d = \frac{2b^3}{27a^2} - \frac{bc}{3a} + d$$

When $p(x) = x^3 - 3x^2 + 2$, $a = 1$, $b = -3$, $c = 0$, and $d = 2$.

$$x_0 = \frac{-(-3)}{3(1)} = 1$$

$$y_0 = \frac{2(-3)^3}{27(1)^2} - \frac{(-3)(0)}{3(1)} + 2 = -2 - 0 + 2 = 0$$

The point of inflection of $p(x) = x^3 - 3x^2 + 2$ is $(x_0, y_0) = (1, 0)$.

80. False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

82. True

$y = \sin(bx)$

Slope: $y' = b\cos(bx)$

$-b \le y' \le b$ (Assume $b > 0$)

84. False. For example, let $f(x) = (x - 2)^4$.

Section 3.5 Limits at Infinity

2. $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$

No vertical asymptotes

Horizontal asymptotes: $y = \pm 2$

Matches (c)

4. $f(x) = 2 + \dfrac{x^2}{x^4 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (a)

6. $f(x) = \dfrac{2x^2 - 3x + 5}{x^2 + 1}$

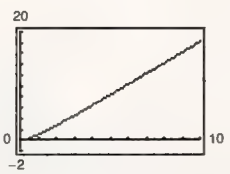
No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (e)

8. $f(x) = \dfrac{2x^2}{x + 1}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1	18.18	198.02	1998.02	19,998	199,998	1,999,998

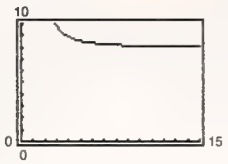


$\lim_{x \to \infty} f(x) = \infty$ (Limit does not exist.)

10. $f(x) = \dfrac{8x}{\sqrt{x^2 - 3}}$

x	10^1	10^2	10^3	10^4	10^5	10^6	10^7
$f(x)$	8.12	8.001	8	8	8	8	8

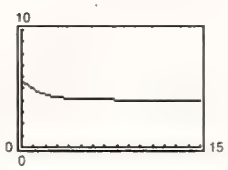
$\lim_{x\to\infty} f(x) = 8$



12. $f(x) = 4 + \dfrac{3}{x^2 + 2}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	5	4.03	4.0003	4.0	4.0	4	4

$\lim_{x\to\infty} f(x) = 4$



14. (a) $h(x) = \dfrac{f(x)}{x} = \dfrac{5x^2 - 3x + 7}{x} = 5x - 3 + \dfrac{7}{x}$

$\lim_{x\to\infty} h(x) = \infty$ (Limit does not exist)

(b) $h(x) = \dfrac{f(x)}{x^2} = \dfrac{5x^2 - 3x + 7}{x^2} = 5 - \dfrac{3}{x} + \dfrac{7}{x^2}$

$\lim_{x\to\infty} h(x) = 5$

(c) $h(x) = \dfrac{f(x)}{x^3} = \dfrac{5x^2 - 3x + 7}{x^3} = \dfrac{5}{x} - \dfrac{3}{x^2} + \dfrac{7}{x^3}$

$\lim_{x\to\infty} h(x) = 0$

16. (a) $\lim_{x\to\infty} \dfrac{3 - 2x}{3x^3 - 1} = 0$

(b) $\lim_{x\to\infty} \dfrac{3 - 2x}{3x - 1} = -\dfrac{2}{3}$

(c) $\lim_{x\to\infty} \dfrac{3 - 2x^2}{3x - 1} = -\infty$ (Limit does not exist)

18. (a) $\lim_{x\to\infty} \dfrac{5x^{3/2}}{4x^2 + 1} = 0$

(b) $\lim_{x\to\infty} \dfrac{5x^{3/2}}{4x^{3/2} + 1} = \dfrac{5}{4}$

(c) $\lim_{x\to\infty} \dfrac{5x^{3/2}}{4\sqrt{x} + 1} = \infty$ (Limit does not exist)

20. $\lim_{x\to\infty} \dfrac{3x^3 + 2}{9x^3 - 2x^2 + 7} = \dfrac{3}{9} = \dfrac{1}{3}$

22. $\lim_{x\to\infty} \left(4 + \dfrac{3}{x}\right) = 4 + 0 = 4$

24. $\lim_{x\to\infty} \left(\dfrac{1}{2}x - \dfrac{4}{x^2}\right) = -\infty$ (Limit does not exist)

26. $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2 + 1}} = \lim_{x\to-\infty} \dfrac{1}{\left(\dfrac{\sqrt{x^2 + 1}}{-\sqrt{x^2}}\right)}$ $\left(\text{for } x < 0, x = -\sqrt{x^2}\right)$

$= \lim_{x\to-\infty} \dfrac{-1}{\sqrt{x + (1/x)}} = -1$

28. $\lim_{x\to-\infty}\dfrac{-3x+1}{\sqrt{x^2+x}} = \lim_{x\to-\infty}\dfrac{-3+(1/x)}{\dfrac{\sqrt{x^2+x}}{-\sqrt{x^2}}}$, (for $x < 0$ we have $-\sqrt{x^2} = x$)

$$= \lim_{x\to-\infty}\frac{3-(1/x)}{\sqrt{1+(1/x)}} = 3$$

30. $\lim_{x\to\infty}\dfrac{x-\cos x}{x} = \lim_{x\to\infty}\left(1-\dfrac{\cos x}{x}\right)$

$$= 1 - 0 = 1$$

Note:

$\lim_{x\to\infty}\dfrac{\cos x}{x} = 0$ by the Squeeze Theorem since

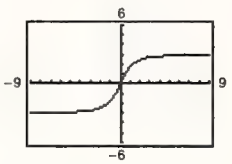
$$-\frac{1}{x} \le \frac{\cos x}{x} \le \frac{1}{x}.$$

32. $\lim_{x\to\infty}\cos\left(\dfrac{1}{x}\right) = \cos 0 = 1$

34. $f(x) = \dfrac{3x}{\sqrt{x^2+2}}$

$\lim_{x\to\infty} f(x) = 3$

$\lim_{x\to-\infty} f(x) = -3$



Therefore, $y = 3$ and $y = -3$ are both horizontal asymptotes.

36. $\lim_{x\to\infty} x\tan\dfrac{1}{x} = \lim_{t\to0^+}\dfrac{\tan t}{t} = \lim_{t\to0^+}\left[\dfrac{\sin t}{t}\cdot\dfrac{1}{\cos t}\right]$

$$= (1)(1) = 1$$

(Let $x = 1/t$.)

38. $\lim_{x\to\infty}\left(2x-\sqrt{4x^2+1}\right) = \lim_{x\to\infty}\left[\left(2x-\sqrt{4x^2+1}\right)\cdot\dfrac{2x+\sqrt{4x^2+1}}{2x+\sqrt{4x^2+1}}\right] = \lim_{x\to\infty}\dfrac{-1}{2x+\sqrt{4x^2+1}} = 0$

40. $\lim_{x\to-\infty}\left(3x+\sqrt{9x^2-x}\right) = \lim_{x\to-\infty}\left[\left(3x+\sqrt{9x^2-x}\right)\cdot\dfrac{3x-\sqrt{9x^2-x}}{3x-\sqrt{9x^2-x}}\right]$

$$= \lim_{x\to-\infty}\frac{x}{3x-\sqrt{9x^2-x}}$$

$$= \lim_{x\to-\infty}\frac{1}{3-\dfrac{\sqrt{9x^2-x}}{-\sqrt{x^2}}} \quad (\text{for } x < 0 \text{ we have } x = -\sqrt{x^2})$$

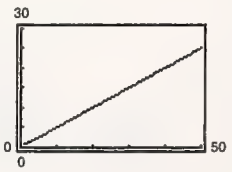
$$= \lim_{x\to-\infty}\frac{1}{3+\sqrt{9-(1/x)}} = \frac{1}{6}$$

42.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1.0	5.1	50.1	500.1	5000.1	50,000.1	500,000.1

$$\lim_{x\to\infty}\frac{x^2-x\sqrt{x^2-x}}{1}\cdot\frac{x^2+x\sqrt{x^2-x}}{x^2+x\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{x^3}{x^2+x\sqrt{x^2-x}} = \infty$$

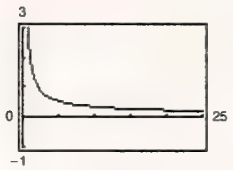
Limit does not exist.



44.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	2.000	0.348	0.101	0.032	0.010	0.003	0.001

$$\lim_{x\to\infty}\frac{x+1}{x\sqrt{x}}=0$$



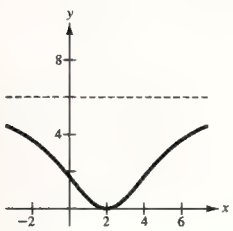
46. $x = 2$ is a critical number.

$f'(x) < 0$ for $x < 2$.

$f'(x) > 0$ for $x > 2$.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 6$$

For example, let $f(x) = \dfrac{-6}{0.1(x-2)^2+1} + 6$.



48. (a) The function is even: $\lim_{x\to-\infty} f(x) = 5$

(b) The function is odd: $\lim_{x\to-\infty} f(x) = -5$

50. $y = \dfrac{x-3}{x-2}$

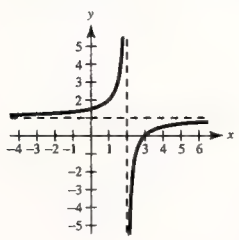
Intercepts: $(3, 0), \left(0, \dfrac{3}{2}\right)$

Symmetry: none

Horizontal asymptote: $y = 1$ since

$$\lim_{x\to-\infty}\frac{x-3}{x-2} = 1 = \lim_{x\to\infty}\frac{x-3}{x-2}.$$

Discontinuity: $x = 2$ (Vertical asymptote)



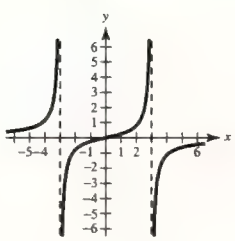
52. $y = \dfrac{2x}{9-x^2}$

Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = \pm 3$



54. $y = \dfrac{x^2}{x^2-9}$

Intercept: $(0, 0)$

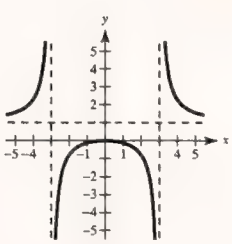
Symmetry: y-axis

Horizontal asymptote: $y = 1$ since

$$\lim_{x\to-\infty}\frac{x^2}{x^2-9} = 1 = \lim_{x\to\infty}\frac{x^2}{x^2-9}.$$

Discontinuities: $x = \pm 3$ (Vertical asymptotes)

Relative maximum: $(0, 0)$



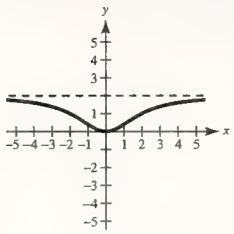
56. $y = \dfrac{2x^2}{x^2 + 4}$

Intercept: $(0, 0)$

Symmetry: y-axis

Horizontal asymptote: $y = 2$

Relative minimum: $(0, 0)$



58. $x^2y = 4$

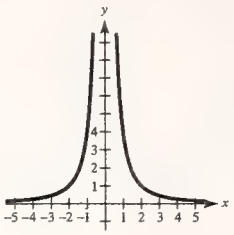
Intercepts: none

Symmetry: y-axis

Horizontal asymptote: $y = 0$ since

$$\lim_{x \to -\infty} \frac{4}{x^2} = 0 = \lim_{x \to \infty} \frac{4}{x^2}.$$

Discontinuity: $x = 0$ (Vertical asymptote)



60. $y = \dfrac{2x}{1 - x^2}$

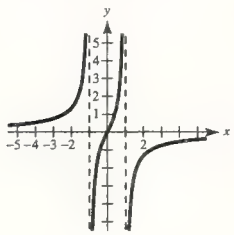
Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$ since

$$\lim_{x \to -\infty} \frac{2x}{1 - x^2} = 0 = \lim_{x \to \infty} \frac{2x}{1 - x^2}.$$

Discontinuities: $x = \pm 1$ (Vertical asymptotes)



62. $y = 1 + \dfrac{1}{x}$

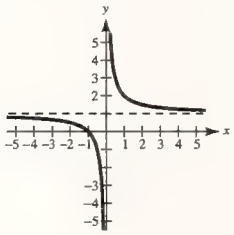
Intercept: $(-1, 0)$

Symmetry: none

Horizontal asymptote: $y = 1$ since

$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right) = 1 = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right).$$

Discontinuity: $x = 0$ (Vertical asymptote)



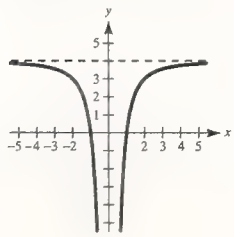
64. $y = 4\left(1 - \dfrac{1}{x^2}\right)$

Intercepts: $(\pm 1, 0)$

Symmetry: y-axis

Horizontal asymptote: $y = 4$

Vertical asymptote: $x = 0$



66. $y = \dfrac{x}{\sqrt{x^2 - 4}}$

Domain: $(-\infty, -2), (2, \infty)$

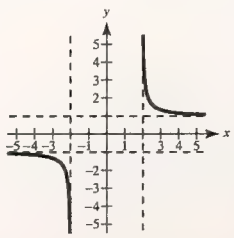
Intercepts: none

Symmetry: origin

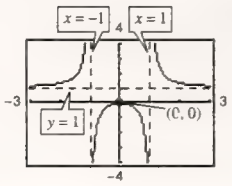
Horizontal asymptotes: $y = \pm 1$ since

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^2 - 4}} = 1, \quad \lim_{x \to -\infty} \frac{x}{\sqrt{x^2 - 4}} = -1.$$

Vertical asymptotes: $x = \pm 2$ (discontinuities)



68. $f(x) = \dfrac{x^2}{x^2 - 1}$



$$f'(x) = \frac{(x^2 - 1)(2x) - x^2(2x)}{(x^2 - 1)^2} = \frac{-2x}{(x^2 - 1)^2} = 0 \text{ when } x = 0.$$

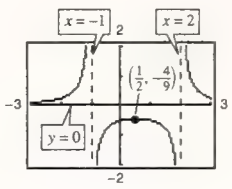
$$f''(x) = \frac{(x^2 - 1)^2(-2) + 2x(2)(x^2 - 1)(2x)}{(x^2 - 1)^4} = \frac{2(3x^2 + 1)}{(x^2 - 1)^3}$$

Since $f''(0) < 0$, then $(0, 0)$ is a relative maximum. Since $f''(x) \neq 0$, nor is it undefined in the domain of f, there are no points of inflection.

Vertical asymptotes: $x = \pm 1$

Horizontal asymptote: $y = 1$

70. $f(x) = \dfrac{1}{x^2 - x - 2} = \dfrac{1}{(x + 1)(x - 2)}$



$$f'(x) = \frac{-(2x - 1)}{(x^2 - x - 2)^2} = 0 \text{ when } x = \frac{1}{2}.$$

$$f''(x) = \frac{(x^2 - x - 2)^2(-2) + (2x - 1)(2)(x^2 - x - 2)(2x - 1)}{(x^2 - x - 2)^4}$$

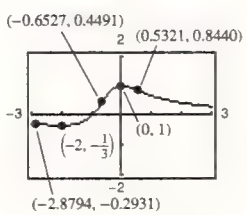
$$= \frac{6(x^2 - x + 1)}{(x^2 - x - 2)^3}$$

Since $f''\left(\frac{1}{2}\right) < 0$, then $\left(\frac{1}{2}, -\frac{4}{9}\right)$ is a relative maximum. Since $f''(x) \neq 0$, nor is it undefined in the domain of f, there are no points of inflection.

Vertical asymptotes: $x = -1, x = 2$

Horizontal asymptote: $y = 0$

72. $f(x) = \dfrac{x + 1}{x^2 + x + 1}$



$$f'(x) = \frac{-x(x + 2)}{(x^2 + x + 1)^2} = 0 \text{ when } x = 0, -2.$$

$$f''(x) = \frac{2(x^3 + 3x^2 - 1)}{(x^2 + x + 1)^3} = 0 \text{ when } x \approx 0.5321, -0.6527, -2.8794.$$

$f''(0) < 0$

Therefore, $(0, 1)$ is a relative maximum.

$f''(-2) > 0$

Therefore,

$$\left(-2, -\frac{1}{3}\right)$$

is a relative minimum.

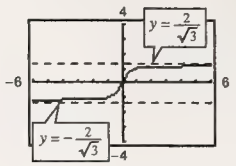
Points of inflection: $(0.5321, 0.8440)$, $(-0.6527, 0.4491)$ and $(-2.8794, -0.2931)$

Horizontal asymptote: $y = 0$

74. $g(x) = \dfrac{2x}{\sqrt{3x^2 + 1}}$

$g'(x) = \dfrac{2}{(3x^2 + 1)^{3/2}}$

$g''(x) = \dfrac{-18x}{(3x^2 + 1)^{5/2}}$



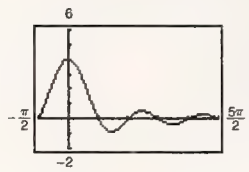
No relative extrema. Point of inflection: $(0, 0)$.

Horizontal asymptotes: $y = \pm\dfrac{2}{\sqrt{3}}$

No vertical asymptotes

76. $f(x) = \dfrac{2 \sin 2x}{x}$ Hole at $(0, 4)$

$f'(x) = \dfrac{4x \cos 2x - 2 \sin 2x}{x^2}$



There are an infinite number of relative extrema. In the interval $(-2\pi, 2\pi)$, you obtain the following.

Relative minima: $(\pm 2.25, -0.869)$, $(\pm 5.45, -0.365)$

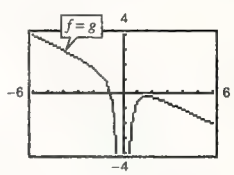
Relative maxima: $(\pm 3.87, 0.513)$

Horizontal asymptote: $y = 0$

No vertical asymptotes

78. $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$, $g(x) = -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2}$

(a)

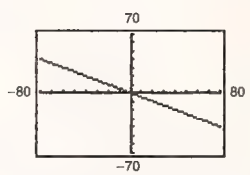


(b) $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$

$= -\left[\dfrac{x^3}{2x^2} - \dfrac{2x^2}{2x^2} + \dfrac{2}{2x^2}\right]$

$= -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2} = g(x)$

(c)

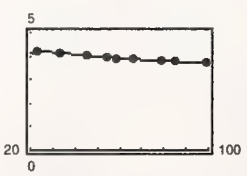


The graph appears as the slant asymptote $y = -\frac{1}{2}x + 1$.

80. $\lim\limits_{v_1/v_2 \to \infty} 100\left[1 - \dfrac{1}{(v_1/v_2)^c}\right] = 100[1 - 0] = 100\%$

82. $y = \dfrac{3.351t^2 + 42.461t - 543.730}{t^2}$

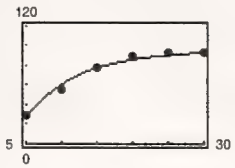
(a)



(b) Yes. $\lim\limits_{t \to \infty} y = 3.351$

84. $S = \dfrac{100t^2}{65 + t^2},\ t > 0$

(a)



(b) Yes. $\lim_{t\to\infty} S = \dfrac{100}{1} = 100$

86. $\lim_{x\to\infty} \dfrac{p(x)}{q(x)} = \lim_{x\to\infty} \dfrac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$

Divide $p(x)$ and $q(x)$ by x^m.

Case 1: If $n < m$: $\lim_{x\to\infty} \dfrac{p(x)}{q(x)} = \lim_{x\to\infty} \dfrac{\dfrac{a_n}{x^{m-n}} + \cdots + \dfrac{a_1}{x^{m-1}} + \dfrac{a_0}{x^m}}{b_m + \cdots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}} = \dfrac{0 + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \dfrac{0}{b_m} = 0$

Case 2: If $m = n$: $\lim_{x\to\infty} \dfrac{p(x)}{q(x)} = \lim_{x\to\infty} \dfrac{a_n + \cdots + \dfrac{a_1}{x^{m-1}} + \dfrac{a_0}{x^m}}{b_m + \cdots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}} = \dfrac{a_n + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \dfrac{a_n}{b_m}.$

Case 3: If $n > m$: $\lim_{x\to\infty} \dfrac{p(x)}{q(x)} = \lim_{x\to\infty} \dfrac{a_n x^{n-m} + \cdots + \dfrac{a_1}{x^{m-1}} + \dfrac{a_0}{x^m}}{b_m + \cdots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}} = \dfrac{\pm\infty + \cdots + 0}{b_m + \cdots + 0} = \pm\infty.$

88. False. Let $y_1 = \sqrt{x+1}$, then $y_1(0) = 1$. Thus, $y_1' = 1/\left(2\sqrt{x+1}\right)$ and $y_1'(0) = 1/2$. Finally,

$$y_1'' = -\frac{1}{4(x+1)^{3/2}} \text{ and } y_1''(0) = -\frac{1}{4}.$$

Let $p = ax^2 + bx + 1$, then $p(0) = 1$. Thus, $p' = 2ax + b$ and $p'(0) = \frac{1}{2} \Rightarrow b = \frac{1}{2}$. Finally, $p'' = 2a$ and $p''(0) = -\frac{1}{4} \Rightarrow a = -\frac{1}{8}$. Therefore,

$$f(x) = \begin{cases} (-1/8)x^2 + (1/2)x + 1, & x < 0 \\ \sqrt{x+1}, & x \geq 0 \end{cases} \text{ and } f(0) = 1,$$

$$f'(x) = \begin{cases} (1/2) - (1/4)x, & x < 0 \\ 1/\left(2\sqrt{x+1}\right), & x > 0 \end{cases} \text{ and } f'(0) = \frac{1}{2}, \text{ and}$$

$$f''(x) = \begin{cases} (-1/4), & x < 0 \\ -1/(4(x+1)^{3/2}), & x > 0 \end{cases} \text{ and } f''(0) = -\frac{1}{4}.$$

$f''(x) < 0$ for all real x, but $f(x)$ *increases* without bound.

Section 3.6 A Summary of Curve Sketching

2. The slope of f approaches ∞ as $x \to 0^-$, and approaches $-\infty$ as $x \to 0^+$. Matches (C)

4. The slope is positive up to approximately $x = 1.5$. Matches (B)

6. (a) x_0, x_2, x_4

(b) x_2, x_3

(c) x_1

(d) x_1

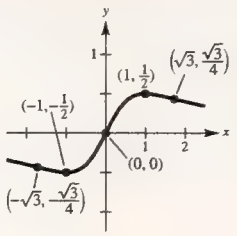
(e) x_2, x_3

8. $y = \dfrac{x}{x^2 + 1}$

$y' = \dfrac{1 - x^2}{(x^2 + 1)^2} = \dfrac{(1 - x)(x + 1)}{(x^2 + 1)^2} = 0$ when $x = \pm 1$.

$y'' = -\dfrac{2x(3 - x^2)}{(x^2 + 1)^3} = 0$ when $x = 0, \pm\sqrt{3}$.

Horizontal asymptote: $y = 0$



	y	y'	y''	Conclusion
$-\infty < x < -\sqrt{3}$		$-$	$-$	Decreasing, concave down
$x = -\sqrt{3}$	$-\frac{\sqrt{3}}{4}$	$-$	0	Point of inflection
$-\sqrt{3} < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	$-\frac{1}{2}$	0	$+$	Relative minimum
$-1 < x < 0$		$+$	$+$	Increasing, concave up
$x = 0$	0	$+$	0	Point of inflection
$0 < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	$\frac{1}{2}$	0	$-$	Relative maximum
$1 < x < \sqrt{3}$		$-$	$-$	Decreasing, concave down
$x = \sqrt{3}$	$\frac{\sqrt{3}}{4}$	$-$	0	Point of inflection
$\sqrt{3} < x < \infty$		$-$	$+$	Decreasing, concave up

10. $y = \dfrac{x^2 + 1}{x^2 - 9}$

$y' = \dfrac{-20x}{(x^2 - 9)^2} = 0$ when $x = 0$

$y'' = \dfrac{60(x^2 + 3)}{(x^2 - 9)^3} < 0$ when $x = 0$

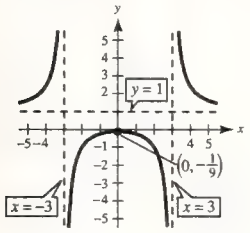
Therefore, $\left(0, -\dfrac{1}{9}\right)$ is a relative maximum.

Intercept: $\left(0, -\dfrac{1}{9}\right)$

Vertical asymptotes: $x = \pm 3$

Horizontal asymptote: $y = 1$

Symmetric about y-axis



12. $f(x) = \dfrac{x + 2}{x} = 1 + \dfrac{2}{x}$

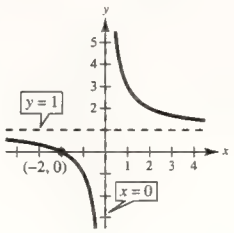
$f'(x) = \dfrac{-2}{x^2} < 0$ when $x \neq 0$.

$f''(x) = \dfrac{4}{x^3} \neq 0$

Intercept: $(-2, 0)$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 1$



14. $f(x) = x + \dfrac{32}{x^2}$

$f'(x) = 1 - \dfrac{64}{x^3} = \dfrac{(x - 4)(x^2 + 4x + 16)}{x^3} = 0$ when $x = 4$.

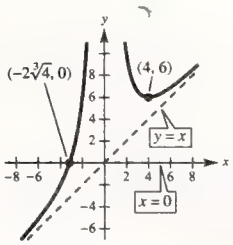
$f''(x) = \dfrac{192}{x^4} > 0$ if $x \neq 0$.

Therefore, (4, 6) is a relative minimum.

Intercept; $\left(-2\sqrt[3]{4}, 0\right)$

Vertical asymptote: $x = 0$

Slant asymptote: $y = x$



16. $f(x) = \dfrac{x^3}{x^2 - 4} = x + \dfrac{4x}{x^2 - 4}$

$f'(x) = \dfrac{x^2(x^2 - 12)}{(x^2 - 4)^2} = 0$ when $x = 0, \pm 2\sqrt{3}$

$f''(x) = \dfrac{8x(x^2 + 12)}{(x^2 - 4)^3} = 0$ when $x = 0$

Intercept: (0, 0)

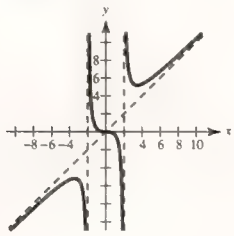
Relative maximum: $\left(-2\sqrt{3}, -3\sqrt{3}\right)$

Relative minimum: $\left(2\sqrt{3}, 3\sqrt{3}\right)$

Inflection point: (0, 0)

Vertical asymptotes: $x = \pm 2$

Slant asymptote: $y = x$



18. $y = \dfrac{2x^2 - 5x + 5}{x - 2} = 2x - 1 + \dfrac{3}{x - 2}$

$y' = 2 - \dfrac{3}{(x-2)^2} = \dfrac{2x^2 - 8x + 5}{(x-2)^2} = 0$ when $x = \dfrac{4 \pm \sqrt{6}}{2}$.

$y'' = \dfrac{6}{(x-2)^3} \neq 0$

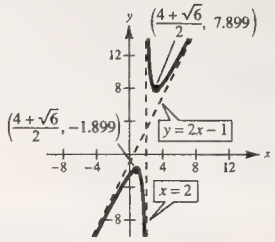
Relative maximum: $\left(\dfrac{4 - \sqrt{6}}{2}, -1.8990\right)$

Relative minimum: $\left(\dfrac{4 + \sqrt{6}}{2}, 7.8990\right)$

Intercept: $(0, -5/2)$

Vertical asymptote: $x = 2$

Slant asymptote: $y = 2x - 1$



20. $g(x) = x\sqrt{9 - x}$ Domain: $x \leq 9$

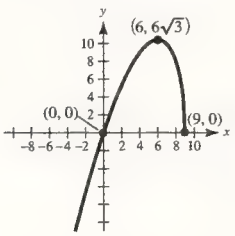
$g'(x) = \dfrac{3(6 - x)}{2\sqrt{9 - x}} = 0$ when $x = 6$

$g''(x) = \dfrac{3(x - 12)}{4(9 - x)^{3/2}} < 0$ when $x = 6$

Relative maximum: $(6, 6\sqrt{3})$

Intercepts: $(0, 0)$, $(9, 0)$

Concave downward on $(-\infty, 9)$



22. $y = x\sqrt{16 - x^2}$ Domain: $-4 \leq x \leq 4$

$y' = \dfrac{2(8 - x^2)}{\sqrt{16 - x^2}} = 0$ when $x = \pm 2\sqrt{2}$

$y'' = \dfrac{2x(x^2 - 24)}{(16 - x^2)^{3/2}} = 0$ when $x = 0$

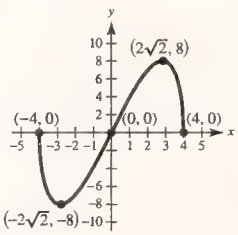
Relative maximum: $(2\sqrt{2}, 8)$

Relative minimum: $(-2\sqrt{2}, -8)$

Intercepts: $(0, 0)$, $(\pm 4, 0)$

Symmetric with respect to the origin

Point of inflection: $(0, 0)$



24. $y = 3(x - 1)^{2/3} - (x - 1)^2$

$y' = \dfrac{2}{(x-1)^{1/3}} - 2(x - 1) = \dfrac{2 - 2(x-1)^{4/3}}{(x-1)^{1/3}} = 0$ when $x = 0, 2$

(y' undefined for $x = 1$)

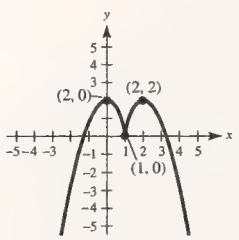
$y'' = \dfrac{-2}{3(x-1)^{4/3}} - 2 < 0$ for all $x \neq 1$

Concave downward on $(-\infty, 1)$ and $(1, \infty)$

Relative maximum: $(0, 2)$, $(2, 2)$

Relative minimum: $(1, 0)$

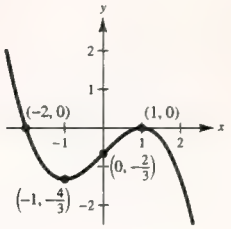
Intercepts: $(0, 2)$, $(1, 0)$, $(-1.280, 0)$, $(3.280, 0)$



26. $y = -\frac{1}{3}(x^3 - 3x + 2)$

$y' = -x^2 + 1 = 0$ when $x = \pm 1$

$y'' = -2x = 0$ when $x = 0$

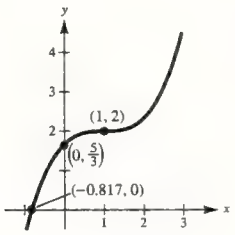


	y	y'	y''	Conclusion
$-\infty < x < -1$		−	+	Decreasing, concave up
$x = -1$	$-\frac{4}{3}$	0	+	Relative minimum
$-1 < x < 0$		+	+	Increasing, concave up
$x = 0$	$-\frac{2}{3}$	+	0	Point of inflection
$0 < x < 1$		+	−	Increasing, concave down
$x = 1$	0	0	−	Relative maximum
$1 < x < \infty$		−	−	Decreasing, concave down

28. $f(x) = \frac{1}{3}(x - 1)^3 + 2$

$f'(x) = (x - 1)^2 = 0$ when $x = 1$.

$f''(x) = 2(x - 1) = 0$ when $x = 1$.



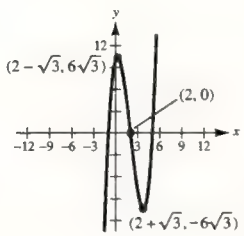
	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < 1$		+	−	Increasing, concave down
$x = 1$	2	0	0	Point of inflection
$1 < x < \infty$		+	+	Increasing, concave up

30. $f(x) = (x + 1)(x - 2)(x - 5)$

$f'(x) = (x + 1)(x - 2) + (x + 1)(x - 5) + (x - 2)(x - 5)$

$= 3(x^2 - 4x + 1) = 0$ when $x = 2 \pm \sqrt{3}$.

$f''(x) = 6(x - 2) = 0$ when $x = 2$.



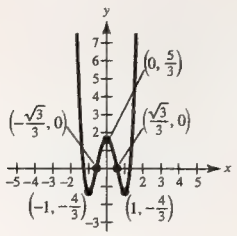
	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < 2 - \sqrt{3}$		+	−	Increasing, concave down
$x = 2 - \sqrt{3}$	$6\sqrt{3}$	0	−	Relative maximum
$2 - \sqrt{3} < x < 2$		−	−	Decreasing, concave down
$x = 2$	0	−	0	Point of inflection
$2 < x < 2 + \sqrt{3}$		−	+	Decreasing, concave up
$x = 2 + \sqrt{3}$	$-6\sqrt{3}$	0	+	Relative minimum
$2 + \sqrt{3} < x < \infty$		+	+	Increasing, concave up

Intercepts: $(0, 10)$, $(-1, 0)$, $(2, 0)$, $(5, 0)$

32. $y = 3x^4 - 6x^2 + \frac{5}{3}$

$y' = 12x^3 - 12x = 12x(x^2 - 1) = 0$ when $x = 0, x = \pm 1$.

$y'' = 36x^2 - 12 = 12(3x^2 - 1) = 0$ when $x = \pm\frac{\sqrt{3}}{3}$.

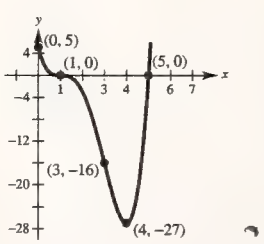


	y	y'	y''	Conclusion
$-\infty < x < -1$		−	+	Decreasing, concave up
$x = -1$	$-4/3$	0	+	Relative minimum
$-1 < x < -\frac{\sqrt{3}}{3}$		+	+	Increasing, concave up
$x = -\frac{\sqrt{3}}{3}$	0	+	0	Point of inflection
$-\frac{\sqrt{3}}{3} < x < 0$		+	−	Increasing, concave down
$x = 0$	5/3	0	−	Relative maximum
$0 < x < \frac{\sqrt{3}}{3}$		−	−	Decreasing, concave down
$x = \frac{\sqrt{3}}{3}$	0	−	0	Point of inflection
$\frac{\sqrt{3}}{3} < x < 1$		−	+	Decreasing, concave up
$x = 1$	$-4/3$	0	+	Relative minimum
$1 < x < \infty$		+	+	Increasing, concave up

34. $f(x) = x^4 - 8x^3 + 18x^2 - 16x + 5$

$f'(x) = 4x^3 - 24x^2 + 36x - 16 = 4(x - 4)(x - 1)^2 = 0$ when $x = 1, x = 4$.

$f''(x) = 12x^2 - 48x + 36 = 12(x - 3)(x - 1) = 0$ when $x = 3, x = 1$.

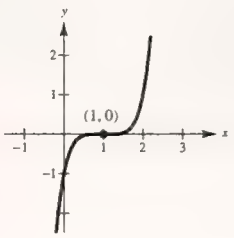


	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < 1$		−	+	Decreasing, concave up
$x = 1$	0	0	0	Point of inflection
$1 < x < 3$		−	−	Decreasing, concave down
$x = 3$	-16	−	0	Point of inflection
$3 < x < 4$		−	+	Decreasing, concave up
$x = 4$	-27	0	+	Relative minimum
$4 < x < \infty$		+	+	Increasing, concave up

36. $y = (x - 1)^5$

$y' = 5(x - 1)^4 = 0$ when $x = 1$.

$y'' = 20(x - 1)^3 = 0$ when $x = 1$.



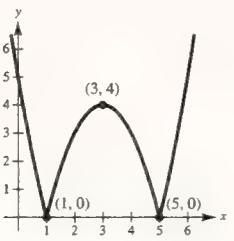
	y	y'	y''	Conclusion
$-\infty < x < 1$		+	−	Increasing, concave down
$x = 1$	0	0	0	Point of inflection
$1 < x < \infty$		+	+	Increasing, concave up

38. $y = |x^2 - 6x + 5|$

$$y' = \frac{2(x - 3)(x^2 - 6x + 5)}{|x^2 - 6x + 5|} = \frac{2(x - 3)(x - 5)(x - 1)}{|(x - 5)(x - 1)|}$$

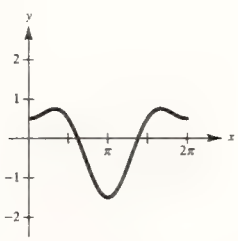
$= 0$ when $x = 3$ and undefined when $x = 1, x = 5$.

$$y'' = \frac{2(x^2 - 6x + 5)}{|x^2 - 6x + 5|} = \frac{2(x - 5)(x - 1)}{|(x - 5)(x - 1)|}$$ undefined when $x = 1, x = 5$.



	y	y'	y''	Conclusion
$-\infty < x < 1$		−	+	Decreasing, concave up
$x = 1$	0	Undefined	Undefined	Relative minimum, point of inflection
$1 < x < 3$		+	−	Increasing, concave down
$x = 3$	4	0	−	Relative maximum
$3 < x < 5$		−	−	Decreasing, concave down
$x = 5$	0	Undefined	Undefined	Relative minimum, point of inflection
$5 < x < \infty$		+	+	Increasing, concave up

40. $y = \cos x - \frac{1}{2}\cos 2x, 0 \le x \le 2\pi$



$y' = -\sin x + \sin 2x = -\sin x(1 - 2\cos x) = 0$ when $x = 0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}$.

$y'' = -\cos x + 2\cos 2x = -\cos x + 2(2\cos^2 x - 1)$

$= 4\cos^2 x - \cos x - 2 = 0$ when $\cos x = \frac{1 \pm \sqrt{33}}{8} \approx 0.8431, -0.5931$.

Therefore, $x \approx 0.5678$ or 5.7154, $x \approx 2.2057$ or 4.0775.

Relative maxima: $\left(\frac{\pi}{3}, \frac{3}{4}\right), \left(\frac{5\pi}{3}, \frac{3}{4}\right)$

Relative minimum: $\left(\pi, -\frac{3}{2}\right)$

Inflection points: $(0.5678, 0.6323), (2.2057, -0.4449), (5.7154, 0.6323), (4.0775, -0.4449)$

42. $y = 2(x - 2) + \cot x,\ 0 < x < \pi$

$y' = 2 - \csc^2 x = 0$ when $x = \frac{\pi}{4}, \frac{3\pi}{4}$

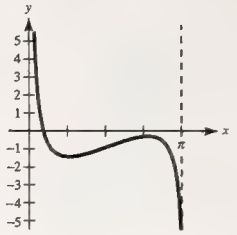
$y'' = 2\csc^2 x \cot x = 0$ when $x = \frac{\pi}{2}$

Relative maximum: $\left(\frac{3\pi}{4}, \frac{3\pi}{2} - 5\right)$

Relative minimum: $\left(\frac{\pi}{4}, \frac{\pi}{2} - 3\right)$

Point of inflection: $\left(\frac{\pi}{2}, \pi - 4\right)$

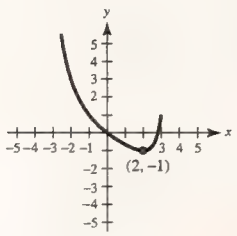
Vertical asymptotes: $x = 0, \pi$



44. $y = \sec^2\left(\frac{\pi x}{8}\right) - 2\tan\left(\frac{\pi x}{8}\right) - 1,\ -3 < x < 3$

$y' = 2\sec^2\left(\frac{\pi x}{8}\right)\tan\left(\frac{\pi x}{8}\right)\left(\frac{\pi}{8}\right) - 2\sec^2\left(\frac{\pi x}{8}\right)\left(\frac{\pi}{8}\right) = 0 \Rightarrow x = 2$

Relative minimum: $(2, -1)$



46. $g(x) = x\cot x,\ -2\pi < x < 2\pi$

$g'(x) = \dfrac{\sin x \cos x - x}{\sin^2 x}$

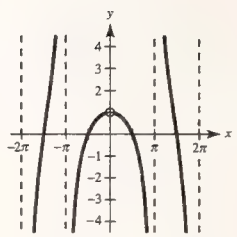
$g'(0)$ does not exist. But $\lim_{x\to 0} x\cot x = \lim_{x\to 0} \dfrac{x}{\tan x} = 1.$

Vertical asymptotes: $x = \pm 2\pi, \pm\pi$

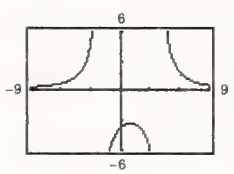
Intercepts: $\left(-\frac{3\pi}{2}, 0\right), \left(-\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right)$

Symmetric with respect to y-axis.

Decreasing on $(0, \pi)$ and $(\pi, 2\pi)$



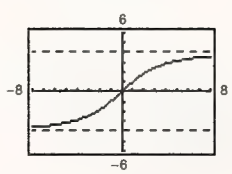
48. $f(x) = 5\left(\dfrac{1}{x-4} - \dfrac{1}{x+2}\right)$



$x = -2, 4$ vertical asymptote

$y = 0$ horizontal asymptote

50. $f(x) = \dfrac{4x}{\sqrt{x^2 + 15}}$



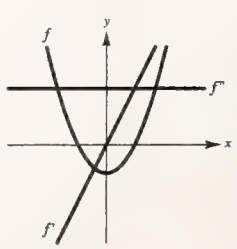
$y = \pm 4$ horizontal asymptotes

$(0, 0)$ point of inflection

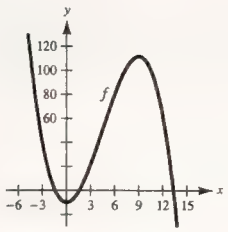
52. f'' is constant.

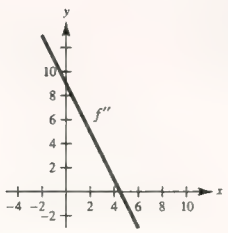
f' is linear.

f is quadratic.



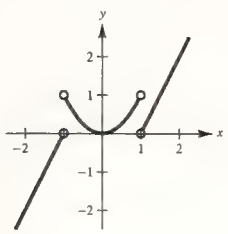
54.

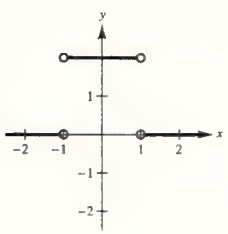




(any vertical translate of f will do)

56.





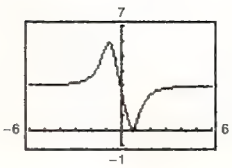
(any vertical translate of the 3 segments of f will do)

58. If $f'(x) = 2$ in $[-5, 5]$, then $f(x) = 2x + 3$ and $f(2) = 7$ is the least possible value of $f(2)$. If $f'(x) = 4$ in $[-5, 5]$, then $f(x) = 4x + 3$ and $f(2) = 11$ is the greatest possible value of $f(2)$.

60. $g(x) = \dfrac{3x^4 - 5x + 3}{x^4 + 1}$

Vertical asymptote: none

Horizontal asymptote: $y = 3$

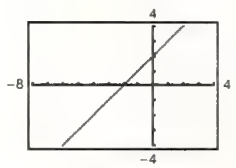


The graph crosses the horizontal asymptote $y = 3$. If a function has a vertical asymptote at $x = c$, the graph would not cross it since $f(c)$ is undefined.

62. $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

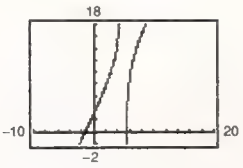
$$= \frac{(x + 2)(x - 1)}{x - 1} = \begin{cases} x + 2, & \text{if } x \neq 1 \\ \text{Undefined}, & \text{if } x = 1 \end{cases}$$

The rational function is not reduced to lowest terms.



hole at $(1, 3)$

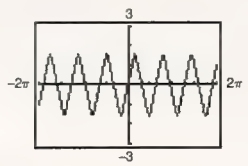
64. $g(x) = \dfrac{2x^2 - 8x - 15}{x - 5} = 2x + 2 - \dfrac{5}{x - 5}$



The graph appears to approach the slant asymptote $y = 2x + 2$.

66. $f(x) = \tan(\sin \pi x)$

(a)



(b) $f(-x) = \tan(\sin(-\pi x)) = \tan(-\sin \pi x)$

$= -\tan(\sin \pi x) = -f(x)$

Symmetry with repect to the origin

(c) Periodic with period 2

(d) On $(-1, 1)$, there is a relative maximum at $\left(\frac{1}{2}, \tan 1\right)$ and a relative minimum at $\left(-\frac{1}{2}, -\tan 1\right)$.

(e) On $(0, 1)$, the graph of f is concave downward.

68. Vertical asymptote: $x = -3$

Horizontal asymptote: none

$$y = \frac{x^2}{x + 3}$$

70. Vertical asymptote: $x = 0$

Slant asymptote: $y = -x$

$$y = -x + \frac{1}{x} = \frac{1 - x^2}{x}$$

72. $f(x) = \frac{1}{2}(ax)^2 - (ax) = \frac{1}{2}(ax)(ax - 2), a \neq 0$

$f'(x) = a^2x - a = a(ax - 1) = 0$ when $x = \frac{1}{a}$.

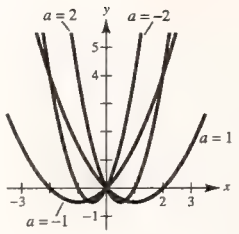
$f''(x) = a^2 > 0$ for all x.

(a) Intercepts: $(0, 0), \left(\frac{2}{a}, 0\right)$

Relative minimum: $\left(\frac{1}{a}, -\frac{1}{2}\right)$

Points of inflection: none

(b)



74. Tangent line at P: $y - y_0 = f'(x_0)(x - x_0)$

(a) Let $y = 0$: $-y_0 = f'(x_0)(x - x_0)$

$$f'(x_0)x = x_0f'(x_0) - y_0$$

$$x = x_0 - \frac{y_0}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

x-intercept: $\left(x_0 - \frac{f(x_0)}{f'(x_0)}, 0\right)$

(b) Let $x = 0$: $y - y_0 = f'(x_0)(-x_0)$

$$y = y_0 - x_0f'(x_0)$$

$$y = f(x_0) - x_0f'(x_0)$$

y-intercept: $(0, f(x_0) - x_0f'(x_0))$

(c) Normal line: $y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$

Let $y = 0$: $-y_0 = -\frac{1}{f'(x_0)}(x - x_0)$

$$-y_0f'(x_0) = -x + x_0$$

$$x = x_0 + y_0f'(x_0) = x_0 + f(x_0)f'(x_0)$$

x-intercept: $(x_0 + f(x_0)f'(x_0), 0)$

(d) Let $x = 0$: $y - y_0 = \frac{-1}{f'(x_0)}(-x_0)$

$$y = y_0 + \frac{x_0}{f'(x_0)}$$

y-intercept: $\left(0, y_0 + \frac{x_0}{f'(x_0)}\right)$

(e) $|BC| = \left|x_0 - \frac{f(x_0)}{f'(x_0)} - x_0\right| = \left|\frac{f(x_0)}{f'(x_0)}\right|$

(f) $|PC|^2 = {y_0}^2 + \left(\frac{f(x_0)}{f'(x_0)}\right) = \frac{f(x_0)^2f'(x_0)^2 + f(x_0)^2}{f'(x_0)^2}$

$$|PC|^2 = \left|\frac{f(x_0)\sqrt{1 + [f'(x_0)]^2}}{f'(x_0)}\right|$$

(g) $|AB| = |x_0 - (x_0 + f(x_0)f'(x_0))| = |f(x_0)f'(x_0)|$

(h) $|AP|^2 = f(x_0)^2f'(x_0)^2 + {y_0}^2$

$$|AP| = |f(x_0)|\sqrt{1 + [f'(x_0)]^2}$$

Section 3.7 Optimization Problems

2. Let x and y be two positive numbers such that $x + y = S$.

$$P = xy = x(S - x) = Sx - x^2$$

$$\frac{dP}{dx} = S - 2x = 0 \text{ when } x = \frac{S}{2}.$$

$$\frac{d^2P}{dx^2} = -2 < 0 \text{ when } x = \frac{S}{2}.$$

P is a maximum when $x = y = S/2$.

4. Let x and y be two positive numbers such that $xy = 192$.

$$S = x + 3y = \frac{192}{y} + 3y$$

$$\frac{dS}{dy} = 3 - \frac{192}{y^2} = 0 \text{ when } y = 8.$$

$$\frac{d^2S}{dy^2} = \frac{384}{y^3} > 0 \text{ when } y = 8.$$

S is minimum when $y = 8$ and $x = 24$.

6. Let x and y be two positive numbers such that $x + 2y = 100$.

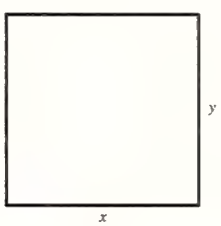
$$P = xy = y(100 - 2y) = 100y - 2y^2$$

$$\frac{dP}{dy} = 100 - 4y = 0 \text{ when } y = 25.$$

$$\frac{d^2P}{dy^2} = -4 < 0 \text{ when } y = 25.$$

P is a maximum when $x = 50$ and $y = 25$.

8. Let x be the length and y the width of the rectangle.



$$2x + 2y = P$$

$$y = \frac{P - 2x}{2} = \frac{P}{2} - x$$

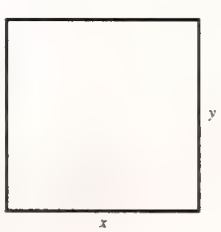
$$A = xy = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2$$

$$\frac{dA}{dx} = \frac{P}{2} - 2x = 0 \text{ when } x = \frac{P}{4}.$$

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = \frac{P}{4}.$$

A is maximum when $x = y = P/4$ units. (A square!)

10. Let x be the length and y the width of the rectangle.



$$xy = A$$

$$y = \frac{A}{x}$$

$$P = 2x + 2y = 2x + 2\left(\frac{A}{x}\right) = 2x + \frac{2A}{x}$$

$$\frac{dP}{dx} = 2 - \frac{2A}{x^2} = 0 \text{ when } x = \sqrt{A}.$$

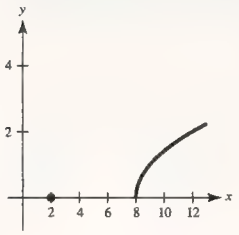
$$\frac{d^2P}{dx^2} = \frac{4A}{x^3} > 0 \text{ when } x = \sqrt{A}.$$

P is minimum when $x = y = \sqrt{A}$ centimeters.

(A square!)

12. $f(x) = \sqrt{x - 8}$, $(2, 0)$

From the graph, it is clear that $(8, 0)$ is the closest point on the graph of f to $(2, 0)$.



14. $f(x) = (x + 1)^2$, $(5, 3)$

$$d = \sqrt{(x - 5)^2 + [(x + 1)^2 - 3]^2}$$
$$= \sqrt{x^2 - 10x + 25 + (x^2 + 2x - 2)^2}$$
$$= \sqrt{x^2 - 10x + 25 + x^4 + 4x^3 - 8x + 4}$$
$$= \sqrt{x^4 + 4x^3 + x^2 - 18x + 29}$$

Since d is smallest when the expression inside the radical is smallest, you need to find the critical numbers of

$$g(x) = x^4 + 4x^3 + x^2 - 18x + 29$$
$$g'(x) = 4x^3 + 12x^2 + 2x - 18$$
$$= 2(x - 1)(2x^2 + 8x + 9)$$

By the First Derivative Test, $x = 1$ yields a minimum. Hence, $(1, 4)$ is closest to $(5, 3)$.

16. $F = \dfrac{v}{22 + 0.02v^2}$

$$\frac{dF}{dv} = \frac{22 - 0.02v^2}{(22 + 0.02v^2)^2}$$

$= 0$ when $v = \sqrt{1100} \approx 33.166$.

By the First Derivative Test, the flow rate on the road is maximized when $v \approx 33$ mph.

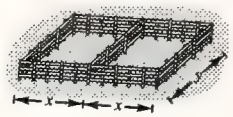
18. $4x + 3y = 200$ is the perimeter. (see figure)

$$A = 2xy = 2x\left(\frac{200 - 4x}{3}\right) = \frac{8}{3}(50x - x^2)$$

$$\frac{dA}{dx} = \frac{8}{3}(50 - 2x) = 0 \text{ when } x = 25.$$

$$\frac{d^2A}{dx^2} = -\frac{16}{3} < 0 \text{ when } x = 25.$$

A is a maximum when $x = 25$ feet and $y = \frac{100}{3}$ feet.

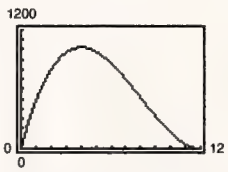


20. (a)

Height, x	Length & Width	Volume
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$
3	$24 - 2(3)$	$3[24 - 2(3)]^2 = 972$
4	$24 - 2(4)$	$4[24 - 2(4)]^2 = 1024$
5	$24 - 2(5)$	$5[24 - 2(5)]^2 = 980$
6	$24 - 2(6)$	$6[24 - 2(6)]^2 = 864$

(b) $V = x(24 - 2x)^2$, $0 < x < 12$

(d)



The maximum volume seems to be 1024.

(c) $$\frac{dV}{dx} = 2x(24 - 2x)(-2) + (24 - 2x)^2 = (24 - 2x)(24 - 6x)$$

$$= 12(12 - x)(4 - x) = 0 \text{ when } x = 12, 4 \text{ (12 is not in the domain).}$$

$$\frac{d^2V}{dx^2} = 12(2x - 16)$$

$$\frac{d^2V}{dx^2} < 0 \text{ when } x = 4.$$

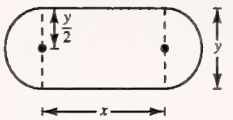
When $x = 4$, $V = 1024$ is maximum.

22. (a) $P = 2x + 2\pi r$

$= 2x + 2\pi\left(\frac{y}{2}\right)$

$= 2x + \pi y = 200$

$\Rightarrow y = \frac{200 - 2x}{\pi} = \frac{2}{\pi}(100 - x)$



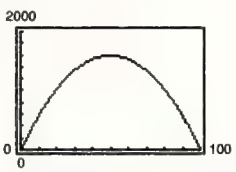
(b)

Length, x	Width, y	Area, xy
10	$\frac{2}{\pi}(100 - 10)$	$(10)\frac{2}{\pi}(100 - 10) \approx 573$
20	$\frac{2}{\pi}(100 - 20)$	$(20)\frac{2}{\pi}(100 - 20) \approx 1019$
30	$\frac{2}{\pi}(100 - 30)$	$(30)\frac{2}{\pi}(100 - 30) \approx 1337$
40	$\frac{2}{\pi}(100 - 40)$	$(40)\frac{2}{\pi}(100 - 40) \approx 1528$
50	$\frac{2}{\pi}(100 - 50)$	$(50)\frac{2}{\pi}(100 - 50) \approx 1592$
60	$\frac{2}{\pi}(100 - 60)$	$(60)\frac{2}{\pi}(100 - 60) = 1528$

The maximum area of the rectangle is approximately 1592 m^2.

(c) $A = xy = x\frac{2}{\pi}(100 - x) = \frac{2}{\pi}(100x - x^2)$

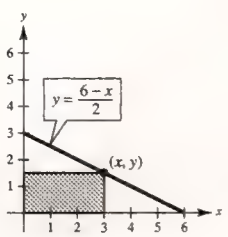
(e)



Maximum area is approximately

1591.55 m^2 ($x = 50$ m).

24. You can see from the figure that $A = xy$ and $y = \frac{6 - x}{2}$.



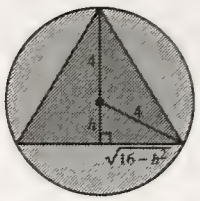
$A = x\left(\frac{6 - x}{2}\right) = \frac{1}{2}(6x - x^2)$.

$\frac{dA}{dx} = \frac{1}{2}(6 - 2x) = 0$ when $x = 3$.

$\frac{d^2A}{dx^2} = -1 < 0$ when $x = 3$.

A is a maximum when $x = 3$ and $y = 3/2$.

26. (a) $A = \frac{1}{2}$ base $\times$ height



$$= \frac{1}{2}(2\sqrt{16-h^2})(4+h)$$

$$= \sqrt{16-h^2}(4+h)$$

$$\frac{dA}{dh} = \frac{1}{2}(16-h^2)^{-1/2}(-2h)(4+h) + (16-h^2)^{1/2}$$

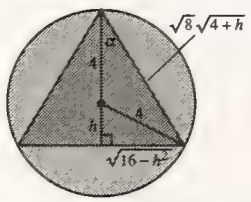
$$= (16-h^2)^{-1/2}[-h(4+h) + (16-h^2)]$$

$$= \frac{-2(h^2+2h-8)}{\sqrt{16-h^2}} = \frac{-2(h+4)(h-2)}{\sqrt{16-h^2}}$$

$\frac{dA}{dh} = 0$ when $h = 2$, which is a maximum by the First Derivative Test.

Hence, the sides are $2\sqrt{16-h^2} = 4\sqrt{3}$, an equilateral triangle. Area $= 12\sqrt{3}$ sq. units.

(b) $\cos\alpha = \dfrac{4+h}{\sqrt{8}\sqrt{4+h}} = \dfrac{\sqrt{4+h}}{\sqrt{8}}$



$$\tan\alpha = \frac{\sqrt{16-h^2}}{4+h}$$

$$\text{Area} = 2\left(\frac{1}{2}\right)\left(\sqrt{16-h^2}\right)(4+h)$$

$$= (4+h)^2\tan\alpha$$

$$= 64\cos^4\alpha\tan\alpha$$

$$A'(\alpha) = 64[\cos^4\alpha\sec^2\alpha + 4\cos^3(-\sin\alpha)\tan\alpha] = 0$$

$$\Rightarrow \cos^4\alpha\sec^2\alpha = 4\cos^3\alpha\sin\alpha\tan\alpha$$

$$1 = 4\cos\alpha\sin\alpha\tan\alpha$$

$$\frac{1}{4} = \sin^2\alpha$$

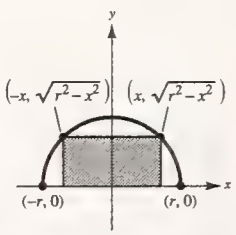
$$\sin\alpha = \frac{1}{2} \Rightarrow \alpha = 30° \text{ and } A = 12\sqrt{3}.$$

(c) Equilateral triangle

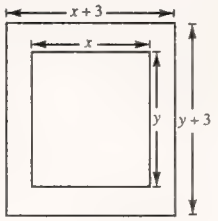
28. $A = 2xy = 2x\sqrt{r^2-x^2}$ (see figure)

$$\frac{dA}{dx} = \frac{2(r^2-2x^2)}{\sqrt{r^2-x^2}} = 0 \text{ when } x = \frac{\sqrt{2}r}{2}.$$

By the First Derivative Test, A is maximum when the rectangle has dimensions $\sqrt{2}r$ by $(\sqrt{2}r)/2$.



30. $xy = 36 \Rightarrow y = \dfrac{36}{x}$



$$A = (x + 3)(y + 3) = (x + 3)\left(\frac{36}{x} + 3\right)$$

$$= 36 + \frac{108}{x} + 3x + 9$$

$$\frac{dA}{dx} = \frac{-108}{x^2} + 3 = 0 \Rightarrow 3x^2 = 108 \Rightarrow x = 6, y = 6$$

Dimensions: 9×9

32. $V = \pi r^2 h = V_0$ cubic units or $h = \dfrac{V_0}{\pi r^2}$

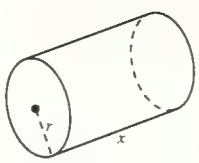
$$S = 2\pi r^2 + 2\pi rh = 2\left(\pi r^2 + \frac{V_0}{r}\right)$$

$$\frac{dS}{dr} = 2\left(2\pi r - \frac{V_0}{r^2}\right) = 0 \text{ when } r = \sqrt[3]{\frac{V_0}{2\pi}} \text{ units.}$$

$$h = \frac{V_0}{\pi\left(\sqrt[3]{V_0/2\pi}\right)^2} = \frac{V_0(2\pi)^{2/3}}{\pi {V_0}^{2/3}} = \frac{2{V_0}^{1/3}}{(2\pi)^{1/3}} = 2r$$

By the First Derivative Test, this will yield the minimum surface area.

34. $V = \pi r^2 x$



$x + 2\pi r = 108 \Rightarrow x = 108 - 2\pi r$ (see figure)

$$V = \pi r^2(108 - 2\pi r) = \pi(108r^2 - 2\pi r^3)$$

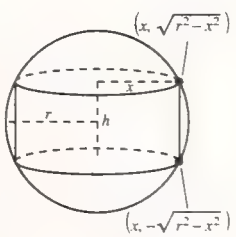
$$\frac{dV}{dr} = \pi(216r - 6\pi r^2) = 6\pi r(36 - \pi r)$$

$$= 0 \text{ when } r = \frac{36}{\pi} \text{ and } x = 36.$$

$$\frac{d^2V}{dr^2} = \pi(216 - 12\pi r) < 0 \text{ when } r = \frac{36}{\pi}.$$

Volume is maximum when $x = 36$ inches and $r = 36/\pi \approx 11.459$ inches.

36. $V = \pi x^2 h = \pi x^2\left(2\sqrt{r^2 - x^2}\right) = 2\pi x^2\sqrt{r^2 - x^2}$ (see figure)



$$\frac{dV}{dx} = 2\pi\left[x^2\left(\frac{1}{2}\right)(r^2 - x^2)^{-1/2}(-2x) + 2x\sqrt{r^2 - x^2}\right]$$

$$= \frac{2\pi x}{\sqrt{r^2 - x^2}}(2r^2 - 3x^2)$$

$$= 0 \text{ when } x = 0 \text{ and } x^2 = \frac{2r^2}{3} \Rightarrow x = \frac{\sqrt{6}r}{3}.$$

By the First Derivative Test, the volume is a maximum when

$$x = \frac{\sqrt{6}r}{3} \text{ and } h = \frac{2r}{\sqrt{3}}.$$

Thus, the maximum volume is

$$V = \pi\left(\frac{2}{3}r^2\right)\left(\frac{2r}{\sqrt{3}}\right) = \frac{4\pi r^3}{3\sqrt{3}}.$$

38. No. The volume will change because the shape of the container changes when squeezed.

40. $V = 3000 = \frac{4}{3}\pi r^3 + \pi r^2 h$

$$h = \frac{3000}{\pi r^2} - \frac{4}{3}r$$

Let k = cost per square foot of the surface area of the sides, then $2k$ = cost per square foot of the hemispherical ends.

$$C = 2k(4\pi r^2) + k(2\pi rh) = k\left[8\pi r^2 + 2\pi r\left(\frac{3000}{\pi r^2} - \frac{4}{3}r\right)\right] = k\left[\frac{16}{3}\pi r^2 + \frac{6000}{r}\right]$$

$$\frac{dC}{dr} = k\left[\frac{32}{3}\pi r - \frac{6000}{r^2}\right] = 0 \text{ when } r = \sqrt[3]{\frac{1125}{2\pi}} \approx 5.636 \text{ feet and } h \approx 22.545 \text{ feet.}$$

By the Second Derivative Test, we have

$$\frac{d^2C}{dr^2} = k\left[\frac{32}{3}\pi + \frac{12{,}000}{r^3}\right] > 0 \text{ when } r = \sqrt[3]{\frac{1125}{2\pi}}.$$

Therefore, these dimensions will produce a minimum cost.

42. (a) Let x be the side of the triangle and y the side of the square.

$$A = \frac{3}{4}\left(\cot\frac{\pi}{3}\right)x^2 + \frac{4}{4}\left(\cot\frac{\pi}{4}\right)y^2 \text{ where } 3x + 4y = 20$$

$$= \frac{\sqrt{3}}{4}x^2 + \left(5 - \frac{3}{4}x\right)^2,\ 0 \le x \le \frac{20}{3}.$$

$$A' = \frac{\sqrt{3}}{2}x + 2\left(5 - \frac{3}{4}x\right)\left(-\frac{3}{4}\right) = 0$$

$$x = \frac{60}{4\sqrt{3} + 9}$$

When $x = 0, A = 25$, when $x = 60/(4\sqrt{3} + 9)$, $A \approx 10.847$, and when $x = 20/3, A \approx 19.245$. Area is maximum when all 20 feet are used on the square.

(b) Let x be the side of the square and y the side of the pentagon.

$$A = \frac{4}{4}\left(\cot\frac{\pi}{4}\right)x^2 + \frac{5}{4}\left(\cot\frac{\pi}{5}\right)y^2 \text{ where } 4x + 5y = 20$$

$$= x^2 + 1.7204774\left(4 - \frac{4}{5}x\right)^2,\ 0 \le x \le 5.$$

$$A' = 2x - 2.75276384\left(4 - \frac{4}{5}x\right) = 0$$

$$x \approx 2.62$$

When $x = 0, A \approx 27.528$, when $x \approx 2.62, A \approx 13.102$, and when $x = 5, A \approx 25$. Area is maximum when all 20 feet are used on the pentagon.

(c) Let x be the side of the pentagon and y the side of the hexagon.

$$A = \frac{5}{4}\left(\cot\frac{\pi}{5}\right)x^2 + \frac{6}{4}\left(\cot\frac{\pi}{6}\right)y^2 \text{ where } 5x + 6y = 20$$

$$= \frac{5}{4}\left(\cot\frac{\pi}{5}\right)x^2 + \frac{3}{2}(\sqrt{3})\left(\frac{20 - 5x}{6}\right)^2,\ 0 \le x \le 4.$$

$$A' = \frac{5}{2}\left(\cot\frac{\pi}{5}\right)x + 3\sqrt{3}\left(-\frac{5}{6}\right)\left(\frac{20 - 5x}{6}\right) = 0$$

$$x \approx 2.0475$$

When $x = 0, A \approx 28.868$, when $x \approx 2.0475$, $A \approx 14.091$, and when $x = 4, A \approx 27.528$. Area is maximum when all 20 feet are used on the hexagon

(d) Let x be the side of the hexagon and r the radius of the circle.

$$A = \frac{6}{4}\left(\cot\frac{\pi}{6}\right)x^2 + \pi r^2 \text{ where } 6x + 2\pi r = 20$$

$$= \frac{3\sqrt{3}}{2}x^2 + \pi\left(\frac{10}{\pi} - \frac{3x}{\pi}\right)^2,\ 0 \le x \le \frac{10}{3}.$$

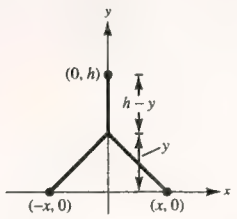
$$A' = 3\sqrt{3} - 6\left(\frac{10}{\pi} - \frac{3x}{\pi}\right) = 0$$

$$x \approx 1.748$$

When $x = 0, A \approx 31.831$, when $x \approx 1.748, A \approx 15.138$, and when $x = 10/3, A \approx 28.868$. Area is maximum when all 20 feet are used on the circle.

In general, using all of the wire for the figure with more sides will enclose the most area.

44. Let A be the amount of the power line.



$$A = h - y + 2\sqrt{x^2 + y^2}$$

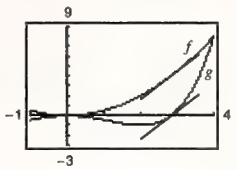
$$\frac{dA}{dy} = -1 + \frac{2y}{\sqrt{x^2 + y^2}} = 0 \text{ when } y = \frac{x}{\sqrt{3}}.$$

$$\frac{d^2A}{dy^2} = \frac{2x^2}{(x^2 + y^2)^{3/2}} > 0 \text{ for } y = \frac{x}{\sqrt{3}}.$$

The amount of power line is minimum when $y = x/\sqrt{3}$.

46. $f(x) = \frac{1}{2}x^2 \qquad g(x) = \frac{1}{16}x^4 - \frac{1}{2}x^2$ on $[0, 4]$

(a)



(b) $d(x) = f(x) - g(x) = \frac{1}{2}x^2 - \left(\frac{1}{16}x^4 - \frac{1}{2}x^2\right) = x^2 - \frac{1}{16}x^4$

$d'(x) = 2x - \frac{1}{4}x^3 = 0 \Rightarrow 8x = x^3$

$\Rightarrow x = 0, 2\sqrt{2}$ (in $[0, 4]$)

The maximum distance is $d = 4$ when $x = 2\sqrt{2}$.

(c) $f'(x) = x$, Tangent line at $\left(2\sqrt{2}, 4\right)$ is

$$y - 4 = 2\sqrt{2}\left(x - 2\sqrt{2}\right)$$

$$y = 2\sqrt{2}x - 4.$$

$g'(x) = \frac{1}{4}x^3 - x$, Tangent line at $\left(2\sqrt{2}, 0\right)$ is

$$y - 0 = \left(\tfrac{1}{4}\left(2\sqrt{2}\right)^3 - 2\sqrt{2}\right)\left(x - 2\sqrt{2}\right)$$

$$y = 2\sqrt{2}x - 8.$$

The tangent lines are parallel and 4 vertical units apart.

(d) The tangent lines will be parallel. If $d(x) = f(x) - g(x)$, then $d'(x) = 0 = f'(x) - g'(x)$ implies that $f'(x) = g'(x)$ at the point x where the distance is maximum.

48. Let F be the illumination at point P which is x units from source 1.

$$F = \frac{kI_1}{x^2} + \frac{kI_2}{(d - x)^2}$$

$$\frac{dF}{dx} = \frac{-2kI_1}{x^3} + \frac{2kI_2}{(d - x)^3} = 0 \text{ when } \frac{2kI_1}{x^3} = \frac{2kI_2}{(d - x)^3}.$$

$$\frac{\sqrt[3]{I_1}}{\sqrt[3]{I_2}} = \frac{x}{d - x}$$

$$(d - x)\sqrt[3]{I_1} = x\sqrt[3]{I_2}$$

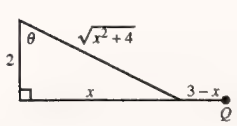
$$d\sqrt[3]{I_1} = x\left(\sqrt[3]{I_1} + \sqrt[3]{I_2}\right)$$

$$x = \frac{d\sqrt[3]{I_1}}{\sqrt[3]{I_1} + \sqrt[3]{I_2}}$$

$$\frac{d^2F}{dx^2} = \frac{6kI_1}{x^4} + \frac{6kI_2}{(d - x)^4} > 0 \text{ when } x = \frac{d\sqrt[3]{I_1}}{\sqrt[3]{I_1} + \sqrt[3]{I_2}}.$$

This is the minimum point.

50. (a) $T = \dfrac{\sqrt{x^2 + 4}}{2} + \dfrac{(3 - x)}{4}$



(b) $$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2 + 4}} - \frac{1}{4} = 0$$

$$\frac{x}{\sqrt{x^2 + 4}} = \frac{1}{2}$$

$$2x^2 = x^2 + 4$$

$$x^2 = 4$$

$$x = 2$$

$$T(2) = \sqrt{2} + \frac{1}{4} \text{ hours}$$

—CONTINUED—

50. —CONTINUED—

(c) $$T = \frac{\sqrt{x^2+4}}{v_1} + \frac{(3-x)}{v_2}$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2+4}} - \frac{1}{v_2} = 0$$

$$\frac{x}{\sqrt{x^2+4}} = \frac{v_1}{v_2}$$

$$\sin\theta = \frac{v_1}{v_2}$$

θ depends on $\frac{v_1}{v_2}$ only.

(d) $\text{Cost} = \sqrt{x^2+4}\,C_1 + (3-x)C_2$

$$= \frac{\sqrt{x^2+4}}{(1/C_1)} + \frac{(3-x)}{(1/C_2)}$$

From above, $\sin\theta = \dfrac{1/C_1}{1/C_2} = \dfrac{C_2}{C_1}$

52. $$T = \frac{\sqrt{x^2+d_1^2}}{v_1} + \frac{\sqrt{d_2^2+(a-x)^2}}{v_2}$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2+d_1^2}} + \frac{x-a}{v_2\sqrt{d_2^2+(a-x)^2}} = 0$$

Since

$$\frac{x}{\sqrt{x^2+d_1^2}} = \sin\theta_1 \text{ and } \frac{x-a}{\sqrt{d_2^2+(a-x)^2}} = -\sin\theta_2$$

we have

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \Rightarrow \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

Since

$$\frac{d^2T}{dx^2} = \frac{d_1^2}{v_1(x^2+d_1^2)^{3/2}} + \frac{d_2^2}{v_2[d_2^2+(a-x)^2]^{3/2}} > 0$$

this condition yields a minimum time.

54. $$C(x) = 2k\sqrt{x^2+4} + k(4-x)$$

$$C'(x) = \frac{2xk}{\sqrt{x^2+4}} - k = 0$$

$$2x = \sqrt{x^2+4}$$

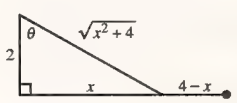
$$4x^2 = x^2+4$$

$$3x^2 = 4$$

$$x = \frac{2}{\sqrt{3}}$$

Or, use Exercise 50(d): $\sin\theta = \dfrac{C_2}{C_1} = \dfrac{1}{2} \Rightarrow \theta = 30°$.

Thus, $x = \dfrac{2}{\sqrt{3}}$.



56. $$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\sqrt{144-r^2}$$

$$\frac{dV}{dr} = \frac{1}{3}\pi\left[r^2\left(\frac{1}{2}\right)(144-r^2)^{-1/2}(-2r) + 2r\sqrt{144-r^2}\right]$$

$$= \frac{1}{3}\pi\left[\frac{288r-3r^3}{\sqrt{144-r^2}}\right] = \pi\left[\frac{r(96-r^2)}{\sqrt{144-r^2}}\right] = 0 \text{ when } r = 0, 4\sqrt{6}.$$

By the First Derivative Test, V is maximum when $r = 4\sqrt{6}$ and $h = 4\sqrt{3}$.

Area of circle: $A = \pi(12)^2 = 144\pi$

Lateral surface area of cone: $S = \pi(4\sqrt{6})\sqrt{(4\sqrt{6})^2+(4\sqrt{3})^2} = 48\sqrt{6}\pi$

Area of sector: $144\pi - 48\sqrt{6}\pi = \dfrac{1}{2}\theta r^2 = 72\theta$

$$\theta = \frac{144\pi - 48\sqrt{6}\pi}{72} = \frac{2\pi}{3}(3-\sqrt{6}) \approx 1.153 \text{ radians or } 66°$$

58. Let d be the amount deposited in the bank, i be the interest rate paid by the bank, and P be the profit.

$P = (0.12)d - id$

$d = ki^2$ (since d is proportional to i^2)

$P = (0.12)(ki^2) - i(ki^2) = k(0.12i^2 - i^3)$

$\frac{dP}{di} = k(0.24i - 3i^2) = 0$ when $i = \frac{0.24}{3} = 0.08$.

$\frac{d^2P}{di^2} = k(0.24 - 6i) < 0$ when $i = 0.08$ (**Note:** $k > 0$).

The profit is a maximum when $i = 8\%$.

60. $P = -\frac{1}{10}s^3 + 6s^2 + 400$

(a) $\frac{dP}{ds} = -\frac{3}{10}s^2 + 12s = -\frac{3}{10}s(s - 40) = 0$ when $x = 0$, $s = 40$.

$\frac{d^2P}{ds^2} = -\frac{3}{5}s + 12$

$\frac{d^2P}{ds^2}(0) > 0 \Rightarrow s = 0$ yields a minimum.

$\frac{d^2P}{ds^2}(40) < 0 \Rightarrow s = 40$ yields a maximum.

The maximum profit occurs when $s = 40$, which corresponds to \$40,000 ($P = \$3{,}600{,}000$).

(b) $\frac{d^2P}{ds^2} = -\frac{3}{5}s + 12 = 0$ when $s = 20$.

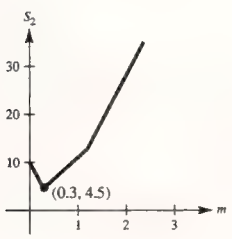
The point of diminishing returns occurs when $s = 20$, which corresonds to \$20,000 being spent on advertising.

62. $S_2 = |4m - 1| + |5m - 6| + |10m - 3|$

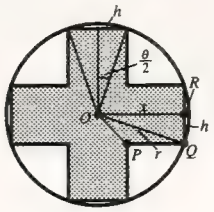
Using a graphing utility, you can see that the minimum occurs when $m = 0.3$.

Line $y = 0.3x$

$S_2 = |4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3| = 4.7$ mi.



64. (a) Label the figure so that $r^2 = x^2 + h^2$.
Then, the area A is 8 times the area of the region given by $OPQR$:



$$A = 8\left[\frac{1}{2}h^2 + (x - h)h\right]$$

$$= 8\left[\frac{1}{2}(r^2 - x^2) + \left(x - \sqrt{r^2 - x^2}\right)\sqrt{r^2 - x^2}\right]$$

$$= 8x\sqrt{r^2 - x^2} + 4x^2 - 4r^2$$

$$A'(x) = 8\sqrt{r^2 - x^2} - \frac{8x^2}{\sqrt{r^2 - x^2}} + 8x = 0$$

$$\frac{8x^2}{\sqrt{r^2 - x^2}} = 8x + 8\sqrt{r^2 - x^2}$$

$$x^2 = x\sqrt{r^2 - x^2} + (r^2 - x^2)$$

$$2x^2 - r^2 = x\sqrt{r^2 - x^2}$$

$$4x^4 - 4x^2r^2 + r^4 = x^2(r^2 - x^2)$$

$5x^4 - 5x^2r^2 + r^4 = 0$ Quadratic in x^2.

$$x^2 = \frac{5r^2 \pm \sqrt{25r^4 - 20r^4}}{10} = \frac{r^2}{10}\left[5 \pm \sqrt{5}\right].$$

Take positive value.

$$x = r\sqrt{\frac{5 + \sqrt{5}}{10}} \approx 0.85065r \quad \text{Critical number}$$

(b) Note that $\sin\frac{\theta}{2} = \frac{h}{r}$ and $\cos\frac{\theta}{2} = \frac{x}{r}$. The area A of the cross equals the sum of two large rectangles minus the common square in the middle.

$$A = 2(2x)(2h) - 4h^2 = 8xh - 4h^2$$

$$= 8r^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2} - 4r^2\sin^2\frac{\theta}{2}$$

$$= 4r^2\left(\sin\theta - \sin^2\frac{\theta}{2}\right)$$

$$A'(\theta) = 4r^2\left(\cos\theta - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = 0$$

$$\cos\theta = \sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\sin\theta$$

$$\tan\theta = 2$$

$$\theta = \arctan(2) \approx 1.10715 \quad \text{or} \quad 63.4°$$

(c) Note that $x^2 = \frac{r^2}{10}(5 + \sqrt{5})$ and $r^2 - x^2 = \frac{r^2}{10}(5 - \sqrt{5})$.

$$A(x) = 8x\sqrt{r^2 - x^2} + 4x^2 - 4r^2$$

$$= 8\left[\frac{r^2}{10}(5 + \sqrt{5})\frac{r^2}{10}(5 - \sqrt{5})\right]^{1/2} + 4\frac{r^2}{10}(5 + \sqrt{5}) - 4r^2$$

$$= 8\left[\frac{r^4}{10}(20)\right]^{1/2} + 2r^2 + \frac{2}{5}\sqrt{5}r^2 - 4r^2$$

$$= \frac{8}{5}r^2\sqrt{5} - 2r^2 + \frac{2\sqrt{5}}{5}r^2$$

$$= 2r^2\left[\frac{4}{5}\sqrt{5} - 1 + \frac{\sqrt{5}}{5}\right] = 2r^2\left(\sqrt{5} - 1\right)$$

Using the angle approach, note that $\tan\theta = 2$, $\sin\theta = \frac{2}{\sqrt{5}}$ and $\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1 - \cos\theta) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)$.

Thus, $A(\theta) = 4r^2\left(\sin\theta - \sin^2\frac{\theta}{2}\right)$

$$= 4r^2\left(\frac{2}{\sqrt{5}} - \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\right)$$

$$= \frac{4r^2\left(\sqrt{5} - 1\right)}{2} = 2r^2\left(\sqrt{5} - 1\right)$$

Section 3.8 Newton's Method

2. $f(x) = 2x^2 - 3$

$f'(x) = 4x$

$x_1 = 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1	-1	4	$-\frac{1}{4}$	$\frac{5}{4}$
2	$\frac{5}{4} = 1.25$	0.125	5.0	0.025	1.225

4. $f(x) = \tan x$

$f'(x) = \sec^2 x$

$x_1 = 0.1$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.1000	0.1003	1.0101	0.0993	0.0007
2	0.0007	0.0007	1.0000	0.0007	0.0000

6. $f(x) = x^5 + x - 1$

$f'(x) = 5x^4 + 1$

Approximation of the zero of f is 0.755.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	-0.4688	1.3125	-0.3571	0.8571
2	0.8571	0.3196	3.6983	0.0864	0.7707
3	0.7707	0.0426	2.7641	0.0154	0.7553
4	0.7553	0.0011	2.6272	0.0004	0.7549

8. $f(x) = x - 2\sqrt{x+1}$

$f'(x) = 1 - \dfrac{1}{\sqrt{x+1}}$

Approximation of the zero of f is 4.8284.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	5	0.1010	0.5918	0.1707	4.8293
2	4.8293	0.0005	0.5858	.00085	4.8284

10. $f(x) = 1 - 2x^3$

$f'(x) = -6x^2$

Approximation of the zero of f is 0.7937.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1	-1	-6	0.1667	0.8333
2	0.8333	-0.1573	-4.1663	0.0378	0.7955
3	0.7955	-0.0068	-3.7969	0.0018	0.7937
4	0.7937	0.0000	-3.7798	0.0000	0.7937

12. $f(x) = \frac{1}{2}x^4 - 3x - 3$

$f'(x) = 2x^3 - 3$

Approximation of the zero of f is -0.8937.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1	0.5	-5	-0.1	-0.9
2	-0.9	0.0281	-4.458	-0.0063	-0.8937
3	-0.8937	0.0001	-4.4276	0.0000	-0.8937

Approximation of the zero of f is 2.0720.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	2	-1	13	-0.0769	2.0769
2	2.0769	0.0725	14.9175	0.0049	2.0720
3	2.0720	-0.0003	14.7910	0.0000	2.0720

14. $f(x) = x^3 - \cos x$

$f'(x) = 3x^2 + \sin x$

Approximation of the zero of f is 0.866.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.9000	0.1074	3.2133	0.0334	0.8666
2	0.8666	0.0034	3.0151	0.0011	0.8655
3	0.8655	0.0001	3.0087	0.0000	0.8655

16. $h(x) = f(x) - g(x) = 3 - x - \frac{1}{x^2 + 1}$

$h'(x) = -1 + \frac{2x}{(x^2 + 1)^2}$

Point of intersection of the graphs of f and g occurs when $x \approx 2.893$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	2.9000	-0.0063	-0.9345	0.0067	2.8933
2	2.8933	0.0000	-0.9341	0.0000	2.8933

18. $h(x) = f(x) - g(x) = x^2 - \cos x$

$h'(x) = 2x + \sin x$

One point of intersection of the graphs of f and g occurs when $x \approx 0.824$. Since $f(x) = x^2$ and $g(x) = \cos x$ are both symmetric with respect to the y-axis, the other point of intersection occurs when $x \approx -0.824$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	0.8000	-0.0567	2.3174	-0.0245	0.8245
2	0.8245	0.0009	2.3832	0.0004	0.8241

20. $f(x) = x^n - a = 0$

$f'(x) = nx^{n-1}$

$$x_{i+1} = x_i - \frac{x_i^n - a}{nx_i^{n-1}}$$

$$= \frac{nx_i^n - x_i^n + a}{nx_i^{n-1}} = \frac{(n-1)x_i^n + a}{nx_i^{n-1}}$$

22. $x_{i+1} = \dfrac{x_i^2 + 5}{2x_i}$

i	1	2	3	4
x_i	2.0000	2.2500	2.2361	2.2361

$\sqrt{5} \approx 2.236$

24. $x_{i+1} = \dfrac{2x_i^3 + 15}{3x_i^2}$

i	1	2	3	4
x_i	2.5000	2.4667	2.4662	2.4662

$\sqrt[3]{15} \approx 2.466$

26. $f(x) = \tan x$

$f'(x) = \sec^2 x$

Approximation of the zero: 3.142

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	3.0000	−0.1425	1.0203	−0.1397	3.1397
2	3.1397	−0.0019	1.0000	−0.0019	3.1416
3	3.1416	0.0000	1.0000	0.0000	3.1416

28. $y = 4x^3 - 12x^2 + 12x - 3 = f(x)$

$y' = 12x^2 - 24x + 12 = f'(x)$

$x_1 = \dfrac{3}{2}$

$f'(x_2) = 0$; therefore, the method fails.

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	$\frac{3}{2}$	$\frac{3}{2}$	3	$\frac{1}{2}$	1
2	1	1	0	—	—

30. $f(x) = 2\sin x + \cos 2x$

$f'(x) = 2\cos x - 2\sin 2x$

$x_1 = \dfrac{3\pi}{2}$

Fails because $f'(x_1) = 0$.

n	x_n	$f(x_n)$	$f'(x_n)$
1	$\frac{3\pi}{2}$	−3	0

32. Newton's Method could fail if $f'(c) \approx 0$, or if the initial value x_1 is far from c.

34. Let $g(x) = f(x) - x = \cot x - x$

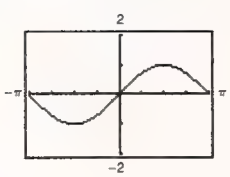
$g'(x) = -\csc^2 x - 1$.

The fixed point is approximately 0.86.

n	x_n	$g(x_n)$	$g'(x_n)$	$\dfrac{g(x_n)}{g'(x_n)}$	$x_n - \dfrac{g(x_n)}{g'(x_n)}$
1	1.0000	−0.3579	−2.4123	0.1484	0.8516
2	0.8516	0.0240	−2.7668	−0.0087	0.8603
3	0.8603	0.0001	−2.7403	0.0000	0.8603

36. $f(x) = \sin x$, $f'(x) = \cos x$

(a)



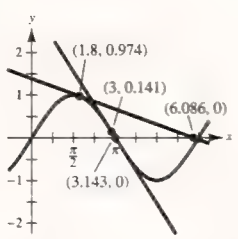
(b) $x_1 = 1.8$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 6.086$$

(c) $x_1 = 3$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 3.143$$

(d)



The x-intercepts correspond to the values resulting from the first iteration of Newton's Method.

(e) If the initial guess x_1 is not "close to" the desired zero of the function, the x-intercept of the tangent line may approximate another zero of the function.

38. (a) $x_{n+1} = x_n(2 - 3x_n)$

i	1	2	3	4
x_i	0.3000	0.3300	0.3333	0.3333

$\frac{1}{3} \approx 0.333$

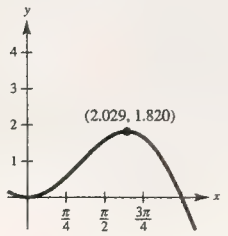
(b) $x_{n+1} = x_n(2 - 11x_n)$

i	1	2	3	4
x_i	0.1000	0.0900	0.0909	0.0909

$\frac{1}{11} \approx 0.091$

40. $f(x) = x \sin x$, $(0, \pi)$

$f'(x) = x \cos x + \sin x = 0$



Letting $F(x) = f'(x)$, we can use Newton's Method as follows.

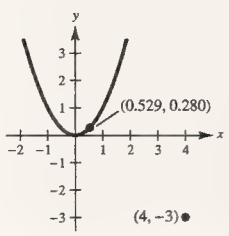
$[F'(x) = 2 \cos x - x \sin x]$

n	x_n	$F(x_n)$	$F'(x_n)$	$\frac{F(x_n)}{F'(x_n)}$	$x_n - \frac{F(x_n)}{F'(x_n)}$
1	2.0000	0.0770	−2.6509	−0.0290	2.0290
2	2.0290	−0.0007	−2.7044	0.0002	2.0288

Approximation to the critical number: 2.029

42. $y = f(x) = x^2$, $(4, -3)$

$d = \sqrt{(x-4)^2 + (y+3)^2} = \sqrt{(x-4)^2 + (x^2+3)^2} = \sqrt{x^4 + 7x^2 - 8x + 25}$



d is minimum when $D = x^4 + 7x^2 - 8x + 25$ is minimum.

$g(x) = D' = 4x^3 + 14x - 8$

$g'(x) = 12x^2 + 14$

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	0.5000	−0.5000	17.0000	−0.0294	0.5294
2	0.5294	0.0051	17.3632	0.0003	0.5291
3	0.5291	−0.0001	17.3594	0.0000	0.5291

$x \approx 0.529$

Point closest to $(4, -3)$ is approximately $(0.529, 0.280)$.

44. Maximize: $C = \frac{3t^2 + t}{50 + t^3}$

$$C' = \frac{-3t^4 - 2t^3 + 300t + 50}{(50 + t^3)^2} = 0$$

Let $f(x) = 3t^4 + 2t^3 - 300t - 50$

$f'(x) = 12t^3 + 6t^2 - 300$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	4.5000	12.4375	915.0000	0.0136	4.4864
2	4.4864	0.0658	904.3822	0.0001	4.4863

Since $f(4) = -354$ and $f(5) = 575$, the solution is in the interval $(4, 5)$.

Approximation: $t \approx 4.486$ hours

46. $170 = 0.808x^3 - 17.974x^2 + 71.248x + 110.843,\ 1 \le x \le 5$

Let $f(x) = 0.808x^3 - 17.974x^2 + 71.248x - 59.157$

$f'(x) = 2.424x^2 - 35.948x + 71.248.$

From the graph, choose $x_1 = 1$ and $x_1 = 3.5$. Apply Newton's Method.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.0000	−5.0750	37.7240	−0.1345	1.1345
2	1.1345	−0.2805	33.5849	−0.0084	1.1429
3	1.1429	0.0006	33.3293	0.0000	1.1429

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.5000	4.6725	−24.8760	−0.1878	3.6878
2	3.6878	−0.3286	−28.3550	0.0116	3.6762
3	3.6762	−0.0009	−28.1450	0.0000	3.6762

The zeros occur when $x \approx 1.1429$ and $x \approx 3.6762$. These approximately correspond to engine speeds of 1143 rev/min and 3676 rev/min.

48. True

50. True

52. $f(x) = \sqrt{4 - x^2}\sin(x - 2)$

Domain: $[-2, 2]$

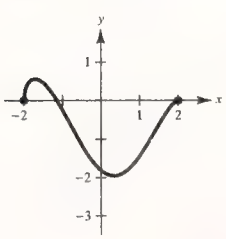
$x = -2$ and $x = 2$ are both zeros.

$f'(x) = \sqrt{4 - x^2}\cos(x - 2) - \frac{x}{\sqrt{4 - x^2}}\sin(x - 2)$

Let $x_1 = -1$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	−1.0000	−0.2444	−1.7962	0.1361	−1.1361
2	−1.1361	−0.0090	−1.6498	0.0055	−1.1416
3	−1.1416	0.0000	−1.6422	0.0000	−1.1416

Zeros: $x = \pm 2, x \approx -1.142$



Section 3.9 Differentials

2. $f(x) = \dfrac{6}{x^2} = 6x^{-2}$

$f'(x) = -12x^{-3} = \dfrac{-12}{x^3}$

Tangent line at $\left(2, \dfrac{3}{2}\right)$:

$$y - \frac{3}{2} = \frac{-12}{8}(x - 2) = \frac{-3}{2}(x - 2)$$

$$y = -\frac{3}{2}x + \frac{9}{2}$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \dfrac{6}{x^2}$	1.6620	1.5151	1.5	1.4851	1.3605
$T(x) = -\dfrac{3}{2}x + \dfrac{9}{2}$	1.65	1.515	1.5	1.485	1.35

4. $f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

Tangent line at $(2, \sqrt{2})$:

$$y - f(2) = f'(2)(x - 2)$$

$$y - \sqrt{2} = \frac{1}{2\sqrt{2}}(x - 2)$$

$$y = \frac{x}{2\sqrt{2}} + \frac{1}{\sqrt{2}}$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \sqrt{x}$	1.3784	1.4107	1.4142	1.4177	1.4491
$T(x) = \dfrac{x}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}$	1.3789	1.4107	1.4142	1.4177	1.4496

6. $f(x) = \csc x$

$f'(x) = -\csc x \cot x$

Tangent line at $(2, \csc 2)$: $y - f(2) = f'(2)(x - 2)$

$$y - \csc 2 = (-\csc 2 \cot 2)(x - 2)$$

$$y = (-\csc 2 \cot 2)(x - 2) + \csc 2$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \csc x$	1.0567	1.0948	1,0998	1.1049	1.1585
$T(x) = (-\csc 2 \cot 2)(x - 2) + \csc 2$	1.0494	1.0947	1.0998	1.1048	1.1501

8. $y = f(x) = 1 - 2x^2, f'(x) = -4x, x = 0, \Delta x = dx = -0.1$

$$\Delta y = f(x + \Delta x) - f(x)$$
$$= f(-0.1) - f(0)$$
$$= [1 - 2(-0.1)^2] - [1 - 2(0)^2] = -0.02$$

$$dy = f'(x)\,dx$$
$$= f'(0)(-0.1)$$
$$= (0)(-0.1) = 0$$

10. $y = f(x) = 2x + 1, f'(x) = 2, x = 2, \Delta x = dx = 0.01$

$$\Delta y = f(x + \Delta x) - f(x)$$
$$= f(2.01) - f(2)$$
$$= [2(2.01) + 1] - [2(2) + 1] = 0.02$$

$$dy = f'(x)\,dx$$
$$= f'(2)(0.01)$$
$$= 2(0.01) = 0.02$$

12. $y = 3x^{2/3}$

$$dy = 2x^{-1/3}dx = \frac{2}{x^{1/3}}dx$$

14. $y = \sqrt{9 - x^2}$

$$dy = \frac{1}{2}(9 - x^2)^{-1/2}(-2x)dx = \frac{-x}{\sqrt{9 - x^2}}dx$$

16. $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

$$dy = \left(\frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}\right)dx = \frac{x - 1}{2x\sqrt{x}}dx$$

18. $y = x \sin x$

$$dy = (x \cos x + \sin x)\, dx$$

20. $y = \dfrac{\sec^2 x}{x^2 + 1}$

$$dy = \left[\frac{(x^2 + 1)2 \sec^2 x \tan x - \sec^2 x(2x)}{(x^2 + 1)^2}\right]dx$$

$$= \left[\frac{2 \sec^2 x(x^2 \tan x + \tan x - x)}{(x^2 + 1)^2}\right]dx$$

22. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$$\approx 1 + (-1)(-0.1) = 1.1$$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$$\approx 1 + (-1)(0.04) = 0.96$$

24. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$$\approx 1 + 0(-0.1) = 1$$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$$\approx 1 + 0(0.04) = 1$$

26. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$$\approx 8 + (3)(-0.07) = 7.79$$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$$\approx 8 + (3)(0.1) = 8.3$$

28. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$$\approx 8 + 5(-0.07) = 7.65$$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$$\approx 8 + 5(0.1) = 8.5$$

30. $A = \frac{1}{2}bh,\ b = 36,\ h = 50$

$db = dh = \pm 0.25$

$dA = \frac{1}{2}b\, dh + \frac{1}{2}h\, db$

$\Delta A \approx dA = \frac{1}{2}(36)(\pm 0.25) + \frac{1}{2}(50)(\pm 0.25)$

$= \pm 10.75$ square centimeters

32. $x = 12$ inches

$\Delta x = dx = \pm 0.03$ inch

(a) $V = x^3$

$dV = 3x^2\, dx = 3(12)^2(\pm 0.03)$

$= \pm 12.96$ cubic inches

(b) $S = 6x^2$

$dS = 12x\, dx = 12(12)(\pm 0.03)$

$= \pm 4.32$ square inches

34. (a) $C = 56$ centimeters

$\Delta C = dC = \pm 1.2$ centimeters

$$C = 2\pi r \Rightarrow r = \frac{C}{2\pi}$$

$$A = \pi r^2 = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{1}{4\pi}C^2$$

$$dA = \frac{1}{2\pi}C\, dC = \frac{1}{2\pi}(56)(\pm 1.2) = \frac{33.6}{\pi}$$

$$\frac{dA}{A} = \frac{33.6/\pi}{[1/(4\pi)](56)^2} \approx 0.042857 = 4.2857\%$$

(b) $\dfrac{dA}{A} = \dfrac{(1/2\pi)C\, dC}{(1/4\pi)C^2} = \dfrac{2dC}{C} \leq 0.03$

$$\frac{dC}{C} \leq \frac{0.03}{2} = 0.015 = 1.5\%$$

36. $P = (500x - x^2) - \left(\frac{1}{2}x^2 - 77x + 3000\right)$, x changes from 115 to 120

$dP = (500 - 2x - x + 77)dx = (577 - 3x)\,dx = [577 - 3(115)](120 - 115) = 1160$

Approximate percentage change: $\frac{dP}{P}(100) = \frac{1160}{43517.50}(100) \approx 2.7\%$

38. $V = \frac{4}{3}\pi r^3$, $r = 100$ cm, $dr = 0.2$ cm

$\Delta V \approx dV = 4\pi r^2\,dr = 4\pi(100)^2(0.2) = 8000\pi\ \text{cm}^3$

40. $E = IR$

$R = \frac{E}{I}$

$dR = -\frac{E}{I^2}dI$

$\frac{dR}{R} = \frac{-(E/I^2)dI}{E/I} = -\frac{dI}{I}$

$\left|\frac{dR}{R}\right| = \left|-\frac{dI}{I}\right| = \left|\frac{dI}{I}\right|$

42. See Exercise 41.

$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(9.5\cot\theta)(9.5) = 45.125\cot\theta$

$dA = -45.125\csc^2\theta\,d\theta$

$\left|\frac{dA}{A}\right| = \frac{\csc^2\theta\,d\theta}{\cot\theta} = \frac{d\theta}{\sin\theta\cos\theta}$

$= \frac{0.25°}{(\sin 26.75°)(\cos 26.75°)}$

$\approx \frac{0.0044}{(\sin 0.4669)(\cos 0.4669)}$

$\approx 0.0109 = 1.09\%$ (in radians)

44. $h = 50\tan\theta$

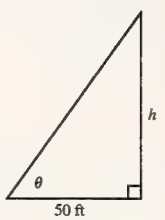
$\theta = 71.5° = 1.2479$ radians

$dh = 50\sec^2\theta \cdot d\theta$

$\left|\frac{dh}{x}\right| = \left|\frac{50\sec^2(1.2479)}{50\tan(1.2479)}d\theta\right| \le 0.06$

$\left|\frac{9.9316}{2.9886}d\theta\right| \le 0.06$

$|d\theta| \le 0.018$



46. Let $f(x) = \sqrt[3]{x}$, $x = 27$, $dx = -1$

$f(x + \Delta x) \approx f(x) + f'(x)dx = \sqrt[3]{x} + \frac{1}{3\sqrt[3]{x^2}}dx$

$\sqrt[3]{26} \approx \sqrt[3]{27} + \frac{1}{3\sqrt[3]{27^2}}(-1) = 3 - \frac{1}{27} \approx 2.9630$

Using a calculator, $\sqrt[3]{26} \approx 2.9625$

48. Let $f(x) = x^3$, $x = 3$, $dx = -0.01$.

$f(x + \Delta x) \approx f(x) + f'(x)\,dx = x^3 + 3x^2\,dx$

$f(x + \Delta x) = (2.99)^3 \approx 3^3 + 3(3)^2(-0.01) = 27 - 0.27 = 26.73$

Using a calculator: $(2.99)^3 \approx 26.7309$

50. Let $f(x) = \tan x$, $x = 0$, $dx = 0.05$, $f'(x) = \sec^2 x$.

Then

$$f(0.05) \approx f(0) + f'(0)dx$$

$$\tan 0.05 \approx \tan 0 + \sec^2 0(0.05) = 0 + 1(0.05).$$

52. Propagated error $= f(x + \Delta x) - f(x)$,

relative error $= \left|\frac{dy}{y}\right|$, and the percent error $= \left|\frac{dy}{y}\right| \times 100$.

54. True, $\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = a$

56. False

Let $f(x) = \sqrt{x}$, $x = 1$, and $\Delta x = dx = 3$. Then

$$\Delta y = f(x + \Delta x) - f(x) = f(4) - f(1) = 1$$

and

$$dy = f'(x)\,dx = \frac{1}{2\sqrt{1}}(3) = \frac{3}{2}.$$

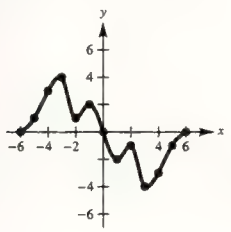
Thus, $dy > \Delta y$ in this example.

Review Exercises for Chapter 3

2. (a) $f(4) = -f(-4) = -3$

(b) $f(-3) = -f(3) = -(-4) = 4$

(c)



At least six critical numbers on $(-6, 6)$.

(d) Yes. Since $f(-2) = -f(2) = -(-1) = 1$ and $f(1) = -f(-1) = -2$, the Mean Value says that there exists at least one value c in $(-2, 1)$ such that

$$f'(c) = \frac{f(1) - f(-2)}{1 - (-2)} = \frac{-2 - 1}{1 + 2} = -1.$$

(e) No, $\lim_{x \to 0} f(x)$ exists because f is continuous at $(0, 0)$.

(f) Yes, f is differentiable at $x = 2$.

4. $f(x) = \frac{x}{\sqrt{x^2 + 1}}$, $[0, 2]$

$$f'(x) = x\left[-\frac{1}{2}(x^2 + 1)^{-3/2}(2x)\right] + (x^2 + 1)^{-1/2}$$

$$= \frac{1}{(x^2 + 1)^{3/2}}$$

No critical numbers

Left endpoint: $(0, 0)$ Minimum

Right endpoint: $\left(2, 2/\sqrt{5}\right)$ Maximum

6. No. f is not differentiable at $x = 2$.

8. No; the function is discontinuous at $x = 0$ which is in the interval $[-2, 1]$.

10. $f(x) = \frac{1}{x}$, $1 \le x \le 4$

$$f'(x) = -\frac{1}{x^2}$$

$$\frac{f(b) - f(a)}{b - a} = \frac{(1/4) - 1}{4 - 1} = \frac{-3/4}{3} = -\frac{1}{4}$$

$$f'(c) = \frac{-1}{c^2} = -\frac{1}{4}$$

$$c = 2$$

12. $f(x) = \sqrt{x} - 2x$, $0 \le x \le 4$

$$f'(x) = \frac{1}{2\sqrt{x}} - 2$$

$$\frac{f(b) - f(a)}{b - a} = \frac{-6 - 0}{4 - 0} = -\frac{3}{2}$$

$$f'(c) = \frac{1}{2\sqrt{c}} - 2 = -\frac{3}{2}$$

$$c = 1$$

14. $f(x) = 2x^2 - 3x + 1$

$f'(x) = 4x - 3$

$$\frac{f(b) - f(a)}{b - a} = \frac{21 - 1}{4 - 0} = 5$$

$f'(c) = 4c - 3 = 5$

$c = 2 =$ Midpoint of $[0, 4]$

16. $g(x) = (x + 1)^3$

$g'(x) = 3(x + 1)^2$

Critical number: $x = -1$

Interval	$-\infty < x < -1$	$-1 < x < \infty$
Sign of $g'(x)$	$g'(x) > 0$	$g'(x) > 0$
Conclusion	Increasing	Increasing

18. $f(x) = \sin x + \cos x, \quad 0 \le x \le 2\pi$

$f'(x) = \cos x - \sin x$

Critical numbers: $x = \frac{\pi}{4}, x = \frac{5\pi}{4}$

Interval	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < 2\pi$
Sign of $f'(x)$	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion	Increasing	Decreasing	Increasing

20. $g(x) = \frac{3}{2}\sin\left(\frac{\pi x}{2} - 1\right), \quad [0, 4]$

$$g'(x) = \frac{3}{2}\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi x}{2} - 1\right)$$

$= 0$ when $x = 1 + \frac{2}{\pi}, 3 + \frac{2}{\pi}$

Relative maximum: $\left(1 + \frac{2}{\pi}, \frac{3}{2}\right)$

Relative minimum: $\left(3 + \frac{2}{\pi}, -\frac{3}{2}\right)$

Test Interval	$0 < x < 1 + \frac{2}{\pi}$	$1 + \frac{2}{\pi} < x < 3 + \frac{2}{\pi}$	$3 + \frac{2}{\pi} < x < 4$
Sign of $g'(x)$	$g'(x) > 0$	$g'(x) < 0$	$g'(x) > 0$
Conclusion	Increasing	Decreasing	Increasing

22. (a) $y = A\sin\left(\sqrt{k/m}\,t\right) + B\cos\left(\sqrt{k/m}\,t\right)$

$$y' = A\sqrt{k/m}\cos\left(\sqrt{k/m}\,t\right) - B\sqrt{k/m}\sin\left(\sqrt{k/m}\,t\right)$$

$$= 0 \text{ when } \frac{\sin\sqrt{k/m}\,t}{\cos\sqrt{k/m}\,t} = \frac{A}{B} \Rightarrow \tan\left(\sqrt{k/m}\,t\right) = \frac{A}{B}.$$

Therefore,

$$\sin\left(\sqrt{k/m}\,t\right) = \frac{A}{\sqrt{A^2 + B^2}}$$

$$\cos\left(\sqrt{k/m}\,t\right) = \frac{B}{\sqrt{A^2 + B^2}}.$$

When $v = y' = 0$,

$$y = A\left(\frac{A}{\sqrt{A^2 + B^2}}\right) + B\left(\frac{B}{\sqrt{A^2 + B^2}}\right) = \sqrt{A^2 + B^2}.$$

(b) Period: $\frac{2\pi}{\sqrt{k/m}}$

Frequency: $\frac{1}{2\pi/\sqrt{k/m}} = \frac{1}{2\pi}\sqrt{k/m}$

24. $f(x) = (x + 2)^2(x - 4) = x^3 - 12x - 16$

$f'(x) = 3x^2 - 12$

$f''(x) = 6x = 0$ when $x = 0$.

Point of inflection: $(0, -16)$

Test Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave downward	Concave upward

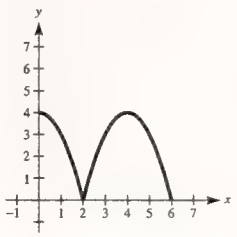
26. $h(t) = t - 4\sqrt{t + 1}$ Domain: $[-1, \infty)$

$$h'(t) = 1 - \frac{2}{\sqrt{t + 1}} = 0 \Rightarrow t = 3$$

$$h''(t) = \frac{1}{(t + 1)^{3/2}}$$

$h''(3) = \frac{1}{8} > 0$ $(3, -5)$ is a relative minimum.

28.



30. $C = \left(\frac{Q}{x}\right)s + \left(\frac{x}{2}\right)r$

$$\frac{dC}{dx} = -\frac{Qs}{x^2} + \frac{r}{2} = 0$$

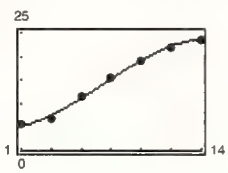
$$\frac{Qs}{x^2} = \frac{r}{2}$$

$$x^2 = \frac{2Qs}{r}$$

$$x = \sqrt{\frac{2Qs}{r}}$$

32. (a) $S = -0.1222t^3 + 1.3655t^2 - 0.9052t + 4.8429$

(b)



(c) $S'(t) = 0$ when $t = 3.7$. This is a maximum by the First Derivative Test.

(d) No, because the t^3 coefficient term is negative.

34. $\lim_{x \to \infty} \frac{2x}{3x^2 + 5} = \lim_{x \to \infty} \frac{2/x}{3 + 5/x^2} = 0$

36. $\lim_{x \to \infty} \frac{3x}{\sqrt{x^2 + 4}} = \lim_{x \to \infty} \frac{3}{\sqrt{1 + 4/x^2}} = 3$

38. $g(x) = \frac{5x^2}{x^2 + 2}$

$$\lim_{x \to \infty} \frac{5x^2}{x^2 + 2} = \lim_{x \to \infty} \frac{5}{1 + (2/x^2)} = 5$$

Horizontal asymptote: $y = 5$

40. $f(x) = \frac{3x}{\sqrt{x^2 + 2}}$

$$\lim_{x \to \infty} \frac{3x}{\sqrt{x^2 + 2}} = \lim_{x \to \infty} \frac{3x/x}{\sqrt{x^2 + 2}/\sqrt{x^2}}$$

$$= \lim_{x \to \infty} \frac{3}{\sqrt{1 + (2/x^2)}} = 3$$

$$\lim_{x \to -\infty} \frac{3x}{\sqrt{x^2 + 2}} = \lim_{x \to -\infty} \frac{3x/x}{\sqrt{x^2 + 2}/(-\sqrt{x^2})}$$

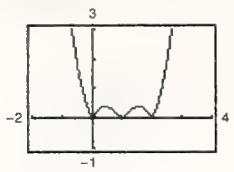
$$= \lim_{x \to -\infty} \frac{3}{-\sqrt{1 + (2/x^2)}} = -3$$

Horizontal asymptotes: $y = \pm 3$

42. $f(x) = |x^3 - 3x^2 + 2x| = |x(x-1)(x-2)|$

Relative minima: $(0, 0), (1, 0), (2, 0)$

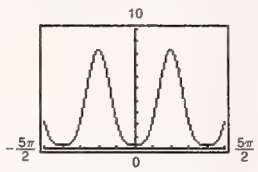
Relative maxima: $(1.577, 0.38), (0.423, 0.38)$



44. $g(x) = \dfrac{\pi^2}{3} - 4\cos x + \cos 2x$

Relative minima: $(2\pi k, 0.29)$ where k is any integer.

Relative maxima: $((2k-1)\pi, 8.29)$ where k is any integer.



46. $f(x) = 4x^3 - x^4 = x^3(4 - x)$

Domain: $(-\infty, \infty)$; Range: $(-\infty, 27)$

$f'(x) = 12x^2 - 4x^3 = 4x^2(3 - x) = 0$ when $x = 0, 3$.

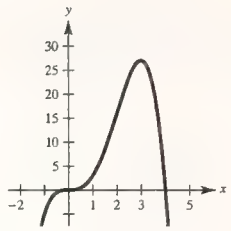
$f''(x) = 24x - 12x^2 = 12x(2 - x) = 0$ when $x = 0, 2$.

$f''(3) < 0$

Therefore, $(3, 27)$ is a relative maximum.

Points of inflection: $(0, 0), (2, 16)$

Intercepts: $(0, 0), (4, 0)$



48. $f(x) = (x^2 - 4)^2$

Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$f'(x) = 4x(x^2 - 4) = 0$ when $x = 0, \pm 2$.

$f''(x) = 4(3x^2 - 4) = 0$ when $x = \pm\dfrac{2\sqrt{3}}{3}$.

$f''(0) < 0$

Therefore, $(0, 16)$ is a relative maximum.

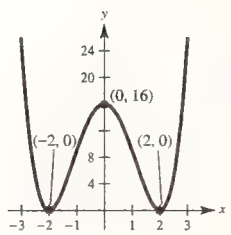
$f''(\pm 2) > 0$

Therefore, $(\pm 2, 0)$ are relative minima.

Points of inflection: $\left(\pm 2\sqrt{3}/3, 64/9\right)$

Intercepts: $(-2, 0), (0, 16), (2, 0)$

Symmetry with respect to y-axis



50. $f(x) = (x - 3)(x + 2)^3$

Domain: $(-\infty, \infty)$; Range: $\left[-\frac{16,875}{256}, \infty\right)$

$f'(x) = (x - 3)(3)(x + 2)^2 + (x + 2)^3$

$= (4x - 7)(x + 2)^2 = 0$ when $x = -2, \frac{7}{4}$.

$f''(x) = (4x - 7)(2)(x + 2) + (x + 2)^2(4)$

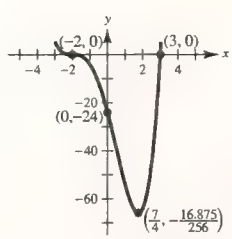
$= 6(2x - 1)(x + 2) = 0$ when $x = -2, \frac{1}{2}$.

$f''\left(\frac{7}{4}\right) > 0$

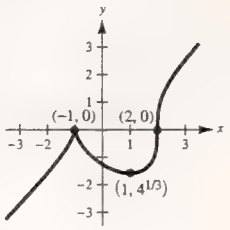
Therefore, $\left(\frac{7}{4}, -\frac{16,875}{256}\right)$ is a relative minimum.

Points of inflection: $(-2, 0), \left(\frac{1}{2}, -\frac{625}{16}\right)$

Intercepts: $(-2, 0), (0, -24), (3, 0)$



52. $f(x) = (x - 2)^{1/3}(x + 1)^{2/3}$



Graph of Exercise 39 translated 2 units to the right (x replaces by $x - 2$).

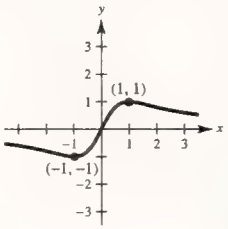
$(-1, 0)$ is a relative maximum.

$\left(1, -\sqrt[3]{4}\right)$ is a relative minimum.

$(2, 0)$ is a point of inflection.

Intercepts: $(-1, 0), (2, 0)$

54. $f(x) = \dfrac{2x}{1 + x^2}$



Domain: $(-\infty, \infty)$; Range: $[-1, 1]$

$f'(x) = \dfrac{2(1 - x)(1 + x)}{(1 + x^2)^2} = 0$ when $x = \pm 1$.

$f''(x) = \dfrac{-2x(3 - x^2)}{(1 + x^2)^3} = 0$ when $x = 0, \pm\sqrt{3}$.

$f''(1) < 0$

Therefore, $(1, 1)$ is a relative maximum.

$f''(-1) > 0$

Therefore, $(-1, -1)$ is a relative minimum.

Points of inflection: $\left(-\sqrt{3}, -\sqrt{3}/2\right), (0, 0), \left(\sqrt{3}, \sqrt{3}/2\right)$

Intercept: $(0, 0)$

Symmetric with respect to the origin

Horizontal asymptote: $y = 0$

56. $f(x) = \dfrac{x^2}{1 + x^4}$

Domain: $(-\infty, \infty)$; Range: $\left[0, \dfrac{1}{2}\right]$

$f'(x) = \dfrac{(1 + x^4)(2x) - x^2(4x^3)}{(1 + x^4)^2} = \dfrac{2x(1 - x)(1 + x)(1 + x^2)}{(1 + x^4)^2} = 0$ when $x = 0, \pm 1$.

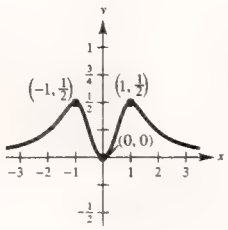
$f''(x) = \dfrac{(1 + x^4)^2(2 - 10x^4) - (2x - 2x^5)(2)(1 + x^4)(4x^3)}{(1 + x^4)^4} = \dfrac{2(1 - 12x^4 + 3x^8)}{(1 + x^4)^3} = 0$ when $x = \pm\sqrt[4]{\dfrac{6 \pm \sqrt{33}}{3}}$.

$f''(\pm 1) < 0$

Therefore, $\left(\pm 1, \dfrac{1}{2}\right)$ are relative maxima.

$f''(0) > 0$

Therefore, $(0, 0)$ is a relative minimum.



Points of inflection: $\left(\pm\sqrt[4]{\dfrac{6 - \sqrt{33}}{3}}, 0.29\right), \left(\pm\sqrt[4]{\dfrac{6 + \sqrt{33}}{3}}, 0.40\right)$

Intercept: $(0, 0)$

Symmetric to the y-axis

Horizontal asymptote: $y = 0$

58. $f(x) = x^2 + \frac{1}{x} = \frac{x^3 + 1}{x}$

Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, \infty)$

$f'(x) = 2x - \frac{1}{x^2} = \frac{2x^3 - 1}{x^2} = 0$ when $x = \frac{1}{\sqrt[3]{2}}$.

$f''(x) = 2 + \frac{2}{x^3} = \frac{2(x^3 + 1)}{x^3} = 0$ when $x = -1$.

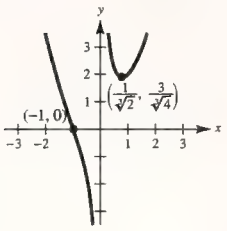
$f''\left(\frac{1}{\sqrt[3]{2}}\right) > 0$

Therefore, $\left(\frac{1}{\sqrt[3]{2}}, \frac{3}{\sqrt[3]{4}}\right)$ is a relative minimum.

Point of inflection: $(-1, 0)$

Intercept: $(-1, 0)$

Vertical asymptote: $x = 0$

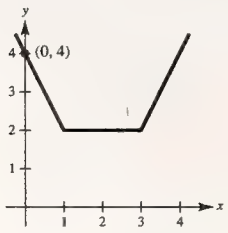


60. $f(x) = |x - 1| + |x - 3| = \begin{cases} -2x + 4, & x \le 1 \\ 2, & 1 < x \le 3 \\ 2x - 4, & x > 3 \end{cases}$

Domain: $(-\infty, \infty)$

Range: $[2, \infty)$

Intercept: $(0, 4)$

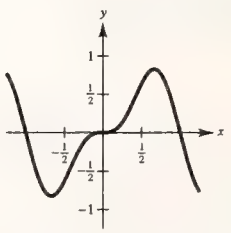


62. $f(x) = \frac{1}{\pi}(2 \sin \pi x - \sin 2\pi x)$

Domain: $[-1, 1]$; Range: $\left[\frac{-3\sqrt{3}}{2\pi}, \frac{3\sqrt{3}}{2\pi}\right]$

$f'(x) = 2(\cos \pi x - \cos 2\pi x) = -2(2 \cos \pi x + 1)(\cos \pi x - 1) = 0$

Critical Numbers: $x = \pm\frac{2}{3}, 0$



$f''(x) = 2\pi(-\sin \pi x + 2 \sin 2\pi x) = 2\pi \sin \pi x(-1 + 4 \cos \pi x) = 0$ when $x = 0, \pm 1, \pm 0.420$.

By the First Derivative Test: $\left(-\frac{2}{3}, \frac{-3\sqrt{3}}{2\pi}\right)$ is a relative minimum.

$\left(\frac{2}{3}, \frac{3\sqrt{3}}{2\pi}\right)$ is a relative maximum.

Points of inflection: $(-0.420, -0.462), (0.420, 0.462), (\pm 1, 0), (0, 0)$

Intercepts: $(-1, 0), (0, 0), (1, 0)$

Symmetric with respect to the origin

64. $f(x) = x^n$, n is a positive integer.

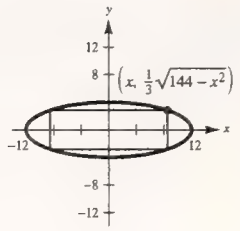
(a) $f'(x) = nx^{n-1}$

The function has a relative minimum at $(0, 0)$ when n is even.

(b) $f''(x) = n(n - 1)x^{n-2}$

The function has a point of inflection at $(0, 0)$ when n is odd and $n \ge 3$.

66. Ellipse: $\frac{x^2}{144} + \frac{y^2}{16} = 1,\ y = \frac{1}{3}\sqrt{144 - x^2}$



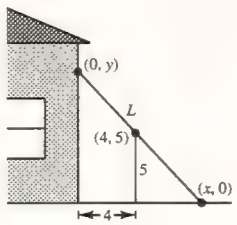
$$A = (2x)\left(\frac{2}{3}\sqrt{144 - x^2}\right) = \frac{4}{3}x\sqrt{144 - x^2}$$

$$\frac{dA}{dx} = \frac{4}{3}\left[\frac{-x^2}{\sqrt{144 - x^2}} + \sqrt{144 - x^2}\right]$$

$$= \frac{4}{3}\left[\frac{144 - 2x^2}{\sqrt{144 - x^2}}\right] = 0 \text{ when } x = \sqrt{72} = 6\sqrt{2}.$$

The dimensions of the rectangle are $2x = 12\sqrt{2}$ by $y = \frac{2}{3}\sqrt{144 - 72} = 4\sqrt{2}$.

68. We have points $(0, y)$, $(x, 0)$, and $(4, 5)$. Thus,



$$m = \frac{y - 5}{0 - 4} = \frac{5 - 0}{4 - x} \text{ or } y = \frac{5x}{x - 4}.$$

Let $f(x) = L^2 = x^2 + \left(\frac{5x}{x - 4}\right)^2$

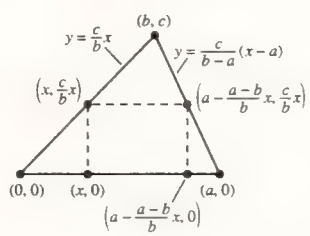
$$f'(x) = 2x + 50\left(\frac{x}{x - 4}\right)\left[\frac{x - 4 - x}{(x - 4)^2}\right] = 0$$

$$x - \frac{100x}{(x - 4)^3} = 0$$

$$x[(x - 4)^3 - 100] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{100}.$$

$$L = \sqrt{x^2 + \frac{25x^2}{(x - 4)^2}} = \frac{x}{x - 4}\sqrt{(x - 4)^2 + 25} = \frac{\sqrt[3]{100} + 4}{\sqrt[3]{100}}\sqrt{100^{2/3} + 25} \approx 12.7 \text{ feet}$$

70. Label triangle with vertices $(0, 0)$, $(a, 0)$, and (b, c). The equations of the sides of the triangle are $y = (c/b)x$ and $y = [c/(b - a)](x - a)$. Let $(x, 0)$ be a vertex of the inscribed rectangle. The coordinates of the upper left vertex are $(x, (c/b)x)$. The y-coordinate of the upper right vertex of the rectangle is $(c/b)x$. Solving for the x-coordinate $\bar{x}$ of the rectangle's upper right vertex, you get



$$\frac{c}{b}x = \frac{c}{b - a}(\bar{x} - a)$$

$$(b - a)x = b(\bar{x} - a)$$

$$\bar{x} = \frac{b - a}{b}x + a = a - \frac{a - b}{b}x.$$

Finally, the lower right vertex is

$$\left(a - \frac{a - b}{b}x, 0\right).$$

Width of rectangle: $a - \frac{a - b}{b}x - x$

Height of rectangle: $\frac{c}{b}x$ (see figure)

$$A = (\text{Width})(\text{Height}) = \left(a - \frac{a - b}{b}x - x\right)\left(\frac{c}{b}x\right) = \left(a - \frac{a}{b}x\right)\frac{c}{b}x$$

$$\frac{dA}{dx} = \left(a - \frac{a}{b}x\right)\frac{c}{b} + \left(\frac{c}{b}x\right)\left(-\frac{a}{b}\right) = \frac{ac}{b} - \frac{2ac}{b^2}x = 0 \text{ when } x = \frac{b}{2}.$$

$$A\left(\frac{b}{2}\right) = \left(a - \frac{a}{b}\frac{b}{2}\right)\left(\frac{c}{b}\frac{b}{2}\right) = \left(\frac{a}{2}\right)\left(\frac{c}{2}\right) = \frac{1}{4}ac = \frac{1}{2}\left(\frac{1}{2}ac\right) = \frac{1}{2}(\text{Area of triangle})$$

72. You can form a right triangle with vertices $(0, y)$, $(0, 0)$, and $(x, 0)$. Choosing a point (a, b) on the hypotenuse (assuming the triangle is in the first quadrant), the slope is

$$m = \frac{y-b}{0-a} = \frac{b-0}{a-x} \Rightarrow y = \frac{-bx}{a-x}.$$

Let $f(x) = L^2 = x^2 + y^2 = x^2 + \left(\frac{-bx}{a-x}\right)^2$.

$$f'(x) = 2x + 2\left(\frac{-bx}{a-x}\right)\left[\frac{-ab}{(a-x)^2}\right]$$

$$\frac{2x[(a-x)^3 + ab^2]}{(a-x)^3} = 0 \text{ when } x = 0,\ a + \sqrt[3]{ab^2}.$$

Choosing the nonzero value, we have $y = b + \sqrt[3]{a^2b}$.

$$L = \sqrt{\left(a + \sqrt[3]{ab^2}\right)^2 + \left(b + \sqrt[3]{a^2b}\right)^2}$$

$$= (a^2 + 3a^{4/3}b^{2/3} + 3a^{2/3}b^{4/3} + b^2)^{1/2}$$

$$= (a^{2/3} + b^{2/3})^{3/2} \text{ meters}$$

74. Using Exercise 73 as a guide we have $L_1 = a \csc\theta$ and $L_2 = b \sec\theta$. Then $dL/d\theta = -a \csc\theta \cot\theta + b \sec\theta \tan\theta = 0$ when

$$\tan\theta = \sqrt[3]{a/b},\ \sec\theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{b^{1/3}},\ \csc\theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{a^{1/3}} \text{ and}$$

$$L = L_1 + L_2 = a \csc\theta + b \sec\theta = a\frac{(a^{2/3} + b^{2/3})^{1/2}}{a^{1/3}} + b\frac{(a^{2/3} + b^{2/3})^{1/2}}{b^{1/3}} = (a^{2/3} + b^{2/3})^{3/2}.$$

This matches the result of Exercise 72.

76. Total cost = (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{500} + 7.50\right)\left(\frac{110}{v}\right) = \frac{11v}{50} + \frac{825}{v}$$

$$\frac{dT}{dv} = \frac{11}{50} - \frac{825}{v^2} = \frac{11v^2 - 41{,}250}{50v^2}$$

$$= 0 \text{ when } v = \sqrt{3750} = 25\sqrt{6} \approx 61.2 \text{ mph.}$$

$$\frac{d^2T}{dv^2} = \frac{1650}{v^3} > 0 \text{ when } v = 25\sqrt{6} \text{ so this value yields a minimum.}$$

78. $f(x) = x^3 + 2x + 1$

From the graph, you can see that $f(x)$ has one real zero.

$f'(x) = 3x^2 + 2$

f changes sign in $[-1, 0]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	−0.5000	−0.1250	2.7500	−0.0455	−0.4545
2	−0.4545	−0.0029	2.6197	−0.0011	−0.4534

On the interval $[-1, 0]$: $x \approx -0.453$.

80. Find the zeros of $f(x) = \sin \pi x + x - 1$.

$f'(x) = \pi \cos \pi x + 1$

From the graph you can see that $f(x)$ has three real zeros.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.2000	−0.2122	3.5416	−0.0599	0.2599
2	0.2599	−0.0113	3.1513	−0.0036	0.2635
3	0.2635	0.0000	3.1253	0.0000	0.2635

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.0000	0.0000	−2.1416	0.0000	1.0000

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.8000	0.2122	3.5416	0.0599	1.7401
2	1.7401	0.0113	3.1513	0.0036	1.7365
3	1.7365	0.0000	3.1253	0.0000	1.7365

The three real zeros of $f(x)$ are $x \approx 0.264$, $x = 1$, and $x \approx 1.737$.

82. $y = \sqrt{36 - x^2}$

$$\frac{dy}{dx} = \frac{1}{2}(36 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{36 - x^2}}$$

$$dy = \frac{-x}{\sqrt{36 - x^2}}dx$$

84. $p = 75 - \frac{1}{4}x$

$$\Delta p = p(8) - p(7)$$

$$= \left(75 - \frac{8}{4}\right) - \left(75 - \frac{7}{4}\right) = -\frac{1}{4}$$

$$dp = -\frac{1}{4}dx = -\frac{1}{4}(1) = -\frac{1}{4}$$

[$\Delta p = dp$ because p is linear]

Problem Solving for Chapter 3

2. (a) $dV = 3x^2dx = 3x^2\Delta x$

$$\Delta V = (x + \Delta x)^3 - x^3 = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

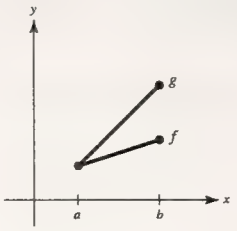
$$\Delta V - dV = 3x(\Delta x)^2 + (\Delta x)^3 = \underbrace{[3x\Delta x + (\Delta x)^2]}_{\varepsilon}\Delta x$$

$$= \varepsilon\Delta x, \text{ where } \varepsilon \to 0 \text{ as } \Delta x \to 0.$$

(b) Let $\varepsilon = \frac{\Delta y}{\Delta x} - f'(x)$. Then $\varepsilon \to 0$ as $\Delta x \to 0$.

Furthermore, $\Delta y - dy = \Delta y - f'(x)dx = \varepsilon\Delta x$.

4. Let $h(x) = g(x) - f(x)$, which is continuous on $[a, b]$ and differentiable on (a, b). $h(a) = 0$ and $h(b) = g(b) - f(b)$.



By the Mean Value Theorem, there exists c in (a, b) such that

$$h'(c) = \frac{h(b) - h(a)}{b - a} = \frac{g(b) - f(b)}{b - a}.$$

Since $h'(c) = g'(c) - f'(c) > 0$ and $b - a > 0$,

$$g(b) - f(b) > 0 \Rightarrow g(b) > f(b).$$

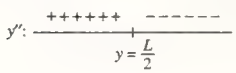
6. (a) $f' = 2ax + b, f'' = 2a \neq 0$. No points of inflection.

(b) $f' = 3ax^2 - 2bx + c, f'' = 6ax + 2b = 0 \Rightarrow x = \dfrac{-b}{3a}$. One point of inflection.

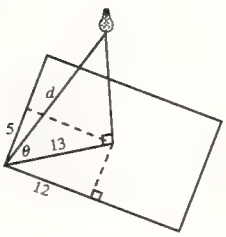
(c) $y' = ky(L - y) = kLy - ky^2$

$y'' = kLy' - 2kyy' = ky'(L - 2y)$

If $y = \dfrac{L}{2}$, then $y'' = 0$ and this is a point of inflection because of the analysis below.



8.



$d = \sqrt{13^2 + x^2}$, $\sin\theta = \dfrac{x}{d}$.

Let A be the amount of illumination at one of the corners, as indicated in the figure. Then

$$A = \frac{kI}{(13^2 + x^2)} \sin\theta = \frac{kIx}{(13^2 + x^2)^{3/2}}$$

$$A'(x) = kI \frac{(x^2 + 169)^{3/2}(1) - x\left(\frac{3}{2}\right)(x^2 + 169)^{1/2}(2x)}{(169 + x^2)^3} = 0$$

$$\Rightarrow (x^2 + 169)^{3/2} = 3x^2(x^2 + 169)^{1/2}$$

$$x^2 + 169 = 3x^2$$

$$2x^2 = 169$$

$$x = \frac{13}{\sqrt{2}} \approx 9.19 \text{ feet}$$

By the First Derivative Test, this is a maximum.

10. Let T be the intersection of PQ and RS. Let MN be the perpendicular to SQ and PR passing through T.

Let $TM = x$ and $TN = b - x$.

$$\frac{SN}{b-x} = \frac{MR}{x} \Rightarrow SN = \frac{b-x}{x}MR$$

$$\frac{NQ}{b-x} = \frac{PM}{x} \Rightarrow NQ = \frac{b-x}{x}PM$$

$$SQ = \frac{b-x}{x}(MR + PM) = \frac{b-x}{x}d$$

$$A(x) = \text{Area} = \frac{1}{2}dx + \frac{1}{2}\left(\frac{b-x}{x}d\right)(b-x) = \frac{1}{2}d\left[x + \frac{(b-x)^2}{x}\right] = \frac{1}{2}d\left[\frac{2x^2 - 2bx + b^2}{x}\right]$$

$$A'(x) = \frac{1}{2}d\left[\frac{x(4x-2b) - (2x^2 - 2bx + b^2)}{x^2}\right]$$

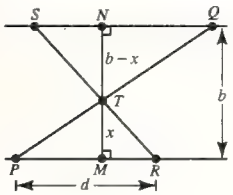
$$A'(x) = 0 \Rightarrow 4x^2 - 2xb = 2x^2 - 2bx + b^2$$

$$2x^2 = b^2$$

$$x = \frac{b}{\sqrt{2}}$$

Hence, we have $SQ = \dfrac{b-x}{x}d = \dfrac{b - (b/\sqrt{2})}{b/\sqrt{2}}d = (\sqrt{2} - 1)d$.

Using the Second Derivative Test, this is a minimum. There is no maximum.



12. (a) Let $M > 0$ be given. Take $N = \sqrt{M}$. Then whenever $x > N = \sqrt{M}$, you have

$$f(x) = x^2 > M.$$

(b) Let $\varepsilon > 0$ be given. Let $M = \sqrt{\dfrac{1}{\varepsilon}}$. Then whenever $x > M = \sqrt{\dfrac{1}{\varepsilon}}$, you have

$$x^2 > \frac{1}{\varepsilon} \Rightarrow \frac{1}{x^2} < \varepsilon \Rightarrow \left|\frac{1}{x^2} - 0\right| < \varepsilon.$$

(c) Let $\varepsilon > 0$ be given. There exists $N > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > N$.

Let $\delta = \dfrac{1}{N}$. Let $x = \dfrac{1}{y}$.

If $0 < y < \delta = \dfrac{1}{N}$, then $\dfrac{1}{x} < \dfrac{1}{N} \Rightarrow x > N$ and

$$|f(x) - L| = \left|f\left(\frac{1}{y}\right) - L\right| < \varepsilon.$$

14. Distance $= \sqrt{4^2 + x^2} + \sqrt{(4-x)^2 + 4^2} = f(x)$

$$f'(x) = \frac{x}{\sqrt{4^2 + x^2}} + \frac{4 - x}{\sqrt{(4-x)^2 + 4^2}} = 0$$

$$x\sqrt{(4-x)^2 + 4^2} = (x - 4)\sqrt{4^2 + x^2}$$

$$x^2[16 - 8x + x^2 + 16] = (x^2 - 8x + 16)(16 + x^2)$$

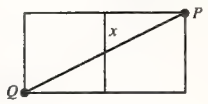
$$32x^2 - 8x^3 + x^4 = x^4 - 8x^3 + 32x^2 - 128x + 256$$

$$128x = 256$$

$$x = 2$$

The bug should head towards the midpoint of the opposite side.

Without Calculus: Imagine opening up the cube:



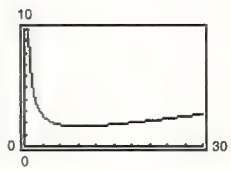
The shortest distance is the line PQ, passing through the midpoint.

16. (a) $s = \dfrac{v\,\dfrac{\text{km}}{\text{hr}}\left(1000\,\dfrac{\text{m}}{\text{km}}\right)}{\left(3600\,\dfrac{\text{sec}}{\text{hr}}\right)} = \dfrac{5}{18}v$

v	20	40	60	80	100
s	5.56	11.11	16.67	22.22	27.78
d	5.1	13.7	27.2	44.2	66.4

$d(t) = 0.071s^2 + 0.389s + 0.727$

(c)



$$T = \frac{1}{s}(0.071s^2 + 0.389s + 0.727) + \frac{5.5}{s}$$

The minimum is attained when $s \approx 9.365$ m/sec.

(b) The distance between the back of the first vehicle and the front of the second vehicle is $d(t)$, the safe stopping distance. The first vehicle passes the given point in $5.5/s$ seconds, and the second vehicle takes $d(s)/s$ more seconds. Hence,

$$T = \frac{d(s)}{s} + \frac{5.5}{s}.$$

(d) $T(s) = 0.071s + 0.389 + \dfrac{6.227}{s}$

$$T'(s) = 0.071 - \frac{6.227}{s^2} \Rightarrow s^2 = \frac{6.227}{0.071}$$

$$\Rightarrow s \approx 9.365 \text{ m/sec}$$

$T(9.365) \approx 1.719$ seconds

$$9.365 \text{ m/sec} \cdot \frac{3600}{1000} = 3.37 \text{ km/hr}$$

(e) $d(9.365) = 10.597$ m

18. (a)

x	0	0.5	1	2
$\sqrt{1+x}$	1	1.2247	1.4142	1.7321
$\frac{1}{2}x + 1$	1	1.25	1.5	2

(b) Let $f(x) = \sqrt{1+x}$. Using the Mean Value Theorem on the interval $[0, x]$, there exists c, $0 < c < x$, satisfying

$$f'(c) = \frac{1}{2\sqrt{1+c}} = \frac{f(x) - f(0)}{x - 0} = \frac{\sqrt{1+x} - 1}{x}.$$

Thus $\sqrt{1+x} = \dfrac{x}{2\sqrt{1+c}} + 1 < \dfrac{x}{2} + 1$ (because $\sqrt{1+c} > 1$).

CHAPTER 4
Integration

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CHAPTER 4
Integration

Section 4.1 Antiderivatives and Indefinite Integration

Solutions to Even-Numbered Exercises

2. $\dfrac{d}{dx}\left(x^4 + \dfrac{1}{x} + C\right) = 4x^3 - \dfrac{1}{x^2}$

4. $\dfrac{d}{dx}\left(\dfrac{2(x^2+3)}{3\sqrt{x}} + C\right) = \dfrac{d}{dx}\left(\dfrac{2}{3}x^{3/2} + 2x^{-1/2} + C\right)$

$= x^{1/2} - x^{-3/2} = \dfrac{x^2 - 1}{x^{3/2}}$

6. $\dfrac{dr}{d\theta} = \pi$

$r = \pi\theta + C$

Check: $\dfrac{d}{d\theta}[\pi\theta + C] = \pi$

8. $\dfrac{dy}{dx} = 2x^{-3}$

$y = \dfrac{2x^{-2}}{-2} + C = \dfrac{-1}{x^2} + c$

Check: $\dfrac{d}{dx}\left[\dfrac{-1}{x^2} + C\right] = 2x^{-3}$

	Given	*Rewrite*	*Integrate*	*Simplify*
10.	$\int \dfrac{1}{x^2}\,dx$	$\int x^{-2}\,dx$	$\dfrac{x^{-1}}{-1} + C$	$-\dfrac{1}{x} + C$
12.	$\int x(x^2+3)\,dx$	$\int (x^3 + 3x)\,dx$	$\dfrac{x^4}{4} + 3\left(\dfrac{x^2}{2}\right) + C$	$\dfrac{1}{4}x^4 + \dfrac{3}{2}x^2 + C$
14.	$\int \dfrac{1}{(3x)^2}dx$	$\dfrac{1}{9}\int x^{-2}dx$	$\dfrac{1}{9}\left(\dfrac{x^{-1}}{-1}\right) + C$	$\dfrac{-1}{9x} + C$

16. $\int (5 - x)dx = 5x - \dfrac{x^2}{2} + C$

Check: $\dfrac{d}{dx}\left[5x - \dfrac{x^2}{2} + C\right] = 5 - x$

18. $\int (4x^3 + 6x^2 - 1)dx = x^4 + 2x^3 - x + C$

Check: $\dfrac{d}{dx}[x^4 + 2x^3 - x + C] = 4x^3 + 6x^2 - 1$

20. $\int (x^3 - 4x + 2)dx = \dfrac{x^4}{4} - 2x^2 + 2x + C$

Check: $\dfrac{d}{dx}\left[\dfrac{x^4}{4} - 2x^2 + 2x + C\right] = x^3 - 4x + 2$

22. $\int \left(\sqrt{x} + \dfrac{1}{2\sqrt{x}}\right)dx = \int \left(x^{1/2} + \dfrac{1}{2}x^{-1/2}\right)dx = \dfrac{x^{3/2}}{3/2} + \dfrac{1}{2}\left(\dfrac{x^{1/2}}{1/2}\right) + C = \dfrac{2}{3}x^{3/2} + x^{1/2} + C$

Check: $\dfrac{d}{dx}\left(\dfrac{2}{3}x^{3/2} + x^{1/2} + C\right) = x^{1/2} + \dfrac{1}{2}x^{-1/2} = \sqrt{x} + \dfrac{1}{2\sqrt{x}}$

24. $\int \left(\sqrt[4]{x^3} + 1\right) dx = \int (x^{3/4} + 1)\, dx = \frac{4}{7}x^{7/4} + x + C$

Check: $\frac{d}{dx}\left(\frac{4}{7}x^{7/4} + x + C\right) = x^{3/4} + 1 = \sqrt[4]{x^3} + 1$

26. $\int \frac{1}{x^4}\, dx = \int x^{-4}\, dx = \frac{x^{-3}}{-3} + C = -\frac{1}{3x^3} + C$

Check: $\frac{d}{dx}\left(-\frac{1}{3x^3} + C\right) = \frac{1}{x^4}$

28. $\int \frac{x^2 + 2x - 3}{x^4} dx = \int (x^{-2} + 2x^{-3} - 3x^{-4}) dx$

$$= \frac{x^{-1}}{-1} + \frac{2x^{-2}}{-2} - \frac{3x^{-3}}{-3} + C$$

$$= \frac{-1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C$$

Check: $\frac{d}{dx}\left[-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C\right] = x^{-2} + 2x^{-3} - 3x^{-4}$

$$= \frac{x^2 + 2x - 3}{x^4}$$

30. $\int (2t^2 - 1)^2\, dt = \int (4t^4 - 4t^2 + 1)\, dt$

$$= \frac{4}{5}t^5 - \frac{4}{3}t^3 + t + C$$

Check: $\frac{d}{dt}\left(\frac{4}{5}t^5 - \frac{4}{3}t^3 + t + C\right) = 4t^4 - 4t^2 + 1$

$$= (2t^2 - 1)^2$$

32. $\int (1 + 3t)t^2\, dt = \int (t^2 + 3t^3)\, dt = \frac{1}{3}t^3 + \frac{3}{4}t^4 + C$

Check: $\frac{d}{dt}\left(\frac{1}{3}t^3 + \frac{3}{4}t^4 + C\right) = t^2 + 3t^3 = (1 + 3t)t^2$

34. $\int 3\, dt = 3t + C$

Check: $\frac{d}{dt}(3t + C) = 3$

36. $\int (t^2 - \sin t)\, dt = \frac{1}{3}t^3 + \cos t + C$

Check: $\frac{d}{dt}\left(\frac{1}{3}t^3 + \cos t + C\right) = t^2 - \sin t$

38. $\int (\theta^2 + \sec^2\theta)\, d\theta = \frac{1}{3}\theta^3 + \tan\theta + C$

Check: $\frac{d}{d\theta}\left(\frac{1}{3}\theta^3 + \tan\theta + C\right) = \theta^2 + \sec^2\theta$

40. $\int \sec y(\tan y - \sec y)\, dy = \int (\sec y \tan y - \sec^2 y)\, dy$

$$= \sec y - \tan y + C$$

Check: $\frac{d}{dy}(\sec y - \tan y + C) = \sec y \tan y - \sec^2 y$

$$= \sec y(\tan y - \sec y)$$

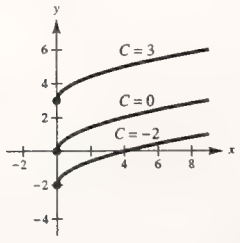
42. $\int \frac{\cos x}{1 - \cos^2 x} dx = \int \frac{\cos x}{\sin^2 x} dx = \int \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) dx$

$$= \int \csc x \cot x\, dx = -\csc x + C$$

Check: $\frac{d}{dx}[-\csc x + C] = \csc x \cot x + \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}$

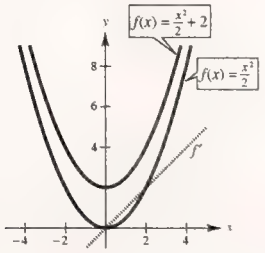
$$= \frac{\cos x}{1 - \cos^2 x}$$

44. $f(x) = \sqrt{x}$



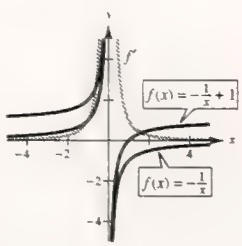
46. $f'(x) = x$

$f(x) = \frac{x^2}{2} + C$



48. $f'(x) = \frac{1}{x^2}$

$f(x) = -\frac{1}{x} + C$



50. $\dfrac{dy}{dx} = 2(x-1) = 2x - 2,\ (3, 2)$

$y = \displaystyle\int 2(x-1)\,dx = x^2 - 2x + C$

$2 = (3)^2 - 2(3) + C \Rightarrow C = -1$

$y = x^2 - 2x - 1$

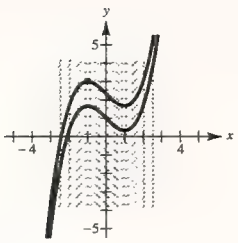
52. $\dfrac{dy}{dx} = -\dfrac{1}{x^2} = -x^{-2},\ (1,3)$

$y = \displaystyle\int -x^{-2}\,dx = \frac{1}{x} + C$

$3 = \dfrac{1}{1} + C \Rightarrow C = 2$

$y = \dfrac{1}{x} + 2,\ x > 0$

54. (a)



(b) $\dfrac{dy}{dx} = x^2 - 1,\ (-1, 3)$

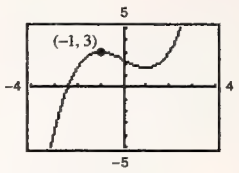
$y = \dfrac{x^3}{3} - x + C$

$3 = \dfrac{(-1)^3}{3} - (-1) + C$

$3 = -\dfrac{1}{3} + 1 + C$

$C = \dfrac{7}{3}$

$y = \dfrac{x^3}{3} - x + \dfrac{7}{3}$



56. $g'(x) = 6x^2,\ g(0) = -1$

$g(x) = \displaystyle\int 6x^2\,dx = 2x^3 + C$

$g(0) = -1 = 2(0)^3 + C \Rightarrow C = -1$

$g(x) = 2x^3 - 1$

58. $f'(s) = 6s - 8s^3,\ f(2) = 3$

$f(s) = \displaystyle\int (6s - 8s^3)ds = 3s^2 - 2s^4 + C$

$f(2) = 3 = 3(2)^2 - 2(2)^4 + C = 12 - 32 + C \Rightarrow C = 23$

$f(s) = 3s^2 - 2s^4 + 23$

60. $f''(x) = x^2$

$f'(0) = 6$

$f(0) = 3$

$f'(x) = \displaystyle\int x^2\,dx = \frac{1}{3}x^3 + C_1$

$f'(0) = 0 + C_1 = 6 \Rightarrow C_1 = 6$

$f'(x) = \dfrac{1}{3}x^3 + 6$

$f(x) = \displaystyle\int \left(\frac{1}{3}x^3 + 6\right) dx = \frac{1}{12}x^4 + 6x + C_2$

$f(0) = 0 + 0 + C_2 = 3 \Rightarrow C_2 = 3$

$f(x) = \dfrac{1}{12}x^4 + 6x + 3$

62. $f''(x) = \sin x$

$f'(0) = 1$

$f(0) = 6$

$f'(x) = \displaystyle\int \sin x\,dx = -\cos x + C_1$

$f'(0) = -1 + C_1 = 1 \Rightarrow C_1 = 2$

$f'(x) = -\cos x + 2$

$f(x) = \displaystyle\int (-\cos x + 2)\,dx = -\sin x + 2x + C_2$

$f(0) = 0 + 0 + C_2 = 6 \Rightarrow C_2 = 6$

$f(x) = -\sin x + 2x + 6$

64. $\dfrac{dP}{dt} = k\sqrt{t},\ 0 \le t \le 10$

$$P(t) = \int kt^{1/2}\,dt = \frac{2}{3}kt^{3/2} + C$$

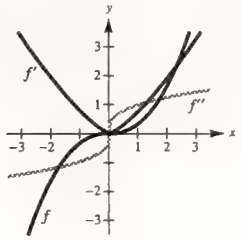
$$P(0) = 0 + C = 500 \Rightarrow C = 500$$

$$P(1) = \frac{2}{3}k + 500 = 600 \Rightarrow k = 150$$

$$P(t) = \frac{2}{3}(150)t^{3/2} + 500 = 100t^{3/2} + 500$$

$$P(7) = 100(7)^{3/2} + 500 \approx 2352 \text{ bacteria}$$

66. Since f'' is negative on $(-\infty, 0)$, f' is decreasing on $(-\infty, 0)$. Since f'' is positive on $(0, \infty)$, f' is increasing on $(0, \infty)$. f' has a relative minimum at $(0, 0)$. Since f' is positive on $(-\infty, \infty)$, f is increasing on $(-\infty, \infty)$.



68. $f''(t) = a(t) = -32 \text{ ft/sec}^2$

$f'(0) = v_0$

$f(0) = s_0$

$$f'(t) = v(t) = \int -32\,dt = -32t + C_1$$

$$f'(0) = 0 + C_1 = v_0 \Rightarrow C_1 = v_0$$

$$f'(t) = -32t + v_0$$

$$f(t) = s(t) = \int (-32t + v_0)\,dt = -16t^2 + v_0t + C_2$$

$$f(0) = 0 + 0 + C_2 = s_0 \Rightarrow C_2 = s_0$$

$$f(t) = -16t^2 + v_0t + s_0$$

70. $v_0 = 16 \text{ ft/sec}$

$s_0 = 64 \text{ ft}$

(a) $$s(t) = -16t^2 + 16t + 64 = 0$$

$$-16(t^2 - t - 4) = 0$$

$$t = \frac{1 \pm \sqrt{17}}{2}$$

Choosing the positive value,

$$t = \frac{1 + \sqrt{17}}{2} \approx 2.562 \text{ seconds.}$$

(b) $$v(t) = s'(t) = -32t + 16$$

$$v\left(\frac{1 + \sqrt{17}}{2}\right) = -32\left(\frac{1 + \sqrt{17}}{2}\right) + 16$$

$$= -16\sqrt{17} \approx -65.970 \text{ ft/sec}$$

72. From Exercise 71, $f(t) = -4.9t^2 + 1600$. (Using the canyon floor as position 0.)

$$f(t) = 0 = -4.9t^2 + 1600$$

$$4.9t^2 = 1600$$

$$t^2 = \frac{1600}{4.9} \Rightarrow t \approx \sqrt{326.53} \approx 18.1 \text{ sec}$$

74. From Exercise 71, $f(t) = -4.9t^2 + v_0t + 2$. If

$$f(t) = 200 = -4.9t^2 + v_0t + 2,$$

then

$$v(t) = -9.8t + v_0 = 0$$

for this t value. Hence, $t = v_0/9.8$ and we solve

$$-4.9\left(\frac{v_0}{9.8}\right)^2 + v_0\left(\frac{v_0}{9.8}\right) + 2 = 200$$

$$\frac{-4.9\,v_0^2}{(9.8)^2} + \frac{v_0^2}{9.8} = 198$$

$$-4.9\,v_0^2 + 9.8\,v_0^2 = (9.8)^2\,198$$

$$4.9\,v_0^2 = (9.8)^2\,198$$

$$v_0^2 = 3880.8 \Rightarrow v_0 \approx 62.3 \text{ m/sec.}$$

76. $\int v\,dv = -GM\int \frac{1}{y^2}\,dy$

$$\frac{1}{2}v^2 = \frac{GM}{y} + C$$

When $y = R$, $v = v_0$.

$$\frac{1}{2}v_0^2 = \frac{GM}{R} + C$$

$$C = \frac{1}{2}v_0^2 - \frac{GM}{R}$$

$$\frac{1}{2}v^2 = \frac{GM}{y} + \frac{1}{2}v_0^2 - \frac{GM}{R}$$

$$v^2 = \frac{2GM}{y} + v_0^2 - \frac{2GM}{R}$$

$$v^2 = v_0^2 + 2GM\left(\frac{1}{y} - \frac{1}{R}\right)$$

78. $x(t) = (t-1)(t-3)^2 \qquad 0 \le t \le 5$

$= t^3 - 7t^2 + 15t - 9$

(a) $v(t) = x'(t) = 3t^2 - 14t + 15 = (3t-5)(t-3)$

$a(t) = v'(t) = 6t - 14$

(b) $v(t) > 0$ when $0 < t < \frac{5}{3}$ and $3 < t < 5$.

(c) $a(t) = 6t - 14 = 0$ when $t = \frac{7}{3}$.

$$v\left(\frac{7}{3}\right) = \left(3\left(\frac{7}{3}\right) - 5\right)\left(\frac{7}{3} - 3\right) = 2\left(-\frac{2}{3}\right) = -\frac{4}{3}$$

80. (a) $a(t) = \cos t$

$$v(t) = \int a(t)\,dt = \int \cos t\,dt = \sin t + C_1 = \sin t \text{ (since } v_0 = 0)$$

$$f(t) = \int v(t)\,dt = \int \sin t\,dt = -\cos t + C_2$$

$$f(0) = 3 = -\cos(0) + C_2 = -1 + C_2 \Rightarrow C_2 = 4$$

$$f(t) = -\cos t + 4$$

(b) $v(t) = 0 = \sin t$ for $t = k\pi$, $k = 0, 1, 2, \ldots$

82. $v(0) = 45$ mph $= 66$ ft/sec

30 mph $= 44$ ft/sec

15 mph $= 22$ ft/sec

$a(t) = -a$

$v(t) = -at + 66$

$s(t) = -\frac{a}{2}t^2 + 66t$ (Let $s(0) = 0$.)

$v(t) = 0$ after car moves 132 ft.

$-at + 66 = 0$ when $t = \frac{66}{a}$.

$$s\left(\frac{66}{a}\right) = -\frac{a}{2}\left(\frac{66}{a}\right)^2 + 66\left(\frac{66}{a}\right)$$

$$= 132 \text{ when } a = \frac{33}{2} = 16.5.$$

$a(t) = -16.5$

$v(t) = -16.5t + 66$

$s(t) = -8.25t^2 + 66t$

(a) $-16.5t + 66 = 44$

$$t = \frac{22}{16.5} \approx 1.333$$

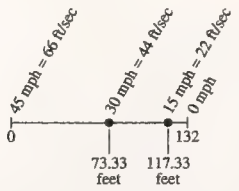
$$s\left(\frac{22}{16.5}\right) \approx 73.33 \text{ ft}$$

(b) $-16.5t + 66 = 22$

$$t = \frac{44}{16.5} \approx 2.667$$

$$s\left(\frac{44}{16.5}\right) \approx 117.33 \text{ ft}$$

(c)



It takes 1.333 seconds to reduce the speed from 45 mph to 30 mph, 1.333 seconds to reduce the speed from 30 mph to 15 mph, and 1.333 seconds to reduce the speed from 15 mph to 0 mph. Each time, less distance is needed to reach the next speed reduction.

84. No, car 2 will be ahead of car 1. If $v_1(t)$ and $v_2(t)$ are the respective velocities, then $\int_0^{30} |v_2(t)|\,dt > \int_0^{30} |v_1(t)|\,dt$.

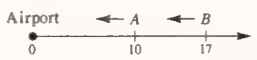
86. (a) $v = 0.6139t^3 - 5.525t^2 + 0.0492t + 65.9881$

(b) $s(t) = \int v(t)dt = \frac{0.6139t^4}{4} - \frac{5.525t^3}{3} + \frac{0.0492t^2}{2} + 65.9881t$

(**Note:** Assume $s(0) = 0$ is initial position)

$s(6) \approx 196.1$ feet

88. Let the aircrafts be located 10 and 17 miles away from the airport, as indicated in the figure.



$v_A(t) = k_A t - 150$ $\qquad$ $v_B = k_B t - 250$

$s_A(t) = \frac{1}{2}k_A t^2 - 150t + 10$ $\qquad$ $s_B = \frac{1}{2}k_B t^2 - 250t + 17$

(a) When aircraft A lands at time t_A you have

$$v_A(t_A) = k_A t_A - 150 = -100 \Rightarrow k_A = \frac{50}{t_A}$$

$$s_A(t_A) = \frac{1}{2}k_A t_A^2 - 150t_A + 10 = 0$$

$$\frac{1}{2}\left(\frac{50}{t_A}\right)t_A^2 - 150t_A = -10$$

$$125t_A = 10$$

$$t_A = \frac{10}{125}.$$

$$k_A = \frac{50}{t_A} = 50\left(\frac{125}{10}\right) = 625 \Rightarrow S_A(t) = S_1(t) = \frac{625}{2}t^2 - 150t + 10$$

Similarly, when aircraft B lands at time t_B you have

$$v_B(t_B) = k_B t_B - 250 = -115 \Rightarrow k_B = \frac{135}{t_B}$$

$$s_B(t_B) = \frac{1}{2}k_B t_B^2 - 250t_B + 17 = 0$$

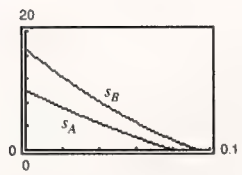
$$\frac{1}{2}\left(\frac{135}{t_B}\right)t_B^2 - 250t_B = -17$$

$$\frac{365}{2}t_B = 17$$

$$t_B = \frac{34}{365}.$$

$$k_B = \frac{135}{t_B} = 135\left(\frac{365}{34}\right) = \frac{49{,}275}{34} \Rightarrow S_B(t) = S_2(t) = \frac{49{,}275}{68}t^2 - 250t + 17$$

(b)



(c) $d = s_B(t) - s_A(t)$

Yes, $d < 3$ for $t > 0.0505$.



90. True

92. True

94. False. f has an infinite number of antiderivatives, each differing by a constant.

96. $\dfrac{d}{dx}\left[[s(x)]^2 + [c(x)]^2\right] = 2s(x)s'(x) + 2c(x)c'(x)$

$= 2s(x)c(x) - 2c(x)s(x)$

$= 0$

Thus, $[s(x)]^2 + [c(x)]^2 = k$ for some constant k. Since, $s(0) = 0$ and $c(0) = 1$, $k = 1$.

Therefore,

$[s(x)]^2 + [c(x)]^2 = 1$.

[Note that $s(x) = \sin x$ and $c(x) = \cos x$ satisfy these properties.]

Section 4.2 Area

2. $\displaystyle\sum_{k=3}^{6} k(k-2) = 3(1) + 4(2) + 5(3) + 6(4) = 50$

4. $\displaystyle\sum_{j=3}^{5} \frac{1}{j} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$

6. $\displaystyle\sum_{i=1}^{4}[(i-1)^2 + (i+1)^3] = (0+8) + (1+27) + (4+64) + (9+125) = 238$

8. $\displaystyle\sum_{i=1}^{15} \frac{5}{1+i}$

10. $\displaystyle\sum_{j=1}^{4}\left[1 - \left(\frac{j}{4}\right)^2\right]$

12. $\displaystyle\frac{2}{n}\sum_{i=1}^{n}\left[1 - \left(\frac{2i}{n} - 1\right)^2\right]$

14. $\displaystyle\frac{1}{n}\sum_{i=0}^{n-1}\sqrt{1 - \left(\frac{i}{n}\right)^2}$

16. $\displaystyle\sum_{i=1}^{15}(2i-3) = 2\sum_{i=1}^{15} i - 3(15)$

$\displaystyle= 2\left[\frac{15(16)}{2}\right] - 45 = 195$

18. $\displaystyle\sum_{i=1}^{10}(i^2-1) = \sum_{i=1}^{10} i^2 - \sum_{i=1}^{10} 1$

$\displaystyle= \left[\frac{10(11)(21)}{6}\right] - 10 = 375$

20. $\displaystyle\sum_{i=1}^{10} i(i^2+1) = \sum_{i=1}^{10} i^3 + \sum_{i=1}^{10} i$

$\displaystyle= \frac{10^2(11)^2}{4} + \left[\frac{10(11)}{2}\right] = 3080$

22. sum seq(x ^ 3 $- 2x$, x, 1, 15, 1) = 14,160 *(TI-82)*

$\displaystyle\sum_{i=1}^{15}(i^3 - 2i) = \frac{(15)^2(15+1)^2}{4} - 2\frac{15(15+1)}{2}$

$\displaystyle= \frac{(15)^2(16)^2}{4} - 15(16) = 14{,}160$

24. $S = [5 + 5 + 4 + 2](1) = 16$

$s = [4 + 4 + 2 + 0](1) = 10$

26. $\displaystyle S = \left[5 + 2 + 1 + \frac{2}{3} + \frac{1}{2}\right] = \frac{55}{6}$

$\displaystyle s = \left[2 + 1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{3}\right] = \frac{9}{2} = 4.5$

28. $\displaystyle S(8) = \left(\sqrt{\frac{1}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{3}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{1} + 2\right)\frac{1}{4}$

$\displaystyle + \left(\sqrt{\frac{5}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{3}{2}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{7}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{2} + 2\right)\frac{1}{4}$

$\displaystyle = \frac{1}{4}\left(16 + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{5}}{2} + \frac{\sqrt{6}}{2} + \frac{\sqrt{7}}{2} + \sqrt{2}\right) \approx 6.038$

$\displaystyle s(8) = (0+2)\frac{1}{4} + \left(\sqrt{\frac{1}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2\right)\frac{1}{4} + \cdots + \left(\sqrt{\frac{7}{4}} + 2\right)\frac{1}{4} \approx 5.685$

30. $S(5) = 1\left(\frac{1}{5}\right) + \sqrt{1-\left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1-\left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1-\left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1-\left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right)$

$= \frac{1}{5}\left[1 + \frac{\sqrt{24}}{5} + \frac{\sqrt{21}}{5} + \frac{\sqrt{16}}{5} + \frac{\sqrt{9}}{5}\right] \approx 0.859$

$s(5) = \sqrt{1-\left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1-\left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1-\left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1-\left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) + 0 \approx 0.659$

32. $\lim_{n\to\infty}\left[\left(\frac{64}{n^3}\right)\frac{n(n+1)(2n+1)}{6}\right] = \frac{64}{6}\lim_{n\to\infty}\left[\frac{2n^3+3n^2+n}{n^3}\right] = \frac{64}{6}(2) = \frac{64}{3}$

34. $\lim_{n\to\infty}\left[\left(\frac{1}{n^2}\right)\frac{n(n+1)}{2}\right] = \frac{1}{2}\lim_{n\to\infty}\left[\frac{n^2+n}{n^2}\right] = \frac{1}{2}(1) = \frac{1}{2}$

36. $\sum_{j=1}^{n}\frac{4j+3}{n^2} = \frac{1}{n^2}\sum_{j=1}^{n}(4j+3) = \frac{1}{n^2}\left[\frac{4n(n+1)}{2} + 3n\right] = \frac{2n+5}{n} = S(n)$

$S(10) = \frac{25}{10} = 2.5$

$S(100) = 2.05$

$S(1000) = 2.005$

$S(10{,}000) = 2.0005$

38. $\sum_{i=1}^{n}\frac{4i^2(i-1)}{n^4} = \frac{4}{n^4}\sum_{i=1}^{n}(i^3-i^2) = \frac{4}{n^4}\left[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\right]$

$= \frac{4}{n^3}\left[\frac{n^3+2n^2+n}{4} - \frac{2n^2+3n+1}{6}\right]$

$= \frac{1}{3n^3}[3n^3+6n^2+3n-4n^2-6n-2]$

$= \frac{1}{3n^3}[3n^3+2n^2-3n-2] = S(n)$

$S(10) = 1.056$

$S(100) = 1.006566$

$S(1000) = 1.00066567$

$S(10{,}000) = 1.000066657$

40. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n}i = \lim_{n\to\infty}\frac{4}{n^2}\left(\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\frac{4}{2}\left(1+\frac{1}{n}\right) = 2$

42. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)^2\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{2}{n^3}\sum_{i=1}^{n}(n+2i)^2$

$= \lim_{n\to\infty}\frac{2}{n^3}\left[\sum_{i=1}^{n}n^2 + 4n\sum_{i=1}^{n}i + 4\sum_{i=1}^{n}i^2\right]$

$= \lim_{n\to\infty}\frac{2}{n^3}\left[n^3 + (4n)\left(\frac{n(n+1)}{2}\right) + \frac{4(n)(n+1)(2n+1)}{6}\right]$

$= 2\lim_{n\to\infty}\left[1 + 2 + \frac{2}{n} + \frac{4}{3} + \frac{2}{n} + \frac{2}{3n^2}\right]$

$= 2\left(1 + 2 + \frac{4}{3}\right) = \frac{26}{3}$

44. $$\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)^3\left(\frac{2}{n}\right) = 2\lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}(n+2i)^3$$

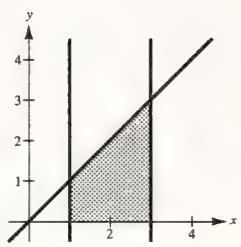
$$= 2\lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}(n^3+6n^2i+12ni^2+8i^3)$$

$$= 2\lim_{n\to\infty}\frac{1}{n^4}\left[n^4+6n^2\left(\frac{n(n+1)}{2}\right)+12n\left(\frac{n(n+1)(2n+1)}{6}\right)+8\left(\frac{n^2(n+1)^2}{4}\right)\right]$$

$$= 2\lim_{n\to\infty}\left(1+3+\frac{3}{n}+4+\frac{6}{n}+\frac{2}{n^2}+2+\frac{4}{n}+\frac{2}{n^2}\right)$$

$$= 2\lim_{n\to\infty}\left(10+\frac{13}{n}+\frac{4}{n^2}\right) = 20$$

46. (a)



(b) $\Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}$

Endpoints:

$$1 < 1+\frac{2}{n} < 1+\frac{4}{n} < \cdots < 1+\frac{2n}{n} = 3$$

$$1 < 1+1\left(\frac{2}{n}\right) < 1+2\left(\frac{2}{n}\right) < \cdots < 1+(n-1)\left(\frac{2}{n}\right) < 1+n\left(\frac{2}{n}\right)$$

(c) Since $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$= \sum_{i=1}^{n} f\left[1+(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[1+(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left[1+i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[1+i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(e)

x	5	10	50	100
$s(n)$	3.6	3.8	3.96	3.98
$S(n)$	4.4	4.2	4.04	4.02

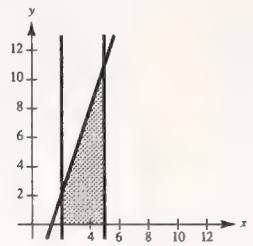
(f) $$\lim_{n\to\infty}\sum_{i=1}^{n}\left[1+(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\left(\frac{2}{n}\right)\left[n+\frac{2}{n}\left(\frac{n(n+1)}{2}-n\right)\right]$$

$$= \lim_{n\to\infty}\left[2+\frac{2n+2}{n}-\frac{4}{n}\right] = \lim_{n\to\infty}\left[4-\frac{2}{n}\right] = 4$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[1+i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{2}{n}\left[n+\left(\frac{2}{n}\right)\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[2+\frac{2(n+1)}{n}\right] = \lim_{n\to\infty}\left[4+\frac{2}{n}\right] = 4$$

48. $y = 3x - 4$ on $[2, 5]$. $\left(\textbf{Note: } \Delta x = \dfrac{5-2}{n} = \dfrac{3}{n}\right)$



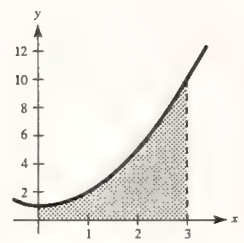
$$S(n) = \sum_{i=1}^{n} f\left(2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \sum_{i=1}^{n}\left[3\left(2 + \frac{3i}{n}\right) - 4\right]\left(\frac{3}{n}\right) = 18 + 3\left(\frac{3}{n}\right)^2 \sum_{i=1}^{n} i - 12$$

$$= 6 + \frac{27}{n^2}\left(\frac{(n+1)n}{2}\right) = 6 + \frac{27}{2}\left(1 + \frac{1}{n}\right)$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 6 + \frac{27}{2} = \frac{39}{2}$$

50. $y = x^2 + 1$ on $[0, 3]$. $\left(\textbf{Note: } \Delta x = \dfrac{3-0}{n} = \dfrac{3}{n}\right)$



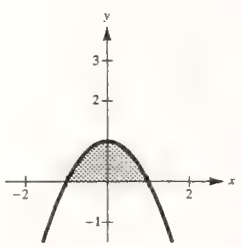
$$S(n) = \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^2 + 1\right]\left(\frac{3}{n}\right)$$

$$= \frac{27}{n^3}\sum_{i=1}^{n} i^2 + \frac{3}{n}\sum_{i=1}^{n} 1$$

$$= \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}(n) = \frac{9}{2}\frac{2n^2 + 3n + 1}{n^2} + 3$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \frac{9}{2}(2) + 3 = 12$$

52. $y = 1 - x^2$ on $[-1, 1]$. Find area of region over the interval $[0, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1}{n}\right)$



$$s(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[1 - \left(\frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right)$$

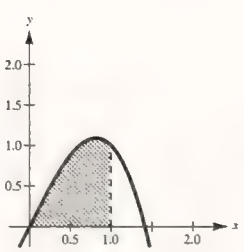
$$= 1 - \frac{1}{n^3}\sum_{i=1}^{n} i^2 = 1 - \frac{n(n+1)(2n+1)}{6n^3} = 1 - \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$\frac{1}{2}\,\text{Area} = \lim_{n\to\infty} s(n) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\text{Area} = \frac{4}{3}$$

54. $y = 2x - x^3$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}\right)$

Since y both increases and decreases on $[0, 1]$, $T(n)$ is neither an upper nor lower sum.



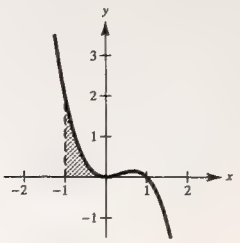
$$T(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[2\left(\frac{i}{n}\right) - \left(\frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right)$$

$$= \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{n(n+1)}{n^2} - \frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]$$

$$= 1 + \frac{1}{n} - \frac{1}{4} - \frac{2}{4n} - \frac{1}{4n^2}$$

$$\text{Area} = \lim_{n\to\infty} T(n) = 1 - \frac{1}{4} = \frac{3}{4}$$

56. $y = x^2 - x^3$ on $[-1, 0]$. $\left(\textbf{Note: } \Delta x = \dfrac{0-(-1)}{n} = \dfrac{1}{n}\right)$



$$s(n) = \sum_{i=1}^{n} f\left(-1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[\left(-1 + \frac{i}{n}\right)^2 - \left(-1 + \frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n}\left[\left(2 - \frac{5i}{n} + \frac{4i^2}{n^2} - \frac{i^3}{n^3}\right)\left(\frac{1}{n}\right) = 2 - \frac{5}{n^2}\sum_{i=1}^{n} i + \frac{4}{n^3}\sum_{i=1}^{n} i^2 - \frac{1}{n^4}\sum_{i=1}^{n} i^3\right.$$

$$= 2 - \frac{5}{2} - \frac{5}{2n} + \frac{4}{3} + \frac{2}{n} + \frac{2}{3n^3} - \frac{1}{4} - \frac{1}{2n} - \frac{1}{4n^2}$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 2 - \frac{5}{2} + \frac{4}{3} - \frac{1}{4} = \frac{7}{12}$$

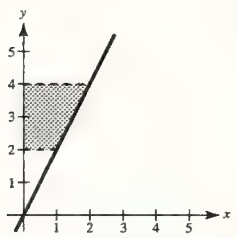
58. $g(y) = \dfrac{1}{2}y$, $2 \le y \le 4$. $\left(\textbf{Note: } \Delta y = \dfrac{4-2}{n} = \dfrac{2}{n}\right)$

$$S(n) = \sum_{i=1}^{n} g\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n} \frac{1}{2}\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \frac{2}{n}\sum_{i=1}^{n}\left(1 + \frac{i}{n}\right)$$

$$= \frac{2}{n}\left[n + \frac{1}{n}\frac{n(n+1)}{2}\right] = 2 + \frac{n+1}{n}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 2 + 1 = 3$$



60. $f(y) = 4y - y^2$, $1 \le y \le 2$. $\left(\textbf{Note: } \Delta y = \dfrac{2-1}{n} = \dfrac{1}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[4\left(1 + \frac{i}{n}\right) - \left(1 + \frac{i}{n}\right)^2\right]$$

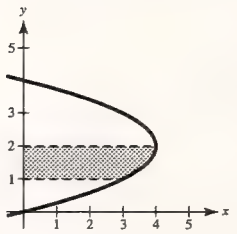
$$= \frac{1}{n}\sum_{i=1}^{n}\left(4 + \frac{4i}{n} - 1 - \frac{2i}{n} - \frac{i^2}{n^2}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(3 + \frac{2i}{n} - \frac{i^2}{n^2}\right)$$

$$= \frac{1}{n}\left[3n + \frac{2}{n}\frac{n(n+1)}{2} - \frac{1}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 3 + \frac{n+1}{n} - \frac{(n+1)(2n+1)}{6}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 3 + 1 - \frac{1}{3} = \frac{11}{3}$$



62. $h(y) = y^3 + 1$, $1 \le y \le 2$ $\left(\textbf{Note: } \Delta y = \dfrac{1}{n}\right)$

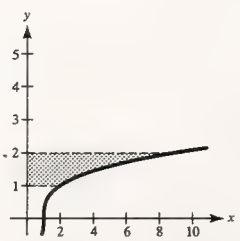
$$S(n) = \sum_{i=1}^{n} h\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n}\left[\left(1 + \frac{i}{n}\right)^3 + 1\right]\frac{1}{n}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(2 + \frac{i^3}{n^3} + \frac{3i^2}{n^2} + \frac{3i}{n}\right)$$

$$= \frac{1}{n}\left[2n + \frac{1}{n^3}\frac{n^2(n+1)^2}{4} + \frac{3}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}\frac{n(n+1)}{2}\right]$$

$$= 2 + \frac{(n+1)^2}{n^24} + \frac{1}{2}\frac{(n+1)(2n+1)}{n^2} + \frac{3(n+1)}{2n}$$



$$\text{Area} = \lim_{n\to\infty} S(n) = 2 + \frac{1}{4} + 1 + \frac{3}{2} = \frac{19}{4}$$

64. $f(x) = x^2 + 4x,\ 0 \le x \le 4,\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = 1,\ c_1 = \frac{1}{2},\ c_2 = \frac{3}{2},\ c_3 = \frac{5}{2},\ c_4 = \frac{7}{2}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4} [c_i^2 + 4c_i](1)$$

$$= \left[\left(\frac{1}{4} + 2\right) + \left(\frac{9}{4} + 6\right) + \left(\frac{25}{4} + 10\right) + \left(\frac{49}{4} + 14\right)\right]$$

$$= 53$$

66. $f(x) = \sin x,\ 0 \le x \le \dfrac{\pi}{2},\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = \frac{\pi}{8},\ c_1 = \frac{\pi}{16},\ c_2 = \frac{3\pi}{16},\ c_3 = \frac{5\pi}{16},\ c_4 = \frac{7\pi}{16}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4} (\sin c_i)\left(\frac{\pi}{8}\right)$$

$$= \frac{\pi}{8}\left(\sin\frac{\pi}{16} + \sin\frac{3\pi}{16} + \sin\frac{5\pi}{16} + \sin\frac{7\pi}{16}\right) \approx 1.006$$

68. $f(x) = \dfrac{8}{x^2 + 1}$ on $[2, 6]$.

n	4	8	12	16	20
Approximate area	2.3397	2.3755	2.3824	2.3848	2.3860

70. $f(x) = \cos\sqrt{x}$ on $[0, 2]$.

n	4	8	12	16	20
Approximate area	1.1041	1.1053	1.1055	1.1056	1.1056

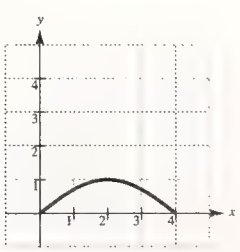
72. See the Definition of Area. Page 259.

74. $f(x) = \sqrt[3]{x},\ 0 \le x \le 8$

n	10	20	50	100	200
$s(n)$	10.998	11.519	11.816	11.910	11.956
$S(n)$	12.598	12.319	12.136	12.070	12.036
$M(n)$	12.040	12.016	12.005	12.002	12.001

(**Note:** exact answer is 12.)

76.

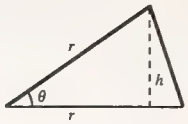


a. $A \approx 3$ square units

78. True. (Theorem 4.3)

80. (a) $\theta = \dfrac{2\pi}{n}$

(b) $\sin\theta = \dfrac{h}{r}$



$$h = r\sin\theta$$

$$A = \frac{1}{2}bh = \frac{1}{2}r(r\sin\theta) = \frac{1}{2}r^2\sin\theta$$

(c) $A_n = n\left(\frac{1}{2}r^2\sin\frac{2\pi}{n}\right) = \frac{r^2n}{2}\sin\frac{2\pi}{n} = \pi r^2\left(\frac{\sin(2\pi/n)}{2\pi/n}\right)$

Let $x = 2\pi/n$. As $n\to\infty$, $x\to 0$.

$$\lim_{n\to\infty} A_n = \lim_{x\to 0}\pi r^2\left(\frac{\sin x}{x}\right) = \pi r^2(1) = \pi r^2$$

82. (a) $\sum_{i=1}^{n} 2i = n(n+1)$

The formula is true for $n = 1$: $2 = 1(1+1) = 2$

Assume that the formula is true for $n = k$:

$$\sum_{i=1}^{k} 2i = k(k+1).$$

Then we have $\sum_{i=1}^{k+1} 2i = \sum_{i=1}^{k} 2i + 2(k+1)$

$$= k(k+1) + 2(k+1)$$

$$= (k+1)(k+2)$$

Which shows that the formula is true for $n = k+1$.

(b) $\sum_{i=1}^{n} i^3 = \dfrac{n^2(n+1)^2}{4}$

The formula is true for $n = 1$ because

$$1^3 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$$

Assume that the formula is true for $n = k$:

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$$

Then we have $\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{(k+1)^2}{4}[k^2 + 4(k+1)]$$

$$= \frac{(k+1)^2}{4}(k+2)^2$$

which shows that the formula is true for $n = k+1$.

Section 4.3 Riemann Sums and Definite Integrals

2. $f(x) = \sqrt[3]{x}$, $y = 0$, $x = 0$, $x = 1$, $c_i = \dfrac{i^3}{n^3}$

$$\Delta x_i = \frac{i^3}{n^3} - \frac{(i-1)^3}{n^3} = \frac{3i^2 - 3i + 1}{n^3}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt[3]{\frac{i^3}{n^3}}\left[\frac{3i^2 - 3i + 1}{n^3}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}(3i^3 - 3i^2 + i)$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left[3\left(\frac{n^2(n+1)^2}{4}\right) - 3\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{3n^4 + 6n^3 + 3n^2}{4} - \frac{2n^3 + 3n^2 + n}{2} + \frac{n^2 + n}{2}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{3n^4}{4} + \frac{n^3}{2} - \frac{n^2}{4}\right] = \lim_{n\to\infty}\left[\frac{3}{4} + \frac{1}{2n} - \frac{1}{4n^2}\right] = \frac{3}{4}$$

4. $y = x$ on $[-2, 3]$. $\left(\textbf{Note: } \Delta x = \dfrac{3-(-2)}{n} = \dfrac{5}{n},\ \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right) = \sum_{i=1}^{n}\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right) = -10 + \frac{25}{n^2}\sum_{i=1}^{n} i$$

$$= -10 + \left(\frac{25}{n^2}\right)\frac{n(n+1)}{2} = -10 + \frac{25}{2}\left(1 + \frac{1}{n}\right) = \frac{5}{2} + \frac{25}{2n}$$

$$\int_{-2}^{3} x\,dx = \lim_{n\to\infty}\left(\frac{5}{2} + \frac{25}{2n}\right) = \frac{5}{2}$$

6. $y = 3x^2$ on $[1, 3]$. $\left(\text{Note: } \Delta x = \dfrac{3-1}{n} = \dfrac{2}{n},\ \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n} 3\left(1 + \frac{2i}{n}\right)^2\left(\frac{2}{n}\right)$$

$$= \frac{6}{n}\sum_{i=1}^{n}\left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right)$$

$$= \frac{6}{n}\left[n + \frac{4}{n}\frac{n(n+1)}{2} + \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 6 + 12\frac{n+1}{n} + 4\frac{(n+1)(2n+1)}{n^2}$$

$$\int_{1}^{3} 3x^2\,dx = \lim_{n\to\infty}\left[6 + \frac{12(n+1)}{n} + \frac{4(n+1)(2n+1)}{n^2}\right]$$

$$= 6 + 12 + 8 = 26$$

8. $y = 3x^2 + 2$ on $[-1, 2]$. $\left(\textbf{Note: } \Delta x = \dfrac{2-(-1)}{n} = \dfrac{3}{n};\ \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} f\left(-1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \frac{3}{n}\sum_{i=1}^{n}\left[3\left(-1 + \frac{3i}{n}\right)^2 + 2\right.$$

$$= \frac{3}{n}\sum_{i=1}^{n}\left[3\left(1 - \frac{6i}{n} + \frac{9i^2}{n^2}\right) + 2\right]$$

$$= \frac{3}{n}\left[3n - \frac{18}{n}\frac{n(n+1)}{2} + \frac{27}{n^2}\frac{n(n+1)(2n+1)}{6} + 2n\right]$$

$$= 15 - \frac{27(n+1)}{n} + \frac{27}{2}\frac{(n+1)(2n+1)}{n^2}$$

$$\int_{-1}^{2} (3x^2 + 2)dx = \lim_{n\to\infty}\left[15 - 27\frac{(n+1)}{n} + \frac{27}{2}\frac{(n+1)(2n+1)}{n^2}\right]$$

$$= 15 - 27 + 27 = 15$$

10. $\lim\limits_{\|\Delta\|\to 0}\sum\limits_{i=1}^{n} 6c_i(4 - c_i)^2\,\Delta x_i = \int_0^4 6x(4-x)^2\,dx$ on the interval $[0, 4]$.

12. $\lim\limits_{\|\Delta\|\to 0}\sum\limits_{i=1}^{n}\left(\dfrac{3}{c_i^2}\right)\Delta x_i = \int_1^3 \dfrac{3}{x^2}\,dx$ on the interval $[1, 3]$.

14. $\int_0^2 (4 - 2x)\,dx$

16. $\int_0^2 x^2\,dx$

18. $\int_{-1}^{1} \dfrac{1}{x^2+1}\,dx$

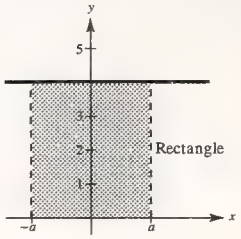
20. $\int_0^{\pi/4} \tan x\,dx$

22. $\int_0^2 (y-2)^2\,dy$

24. Rectangle

$A = bh = 2(4)(a)$

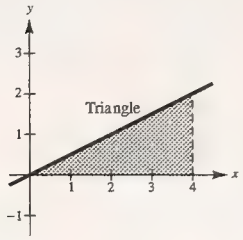
$A = \int_{-a}^{a} 4\,dx = 8a$



26. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(4)(2)$

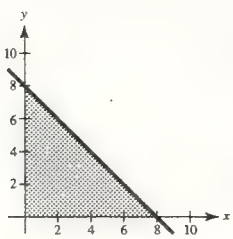
$A = \int_0^4 \frac{x}{2}\,dx = 4$



28. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(8)(8) = 32$

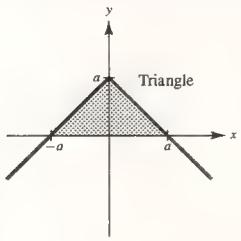
$A = \int_0^8 (8-x)dx = 32$



30. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(2a)a$

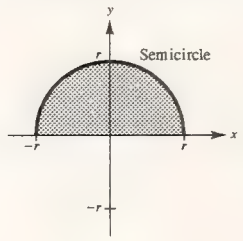
$A = \int_{-a}^{a} (a - |x|)\,dx = a^2$



32. Semicircle

$A = \frac{1}{2}\pi r^2$

$A = \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = \frac{1}{2}\pi r^2$



In Exercises 34–40, $\int_2^4 x^3\,dx = 60$, $\int_2^4 x\,dx = 6$, $\int_2^4 dx = 2$.

34. $\int_2^2 x^3\,dx = 0$

36. $\int_2^4 15\,dx = 15\int_2^4 dx = 15(2) = 30$

38. $\int_2^4 (x^3+4)dx = \int_2^4 x^3\,dx + 4\int_2^4 dx = 60 + 4(2) = 68$

40. $\int_2^4 (6 + 2x - x^3)dx = 6\int_2^4 dx + 2\int_2^4 x\,dx - \int_2^4 x^3\,dx$

$= 6(2) + 2(6) - 60 = -36$

42. (a) $\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx = 4 + (-1) = 3$

(b) $\int_6^3 f(x)\,dx = -\int_3^6 f(x)\,dx = -(-1) = 1$

(c) $\int_3^3 f(x)\,dx = 0$

(d) $\int_3^6 -5f(x)\,dx = -5\int_3^6 f(x)\,dx = -5(-1) = 5$

44. (a) $\int_{-1}^0 f(x)\,dx = \int_{-1}^1 f(x)\,dx - \int_0^1 f(x)\,dx = 0 - 5 = -5$

(b) $\int_0^1 f(x)\,dx - \int_1^0 f(x)\,dx = 5 - (-5) = 10$

(c) $\int_{-1}^1 3f(x)\,dx = 3\int_{-1}^1 f(x)\,dx = 3(0) = 0$

(d) $\int_0^1 3f(x)\,dx = 3\int_0^1 f(x)\,dx = 3(5) = 15$

46. (a) $\int_0^5 [f(x) + 2]\,dx = \int_0^5 f(x)\,dx + \int_0^5 2\,dx = 4 + 10 = 14$ (b) $\int_{-2}^3 f(x + 2)\,dx = \int_0^5 f(x)\,dx = 4$ (Let $u = x + 2$.)

(c) $\int_{-5}^5 f(x)\,dx = 2\int_0^5 f(x)\,dx = 2(4) = 8$ (f even) (d) $\int_{-5}^5 f(x)\,dx = 0$ (f odd)

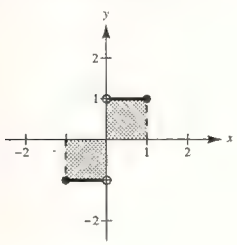
48. The right endpoint approximation will be less than the actual area: <

50. The average of Exercise 39 and Exercise 40 consists of a trapezoidal approximation, and is greater than the exact area: >

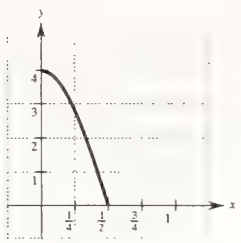
52. $f(x) = |x|/x$ is integrable on $[-1, 1]$, but is not continuous on $[-1, 1]$. There is discontinuity at $x = 0$. To see that

$$\int_{-1}^{1} \frac{|x|}{x}\,dx$$

is integrable, sketch a graph of the region bounded by $f(x) = |x|/x$ and the x-axis for $-1 \le x \le 1$. You see that the integral equals 0.

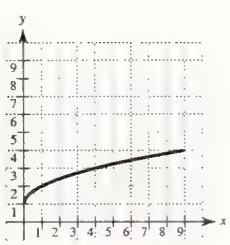


54.



b. $A \approx \frac{4}{3}$ square units

56.



c. Area ≈ 27.

58. $\int_0^3 \frac{5}{x^2 + 1}\,dx$

n	4	8	12	16	20
$L(n)$	7.9224	7.0855	6.8062	6.6662	6.5822
$M(n)$	6.2485	6.2470	7.2460	6.2457	6.2455
$R(n)$	4.5474	5.3980	5.6812	5.8225	5.9072

60. $\int_0^3 x \sin x\,dx$

n	4	8	12	16	20
$L(n)$	2.8186	2.9985	3.0434	3.0631	3.0740
$M(n)$	3.1784	3.1277	3.1185	3.1152	3.1138
$R(n)$	3.1361	3.1573	3.1493	3.1425	3.1375

62. False

$$\int_0^1 x\sqrt{x}\,dx \ne \left(\int_0^1 x\,dx\right)\left(\int_0^1 \sqrt{x}\,dx\right)$$

64. True

66. False

$$\int_{-2}^{4} x\,dx = 6$$

68. $f(x) = \sin x, [0, 2\pi]$

$x_0 = 0, x_1 = \frac{\pi}{4}, x_2 = \frac{\pi}{3}, x_3 = \pi, x_4 = 2\pi$

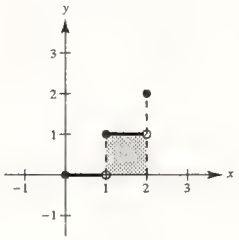
$\Delta x_1 = \frac{\pi}{4}, \Delta x_2 = \frac{\pi}{12}, \Delta x_3 = \frac{2\pi}{3}, \Delta x_4 = \pi$

$c_1 = \frac{\pi}{6}, c_2 = \frac{\pi}{3}, c_3 = \frac{2\pi}{3}, c_4 = \frac{3\pi}{2}$

$$\sum_{i=1}^{4} f(c_i)\,\Delta x_i = f\left(\frac{\pi}{6}\right)\Delta x_1 + f\left(\frac{\pi}{3}\right)\Delta x_2 + f\left(\frac{2\pi}{3}\right)\Delta x_3 + f\left(\frac{3\pi}{2}\right)\Delta x_4$$

$$= \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{12}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{2\pi}{3}\right) + (-1)(\pi) \approx -0.708$$

70. To find $\int_0^2 [\![x]\!]\,dx$, use a geometric approach.



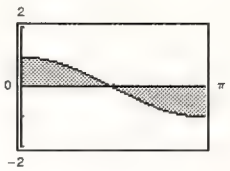
Thus,

$$\int_0^2 [\![x]\!]\,dx = 1(2 - 1) = 1.$$

Section 4.4 The Fundamental Theorem of Calculus

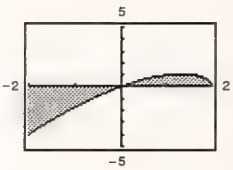
2. $f(x) = \cos x$

$\int_0^{\pi} \cos x\,dx = 0$



4. $f(x) = x\sqrt{2 - x}$

$\int_{-2}^{2} x\sqrt{2 - x}\,dx$ is negative.



6. $\int_2^7 3\,dv = \left[3v\right]_2^7 = 3(7) - 3(2) = 15$

8. $\int_2^5 (-3v + 4)\,dv = \left[-\frac{3}{2}v^2 + 4v\right]_2^5 = \left(-\frac{75}{2} + 20\right) - (-6 + 8) = -\frac{39}{2}$

10. $\int_1^3 (3x^2 + 5x - 4)dx = \left[x^3 + \frac{5x^2}{2} - 4x\right]_1^3 = \left(27 + \frac{45}{2} - 12\right) - \left(1 + \frac{5}{2} - 4\right)$

$= 38$

12. $\int_{-1}^{1} (t^3 - 9t)\,dt = \left[\frac{1}{4}t^4 - \frac{9}{2}t^2\right]_{-1}^{1} = \left(\frac{1}{4} - \frac{9}{2}\right) - \left(\frac{1}{4} - \frac{9}{2}\right) = 0$

14. $\int_{-2}^{-1} \left(u - \frac{1}{u^2}\right) du = \left[\frac{u^2}{2} + \frac{1}{u}\right]_{-2}^{-1} = \left(\frac{1}{2} - 1\right) - \left(2 - \frac{1}{2}\right) = -2$

16. $\displaystyle\int_{-3}^{3} v^{1/3}\,dv = \left[\frac{3}{4}v^{4/3}\right]_{-3}^{3} = \frac{3}{4}\left[\left(\sqrt[3]{-3}\right)^4\right] - \left(\sqrt[3]{-3}\right)^4] = 0$

18. $\displaystyle\int_1^8 \sqrt{\frac{2}{x}}\,dx = \sqrt{2}\int_1^8 x^{-1/2}\,dx = \left[\sqrt{2}(2)x^{1/2}\right]_1^8 = \left[2\sqrt{2x}\right]_1^8 = 8 - 2\sqrt{2}$

20. $\displaystyle\int_0^2 (2-t)\sqrt{t}\,dt = \int_0^2 (2t^{1/2} - t^{3/2})\,dt = \left[\frac{4}{3}t^{3/2} - \frac{2}{5}t^{5/2}\right]_0^2 = \left[\frac{t\sqrt{t}}{15}(20-6t)\right]_0^2 = \frac{2\sqrt{2}}{15}(20-12) = \frac{16\sqrt{2}}{15}$

22. $\displaystyle\int_{-8}^{-1} \frac{x-x^2}{2\sqrt[3]{x}}\,dx = \frac{1}{2}\int_{-8}^{-1} (x^{2/3} - x^{5/3})\,dx$

$$= \frac{1}{2}\left[\frac{3}{5}x^{5/3} - \frac{3}{8}x^{8/3}\right]_{-8}^{-1} = \left[\frac{x^{5/3}}{80}(24-15x)\right]_{-8}^{-1} = -\frac{1}{80}(39) + \frac{32}{80}(144) = \frac{4569}{80}$$

24. $\displaystyle\int_1^4 (3 - |x-3|)\,dx = \int_1^3 [3 + (x-3)]\,dx + \int_3^4 [3 - (x-3)]\,dx$

$$= \int_1^3 x\,dx + \int_3^4 (6-x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_1^3 + \left[6x - \frac{x^2}{2}\right]_3^4$$

$$= \left(\frac{9}{2} - \frac{1}{2}\right) + \left[(24-8) - \left(18 - \frac{9}{2}\right)\right]$$

$$= 4 + 16 - 18 + \frac{9}{2} = \frac{13}{2}$$

26. $\displaystyle\int_0^4 |x^2 - 4x + 3|\,dx = \int_0^1 (x^2-4x+3)\,dx - \int_1^3 (x^2-4x+3)\,dx + \int_3^4 (x^2-4x+3)\,dx$ (split up the integral at the zeros $x = 1, 3$)

$$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_0^1 - \left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3 + \left[\frac{x^3}{3} - 2x^2 + 3x\right]_3^4$$

$$= \left(\frac{1}{3} - 2 + 3\right) - (9 - 18 + 9) + \left(\frac{1}{3} - 2 + 3\right) + \left(\frac{64}{3} - 32 + 12\right) - (9 - 18 + 9)$$

$$= \frac{4}{3} - 0 + \frac{4}{3} + \frac{4}{3} - 0 = 4$$

28. $\displaystyle\int_0^{\pi/4} \frac{1 - \sin^2\theta}{\cos^2\theta}\,d\theta = \int_0^{\pi/4} d\theta = \left[\theta\right]_0^{\pi/4} = \frac{\pi}{4}$

30. $\displaystyle\int_{\pi/4}^{\pi/2} (2 - \csc^2 x)\,dx = \left[2x + \cot x\right]_{\pi/4}^{\pi/2} = (\pi + 0) - \left(\frac{\pi}{2} + 1\right) = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$

32. $\displaystyle\int_{-\pi/2}^{\pi/2} (2t + \cos t)\,dt = \left[t^2 + \sin t\right]_{-\pi/2}^{\pi/2} = \left(\frac{\pi^2}{4} + 1\right) - \left(\frac{\pi^2}{4} - 1\right) = 2$

34. $P = \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2} \sin\theta\,d\theta = \left[-\frac{2}{\pi}\cos\theta\right]_0^{\pi/2} = -\frac{2}{\pi}(0-1) = \frac{2}{\pi} \approx 63.7\%$

36. $A = \displaystyle\int_{-1}^{1} (1 - x^4)\,dx = \left[x - \frac{1}{5}x^5\right]_{-1}^{1} = \frac{8}{5}$

38. $A = \displaystyle\int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$

40. $A = \displaystyle\int_0^{\pi} (x + \sin x)\,dx = \left[\frac{x^2}{2} - \cos x\right]_0^{\pi} = \frac{\pi^2}{2} + 2 = \frac{\pi^2 + 4}{2}$

42. Since $y \geq 0$ on $[0, 8]$,

$$\text{Area} = \int_0^8 (1 + x^{1/3})\,dx = \left[x + \frac{3}{4}x^{4/3}\right]_0^8 = 8 + \frac{3}{4}(16) = 20$$

44. Since $y \geq 0$ on $[0, 3]$,

$$A = \int_0^3 (3x - x^2)\,dx = \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_0^3 = \frac{9}{2}.$$

46. $\displaystyle\int_1^3 \frac{9}{x^3}\,dx = \left[-\frac{9}{2x^2}\right]_1^3 = -\frac{1}{2} + \frac{9}{2} = 4$

$$f(c)(3 - 1) = 4$$

$$\frac{9}{c^3} = 2$$

$$c^3 = \frac{9}{2}$$

$$c = \sqrt[3]{\frac{9}{2}} \approx 1.6510$$

48. $\displaystyle\int_{-\pi/3}^{\pi/3} \cos x\,dx = \left[\sin x\right]_{-\pi/3}^{\pi/3} = \sqrt{3}$

$$f(c)\left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right] = \sqrt{3}$$

$$\cos c = \frac{3\sqrt{3}}{2\pi}$$

$$c \approx \pm 0.5971$$

50. $\displaystyle\frac{1}{3 - 1}\int_1^3 \frac{4(x^2 + 1)}{x^2}\,dx = 2\int_1^3 (1 + x^{-2})\,dx = 2\left[x - \frac{1}{x}\right]_1^3$

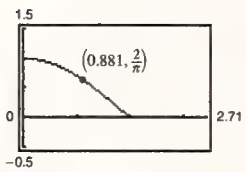
$$= 2\left(3 - \frac{1}{3}\right) = \frac{16}{3}$$

52. $\displaystyle\frac{1}{(\pi/2) - 0}\int_0^{\pi/2} \cos x\,dx = \left[\frac{2}{\pi}\sin x\right]_0^{\pi/2} = \frac{2}{\pi}$

$$\text{Average value} = \frac{2}{\pi}$$

$$\cos x = \frac{2}{\pi}$$

$$x \approx 0.881$$

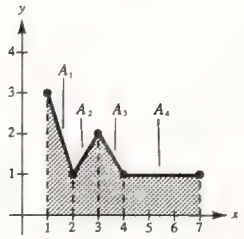


54. (a) $\displaystyle\int_1^7 f(x)\,dx$ = Sum of the areas

$$= A_1 + A_2 + A_3 + A_4$$

$$= \frac{1}{2}(3 + 1) + \frac{1}{2}(1 + 2) + \frac{1}{2}(2 + 1) + (3)(1)$$

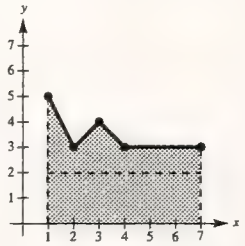
$$= 8$$



(b) $\displaystyle\text{Average value} = \frac{\int_1^7 f(x)\,dx}{7 - 1} = \frac{8}{6} = \frac{4}{3}$

(c) $A = 8 + (6)(2) = 20$

$$\text{Average value} = \frac{20}{6} = \frac{10}{3}$$



56. $\int_2^6 f(x)\,dx = \text{(area or region } B) = \int_0^6 f(x)\,dx - \int_0^2 f(x)\,dx$

$= 3.5 - (-1.5) = 5.0$

58. $\int_0^2 -2f(x)\,dx = -2\int_0^2 f(x)\,dx = -2(-1.5) = 3.0$

60. Average value $= \frac{1}{6}\int_0^6 f(x)\,dx = \frac{1}{6}(3.5) = 0.5833$

62. $\frac{1}{R-0}\int_0^R k(R^2 - r^2)\,dr = \frac{k}{R}\left[R^2 r - \frac{r^3}{3}\right]_0^R = \frac{2kR^2}{3}$

64. $P = 5\left(\sqrt{t} + 30\right)$

(a)

t	1	2	3	4	5	6
P	155	157.071	158.660	160	161.180	162.247

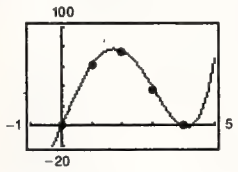
Average profit $\approx \frac{1}{6}(155 + 157.071 + 158.660 + 160 + 161.180 + 162.247) = \frac{954.158}{6} \approx 159.026$

(b) $\frac{1}{6}\int_{0.5}^{6.5} 5\left(\sqrt{t} + 30\right)dt = \frac{1}{6}\left[5\left(\frac{2}{3}t^{3/2} + 30t\right)\right]_{5.0}^{6.5} \approx \frac{954.061}{6} \approx 159.010$

(c) The definite integral yields a better approximation.

66. (a) $R = 2.33t^4 - 14.67t^3 + 3.67t^2 + 70.67t$

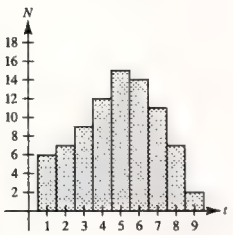
(b)



(c) $\int_0^4 R(t)\,dt = \left[\frac{2.33t^5}{5} - \frac{14.67t^4}{4} + \frac{3.67t^3}{3} + \frac{70.67t^2}{2}\right]_0^4$

$= 181.957$

68. (a) histogram

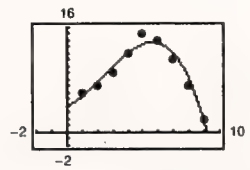


(b) $[6 + 7 + 9 + 12 + 15 + 14 + 11 + 7 + 2]60 = (83)60 = 4980$ customers

(c) Using a graphing utility, you obtain

$N(t) = -0.084175t^3 + 0.63492t^2 + 0.79052 + 4.10317.$

(d)



(e) $\int_0^9 N(t)\,dt \approx 85.162$

The estimated number of customers is $(85.162)(60) \approx 5110$.

(f) Between 3 P.M. and 7 P.M., the number of customers is approximately $\left(\int_3^7 N(t)\,dt\right)(60) \approx (50.28)(60) \approx 3017$.

Hence, $3017/240 \approx 12.6$ per minute.

70. $F(x) = \displaystyle\int_2^x (t^3 + 2t - 2)\,dt = \left[\frac{t^4}{4} + t^2 - 2t\right]_2^x$

$$= \left(\frac{x^4}{4} + x^2 - 2x\right) - (4 + 4 - 4)$$

$$= \frac{x^4}{4} + x^2 - 2x - 4$$

$F(2) = 4 + 4 - 4 - 4 = 0$ $\quad\left[\text{Note: } F(2) = \displaystyle\int_2^2 (t^3 + 2t - 2)\,dt = 0\right]$

$F(5) = \dfrac{625}{4} + 25 - 10 - 4 = 167.25$

$F(8) = \dfrac{8^4}{4} + 64 - 16 - 4 = 1068$

72. $F(x) = \displaystyle\int_2^x \frac{-2}{t^3}\,dt = -\int_2^x 2t^{-3}\,dt = \frac{1}{t^2}\Big]_2^x = \frac{1}{x^2} - \frac{1}{4}$

$F(2) = \dfrac{1}{4} - \dfrac{1}{4} = 0$

$F(5) = \dfrac{1}{25} - \dfrac{1}{4} = -\dfrac{21}{100} = -0.21$

$F(8) = \dfrac{1}{64} - \dfrac{1}{4} = -\dfrac{15}{64}$

74. $F(x) = \displaystyle\int_0^x \sin\theta\,d\theta = -\cos\theta\Big]_0^x = -\cos x + \cos 0 = 1 - \cos x$

$F(2) = 1 - \cos 2 \approx 1.4161$

$F(5) = 1 - \cos 5 \approx 0.7163$

$F(8) = 1 - \cos 8 \approx 1.1455$

76. (a) $\displaystyle\int_0^x t(t^2 + 1)\,dt = \int_0^x (t^3 + t)\,dt = \left[\frac{1}{4}t^4 + \frac{1}{2}t^2\right]_0^x = \frac{1}{4}x^4 + \frac{1}{2}x^2 = \frac{x^2}{4}(x^2 + 2)$

(b) $\dfrac{d}{dx}\left[\dfrac{1}{4}x^4 + \dfrac{1}{2}x^2\right] = x^3 + x = x(x^2 + 1)$

78. (a) $\displaystyle\int_4^x \sqrt{t}\,dt = \left[\frac{2}{3}t^{3/2}\right]_4^x = \frac{2}{3}x^{3/2} - \frac{16}{3} = \frac{2}{3}(x^{3/2} - 8)$

(b) $\dfrac{d}{dx}\left[\dfrac{2}{3}x^{3/2} - \dfrac{16}{3}\right] = x^{1/2} = \sqrt{x}$

80. (a) $\displaystyle\int_{\pi/3}^x \sec t \tan t\,dt = \Big[\sec t\Big]_{\pi/3}^x = \sec x - 2$

(b) $\dfrac{d}{dx}[\sec x - 2] = \sec x \tan x$

82. $F(x) = \displaystyle\int_1^x \frac{t^2}{t^2 + 1}\,dt$

$F'(x) = \dfrac{x^2}{x^2 + 1}$

84. $F(x) = \displaystyle\int_1^x \sqrt[4]{t}\,dt$

$F'(x) = \sqrt[4]{x}$

86. $F(x) = \displaystyle\int_0^x \sec^3 t\,dt$

$F'(x) = \sec^3 x$

88. $F(x) = \int_{-x}^{x} t^3\,dt = \left[\frac{t^4}{4}\right]_{-x}^{x} = 0$

$F'(x) = 0$

Alternate solution

$$F(x) = \int_{-x}^{x} t^3\,dt$$
$$= \int_{-x}^{0} t^3\,dt + \int_{0}^{x} t^3\,dt$$
$$= -\int_{0}^{-x} t^3\,dt + \int_{0}^{x} t^3\,dt$$
$$F'(x) = -(-x)^3(-1) + (x^3) = 0$$

90. $F(x) = \int_{2}^{x^2} t^{-3}\,dt = \left[\frac{t^{-2}}{-2}\right]_{2}^{x^2} = \left[-\frac{1}{2t^2}\right]_{2}^{x^2} = \frac{-1}{2x^4} + \frac{1}{8} \Rightarrow F'(x) = 2x^{-5}$

Alternate solution: $F'(x) = (x^2)^{-3}(2x) = 2x^{-5}$

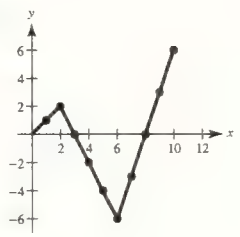
92. $F(x) = \int_{0}^{x^2} \sin\theta^2 d\theta$

$F'(x) = \sin(x^2)^2\,(2x) = 2x\sin x^4$

94. (a)

x	1	2	3	4	5	6	7	8	9	10
$g(x)$	1	2	0	-2	-4	-6	-3	0	3	6

(b)



(c) Minimum of g at $(6, -6)$.

(d) Minimum at $(10, 6)$. Relative maximum at $(2, 2)$.

(e) On $[6, 10]$ g increases at a rate of $\frac{12}{4} = 3$.

(f) Zeros of g: $x = 3, x = 8$.

96. (a) $g(t) = 4 - \frac{4}{t^2}$

$\lim_{t\to\infty} g(t) = 4$

Horizontal asymptote: $y = 4$

(b) $A(x) = \int_{1}^{x}\left(4 - \frac{4}{t^2}\right) dt$

$$= \left[4t + \frac{4}{t}\right]_{1}^{x} = 4x + \frac{4}{x} - 8$$
$$= \frac{4x^2 - 8x + 4}{x} = \frac{4(x-1)^2}{x}$$

$\lim_{x\to\infty} A(x) = \lim_{x\to\infty}\left(4x + \frac{4}{x} - 8\right) = \infty + 0 - 8 = \infty$

The graph of $A(x)$ does not have a horizontal asymptote.

98. True

100. Let $F(t)$ be an antiderivative of $f(t)$. Then,

$$\int_{u(x)}^{v(x)} f(t)\,dt = \Big[F(t)\Big]_{u(x)}^{v(x)} = F(v(x)) - F(u(x))$$
$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\,dt\right] = \frac{d}{dx}[F(v(x)) - F(u(x))]$$
$$= F'(v(x))v'(x) - F'(u(x))u'(x)$$
$$= f(v(x))v'(x) - f(u(x))u'(x).$$

102. $G(x) = \int_0^x \left[s \int_0^s f(t)\,dt \right] ds$

(a) $G(0) = \int_0^0 \left[s \int_0^s f(t)\,dt \right] ds = 0$

(b) Let $F(s) = s\int_0^s f(t)\,dt$.

$G(x) = \int_0^x F(s)\,ds$

$G'(x) = F(x) = x\int_0^x f(t)\,dt$

$G'(0) = 0\int_0^0 f(t)\,dt = 0$

(c) $G''(x) = x \cdot f(x) + \int_0^x f(t)\,dt$

(d) $G''(0) = 0 \cdot f(0) + \int_0^0 f(t)\,dt = 0$

104. $x(t) = (t-1)(t-3)^2 = t^3 - 7t^2 + 15t - 9$

$x'(t) = 3t^2 - 14t + 15$

Using a graphing utility,

$$\text{Total distance} = \int_0^5 |x'(t)|\,dt \approx 27.37 \text{ units}$$

Section 4.5 Integration by Substitution

$\int f(g(x))g'(x)\,dx$	$u = g(x)$	$du = g'(x)\,dx$
2. $\int x^2\sqrt{x^3+1}\,dx$	$x^3 + 1$	$3x^2\,dx$
4. $\int \sec 2x \tan 2x\,dx$	$2x$	$2\,dx$
6. $\int \frac{\cos x}{\sin^2 x}\,dx$	$\sin x$	$\cos x\,dx$

8. $\int (x^2-9)^3(2x)\,dx = \frac{(x^2-9)^4}{4} + C$

Check: $\frac{d}{dx}\left[\frac{(x^2-9)^4}{4} + C\right] = \frac{4(x^2-9)^3}{4}(2x) = (x^2-9)^3(2x)$

10. $\int (1-2x^2)^{1/3}(-4x)\,dx = \frac{3}{4}(1-2x^2)^{4/3} + C$

Check: $\frac{d}{dx}\left[\frac{3}{4}(1-2x^2)^{4/3} + C\right] = \frac{3}{4} \cdot \frac{4}{3}(1-2x^2)^{1/3}(-4x) = (1-2x^2)^{1/3}(-4x)$

12. $\int x^2(x^3+5)^4\,dx = \frac{1}{3}\int (x^3+5)^4(3x^2)\,dx = \frac{1}{3}\frac{(x^3+5)^5}{5} + C = \frac{(x^3+5)^5}{15} + C$

Check: $\frac{d}{dx}\left[\frac{(x^3+5)^5}{15} + C\right] = \frac{5(x^3+5)^4(3x^2)}{15} = (x^3+5)^4x^2$

14. $\int x(4x^2+3)^3\,dx = \frac{1}{8}\int (4x^2+3)^3(8x)\,dx = \frac{1}{8}\left[\frac{(4x^2+3)^4}{4}\right] + C = \frac{(4x^2+3)^4}{32} + C$

Check: $\frac{d}{dx}\left[\frac{(4x^2+3)^4}{32} + C\right] = \frac{4(4x^2+3)^3(8x)}{32} = x(4x^2+3)^3$

16. $\displaystyle\int t^3\sqrt{t^4+5}\,dt = \frac{1}{4}\int (t^4+5)^{1/2}(4t^3)\,dt = \frac{1}{4}\frac{(t^4+5)^{3/2}}{3/2} + C = \frac{1}{6}(t^4+5)^{3/2} + C$

Check: $\displaystyle\frac{d}{dt}\left[\frac{1}{6}(t^4+5)^{3/2} + C\right] = \frac{1}{6}\cdot\frac{3}{2}(t^4+5)^{1/2}(4t^3) = (t^4+5)^{1/2}(t^3)$

18. $\displaystyle\int u^2\sqrt{u^3+2}\,du = \frac{1}{3}\int (u^3+2)^{1/2}(3u^2)\,du = \frac{1}{3}\frac{(u^3+2)^{3/2}}{3/2} + C = \frac{2(u^3+2)^{3/2}}{9} + C$

Check: $\displaystyle\frac{d}{du}\left[\frac{2(u^3+2)^{3/2}}{9} + C\right] = \frac{2}{9}\cdot\frac{3}{2}(u^3+2)^{1/2}(3u^2) = (u^3+2)^{1/2}(u^2)$

20. $\displaystyle\int \frac{x^3}{(1+x^4)^2}\,dx = \frac{1}{4}\int (1+x^4)^{-2}(4x^3)\,dx = -\frac{1}{4}(1+x^4)^{-1} + C = \frac{-1}{4(1+x^4)} + C$

Check: $\displaystyle\frac{d}{dx}\left[\frac{-1}{4(1+x^4)} + C\right] = \frac{1}{4}(1+x^4)^{-2}(4x^3) = \frac{x^3}{(1+x^4)^2}$

22. $\displaystyle\int \frac{x^2}{(16-x^3)^2}\,dx = -\frac{1}{3}\int (16-x^3)^{-2}(-3x^2)\,dx = -\frac{1}{3}\left[\frac{(16-x^3)^{-1}}{-1}\right] + C = \frac{1}{3(16-x^3)} + C$

Check: $\displaystyle\frac{d}{dx}\left[\frac{1}{3(16-x^3)} + C\right] = \frac{1}{3}(-1)(16-x^3)^{-2}(3x^2) = \frac{x^2}{(16-x^3)^2}$

24. $\displaystyle\int \frac{x^3}{\sqrt{1+x^4}}\,dx = \frac{1}{4}\int (1+x^4)^{-1/2}(4x^3)\,dx = \frac{1}{4}\frac{(1+x^4)^{1/2}}{1/2} + C = \frac{\sqrt{1+x^4}}{2} + C$

Check: $\displaystyle\frac{d}{dx}\left[\frac{\sqrt{1+x^4}}{2} + C\right] = \frac{1}{2}\cdot\frac{1}{2}(1+x^4)^{-1/2}(4x^3) = \frac{x^3}{\sqrt{1+x^4}}$

26. $\displaystyle\int\left[x^2 + \frac{1}{(3x)^2}\right]dx = \int\left(x^2 + \frac{1}{9}x^{-2}\right)dx = \frac{x^3}{3} + \frac{1}{9}\left(\frac{x^{-1}}{-1}\right) + C = \frac{x^3}{3} - \frac{1}{9x} + C = \frac{3x^4-1}{9x} + C$

Check: $\displaystyle\frac{d}{dx}\left[\frac{1}{3}x^3 - \frac{1}{9}x^{-1} + C\right] = x^2 + \frac{1}{9}x^{-2} = x^2 + \frac{1}{(3x)^2}$

28. $\displaystyle\int \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2}\int x^{-1/2}\,dx = \frac{1}{2}\left(\frac{x^{1/2}}{1/2}\right) + C = \sqrt{x} + C$

Check: $\displaystyle\frac{d}{dx}\left[\sqrt{x} + C\right] = \frac{1}{2\sqrt{x}}$

30. $\displaystyle\int \frac{t+2t^2}{\sqrt{t}}\,dt = \int (t^{1/2} + 2t^{3/2})\,dt = \frac{2}{3}t^{3/2} + \frac{4}{5}t^{5/2} + C = \frac{2}{15}t^{3/2}(5+6t) + C$

Check: $\displaystyle\frac{d}{dt}\left[\frac{2}{3}t^{3/2} + \frac{4}{5}t^{5/2} + C\right] = t^{1/2} + 2t^{3/2} = \frac{t+2t^2}{\sqrt{t}}$

32. $\displaystyle\int\left(\frac{t^3}{3} + \frac{1}{4t^2}\right)dt = \int\left(\frac{1}{3}t^3 + \frac{1}{4}t^{-2}\right)dt = \frac{1}{3}\left(\frac{t^4}{4}\right) + \frac{1}{4}\left(\frac{t^{-1}}{-1}\right) + C = \frac{1}{12}t^4 - \frac{1}{4t} + C$

Check: $\displaystyle\frac{d}{dt}\left[\frac{1}{12}t^4 - \frac{1}{4t} + C\right] = \frac{1}{3}t^3 + \frac{1}{4t^2}$

34. $\displaystyle\int 2\pi y(8 - y^{3/2})\,dy = 2\pi\int (8y - y^{5/2})\,dy = 2\pi\left(4y^2 - \frac{2}{7}y^{7/2}\right) + C = \frac{4\pi y^2}{7}(14 - y^{3/2}) + C$

Check: $\displaystyle\frac{d}{dy}\left[\frac{4\pi y^2}{7}(14 - y^{3/2}) + C\right] = \frac{d}{dy}\left[2\pi\left(4y^2 - \frac{2}{7}y^{7/2}\right) + C\right] = 16\pi y - 2\pi y^{5/2} = (2\pi y)(8 - y^{3/2})$

36. $\displaystyle y = \int \frac{10x^2}{\sqrt{1 + x^3}}\,dx$

$\displaystyle = \frac{10}{3}\int (1 + x^3)^{-1/2}(3x^2)\,dx$

$\displaystyle = \frac{10}{3}\left[\frac{(1 + x^3)^{1/2}}{1/2}\right] + C$

$\displaystyle = \frac{20}{3}\sqrt{1 + x^3} + C$

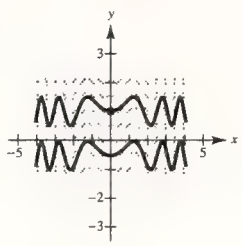
38. $\displaystyle y = \int \frac{x - 4}{\sqrt{x^2 - 8x + 1}}\,dx$

$\displaystyle = \frac{1}{2}\int (x^2 - 8x + 1)^{-1/2}(2x - 8)\,dx$

$\displaystyle = \frac{1}{2}\left[\frac{(x^2 - 8x + 1)^{1/2}}{1/2}\right] + C$

$= \sqrt{x^2 - 8x + 1} + C$

40. (a)



(b) $\displaystyle\frac{dy}{dx} = x\cos x^2$, (0, 1)

$\displaystyle y = \int x\cos x^2\,dx = \frac{1}{2}\int \cos(x^2)2x\,dx$

$\displaystyle = \frac{1}{2}\sin(x^2) + C$

(0, 1): $\displaystyle 1 = \frac{1}{2}\sin(0) + C \Rightarrow C = 1$

$\displaystyle y = \frac{1}{2}\sin(x^2) + 1$

42. $\displaystyle\int 4x^3 \sin x^4\,dx = \int \sin x^4(4x^3)\,dx = -\cos x^4 + C$

44. $\displaystyle\int \cos 6x\,dx = \frac{1}{6}\int (\cos 6x)(6)\,dx = \frac{1}{6}\sin 6x + C$

46. $\displaystyle\int x\sin x^2\,dx = \frac{1}{2}\int (\sin x^2)(2x)\,dx = -\frac{1}{2}\cos x^2 + C$

48. $\displaystyle\int \sec(1 - x)\tan(1 - x)\,dx = -\int [\sec(1 - x)\tan(1 - x)](-1)\,dx = -\sec(1 - x) + C$

50. $\displaystyle\int \sqrt{\tan x}\sec^2 x\,dx = \frac{(\tan x)^{3/2}}{3/2} + C = \frac{2}{3}(\tan x)^{3/2} + C$

52. $\displaystyle\int \frac{\sin x}{\cos^3 x}\,dx = -\int (\cos x)^{-3}(-\sin x)\,dx = -\frac{(\cos x)^{-2}}{-2} + C = \frac{1}{2\cos^2 x} + C = \frac{1}{2}\sec^2 x + C$

54. $\displaystyle\int \csc^2\left(\frac{x}{2}\right)dx = 2\int \csc^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)dx = -2\cot\left(\frac{x}{2}\right) + C$

56. $\displaystyle f(x) = \int \pi\sec \pi x\tan \pi x\,dx = \sec \pi x + C$

Since $f(1/3) = 1 = \sec(\pi/3) + C$, $C = -1$. Thus

$f(x) = \sec \pi x - 1$.

58. $u = 2x + 1, x = \frac{1}{2}(u - 1), dx = \frac{1}{2}du$

$$\int x\sqrt{2x+1}\,dx = \int \frac{1}{2}(u-1)\sqrt{u}\,\frac{1}{2}\,du$$

$$= \frac{1}{4}\int (u^{3/2} - u^{1/2})\,du$$

$$= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C$$

$$= \frac{u^{3/2}}{30}(3u - 5) + C$$

$$= \frac{1}{30}(2x+1)^{3/2}[3(2x+1) - 5] + C$$

$$= \frac{1}{30}(2x+1)^{3/2}(6x - 2) + C$$

$$= \frac{1}{15}(2x+1)^{3/2}(3x - 1) + C$$

60. $u = 2 - x, x = 2 - u, dx = -du$

$$\int (x+1)\sqrt{2-x}\,dx = -\int (3-u)\sqrt{u}\,du$$

$$= -\int (3u^{1/2} - u^{3/2})\,du$$

$$= -\left(2u^{3/2} - \frac{2}{5}u^{5/2}\right) + C$$

$$= -\frac{2u^{3/2}}{5}(5 - u) + C$$

$$= -\frac{2}{5}(2-x)^{3/2}[5 - (2-x)] + C$$

$$= -\frac{2}{5}(2-x)^{3/2}(x+3) + C$$

62. Let $u = x + 4, x = u - 4, du = dx$.

$$\int \frac{2x+1}{\sqrt{x+4}}\,dx = \int \frac{2(u-4)+1}{\sqrt{u}}\,du$$

$$= \int (2u^{1/2} - 7u^{-1/2})\,du$$

$$= \frac{4}{3}u^{3/2} - 14u^{1/2} + C$$

$$= \frac{2}{3}u^{1/2}(2u - 21) + C$$

$$= \frac{2}{3}\sqrt{x+4}[2(x+4) - 21] + C$$

$$= \frac{2}{3}\sqrt{x+4}(2x - 13) + C$$

64. $u = t - 4, t = u + 4, dt = du$

$$\int t\sqrt[3]{t-4}\,dt = \int (u+4)u^{1/3}\,du$$

$$= \int (u^{4/3} + 4u^{1/3})\,du$$

$$= \frac{3}{7}u^{7/3} + 3u^{4/3} + C$$

$$= \frac{3u^{4/3}}{7}(u + 7) + C$$

$$= \frac{3}{7}(t-4)^{4/3}[(t-4) + 7] + C$$

$$= \frac{3}{7}(t-4)^{4/3}(t+3) + C$$

66. Let $u = x^3 + 8, du = 3x^2\,dx$.

$$\int_{-2}^{4} x^2(x^3+8)^2\,dx = \frac{1}{3}\int_{-2}^{4} (x^3+8)^2(3x^2)\,dx = \left[\frac{1}{3}\frac{(x^3+8)^3}{3}\right]_{-2}^{4}$$

$$= \frac{1}{9}[(64+8)^3 - (-8+8)^3] = 41{,}472$$

68. Let $u = 1 - x^2, du = -2x\,dx$.

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int_0^1 (1-x^2)^{1/2}(-2x)\,dx = \left[-\frac{1}{3}(1-x^2)^{3/2}\right]_0^1 = 0 + \frac{1}{3} = \frac{1}{3}$$

70. Let $u = 1 + 2x^2, du = 4x\,dx$.

$$\int_0^2 \frac{x}{\sqrt{1+2x^2}}\,dx = \frac{1}{4}\int_0^2 (1+2x^2)^{-1/2}(4x)\,dx = \left[\frac{1}{2}\sqrt{1+2x^2}\right]_0^2 = \frac{3}{2} - \frac{1}{2} = 1$$

72. Let $u = 4 + x^2$, $du = 2x\,dx$.

$$\int_0^2 x\sqrt[3]{4+x^2}\,dx = \frac{1}{2}\int_0^2 (4+x^2)^{1/3}(2x)\,dx = \left[\frac{3}{8}(4+x^2)^{4/3}\right]_0^2 = \frac{3}{8}(8^{4/3} - 4^{4/3}) = 6 - \frac{3}{2}\sqrt[3]{4} \approx 3.619$$

74. Let $u = 2x - 1$, $du = 2\,dx$, $x = \frac{1}{2}(u+1)$.

When $x = 1$, $u = 1$. When $x = 5$, $u = 9$.

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx = \int_1^9 \frac{1/2(u+1)}{\sqrt{u}}\frac{1}{2}\,du = \frac{1}{4}\int_1^9 (u^{1/2} + u^{-1/2})\,du$$

$$= \frac{1}{4}\left[\frac{2}{3}u^{3/2} + 2u^{1/2}\right]_1^9$$

$$= \frac{1}{4}\left[\left(\frac{2}{3}(27) + 2(3)\right) - \left(\frac{2}{3} + 2\right)\right]$$

$$= \frac{16}{3}$$

76. $$\int_{\pi/3}^{\pi/2} (x + \cos x)\,dx = \left[\frac{x^2}{2} + \sin x\right]_{\pi/3}^{\pi/2} = \left(\frac{\pi^2}{8} + 1\right) - \left(\frac{\pi^2}{18} + \frac{\sqrt{3}}{2}\right) = \frac{5\pi^2}{72} + \frac{2-\sqrt{3}}{2}$$

78. $u = x + 2$, $x = u - 2$, $dx = du$

When $x = -2$, $u = 0$. When $x = 6$, $u = 8$.

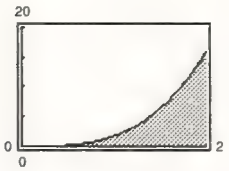
$$\text{Area} = \int_{-2}^{6} x^2\sqrt[3]{x+2}\,dx = \int_0^8 (u-2)^2\sqrt[3]{u}\,du = \int_0^8 (u^{7/3} - 4u^{4/3} + 4u^{1/3})\,du = \left[\frac{3}{10}u^{10/3} - \frac{12}{7}u^{7/3} + 3u^{4/3}\right]_0^8 = \frac{4752}{35}$$

80. $$A = \int_0^{\pi} (\sin x + \cos 2x)\,dx = \left[-\cos x + \frac{1}{2}\sin 2x\right]_0^{\pi} = 2$$

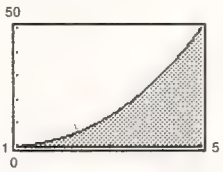
82. Let $u = 2x$, $du = 2\,dx$.

$$\text{Area} = \int_{\pi/12}^{\pi/4} \csc 2x \cot 2x\,dx = \frac{1}{2}\int_{\pi/12}^{\pi/4} \csc 2x \cot 2x(2)\,dx = \left[-\frac{1}{2}\csc 2x\right]_{\pi/12}^{\pi/4} = \frac{1}{2}$$

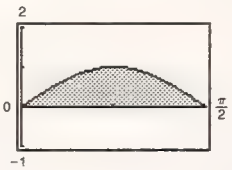
84. $$\int_0^2 x^3\sqrt{x+2}\,dx \approx 7.581$$



86. $$\int_1^5 x^2\sqrt{x-1}\,dx \approx 67.505$$



88. $$\int_0^{\pi/2} \sin 2x\,dx \approx 1.0$$



90. $$\int \sin x \cos x\,dx = \int (\sin x)^1(\cos x\,dx) = \frac{\sin^2 x}{2} + C_1$$

$$\int \sin x \cos x\,dx = -\int (\cos x)^1(-\sin x\,dx) = -\frac{\cos^2 x}{2} + C_2$$

$$-\frac{\cos^2 x}{2} + C_2 = -\frac{(1-\sin^2 x)}{2} + C_2 = \frac{\sin^2 x}{2} - \frac{1}{2} + C_2$$

They differ by a constant: $C_2 = C_1 + \frac{1}{2}$.

92. $f(x) = \sin^2 x \cos x$ is even.

$$\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\, dx = \int_0^{\pi/2} \sin^2 x(\cos x)\, dx$$

$$= 2\left[\frac{\sin^3 x}{3}\right]_0^{\pi/2}$$

$$= \frac{2}{3}$$

94. $f(x) = \sin x \cos x$ is odd.

$$\int_{-\pi/2}^{\pi/2} \sin x \cos x\, dx = 0$$

96. (a) $\displaystyle\int_{-\pi/4}^{\pi/4} \sin x\, dx = 0$ since $\sin x$ is symmetric to the origin.

(b) $\displaystyle\int_{-\pi/4}^{\pi/4} \cos x\, dx = 2\int_0^{\pi/4} \cos x\, dx = \left[2 \sin x\right]_0^{\pi/4} = \sqrt{2}$ since $\cos x$ is symmetric to the y-axis.

(c) $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x\, dx = 2\int_0^{\pi/2} \cos x\, dx = \left[2 \sin x\right]_0^{\pi/2} = 2$

(d) $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x \cos x\, dx = 0$ since $\sin(-x)\cos(-x) = -\sin x \cos x$ and hence, is symmetric to the origin.

98. $\displaystyle\int_{-\pi}^{\pi} (\sin 3x + \cos 3x)\, dx = \int_{-\pi}^{\pi} \sin 3x\, dx + \int_{-\pi}^{\pi} \cos 3x\, dx = 0 + 2\int_0^{\pi} \cos 3x\, dx = \left[\frac{2}{3} \sin 3x\right]_0^{\pi} = 0$

100. If $u = 5 - x^2$, then $du = -2x\, dx$ and $\displaystyle\int x(5 - x^2)^3\, dx = -\frac{1}{2}\int (5 - x^2)^3(-2x)\, dx = -\frac{1}{2}\int u^3\, du$.

102. $\dfrac{dQ}{dt} = k(100 - t)^2$

$$Q(t) = \int k(100 - t)^2\, dt = -\frac{k}{3}(100 - t)^3 + C$$

$$Q(100) = C = 0$$

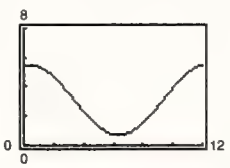
$$Q(t) = -\frac{k}{3}(100 - t)^3$$

$$Q(0) = -\frac{k}{3}(100)^3 = 2{,}000{,}000 \Rightarrow k = -6$$

Thus, $Q(t) = 2(100 - t)^3$. When $t = 50$, $Q(50) =$ \$250,000.

104. $R = 3.121 + 2.399 \sin(0.524t + 1.377)$

(a)



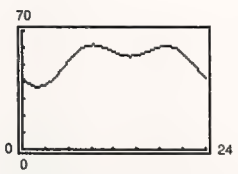
Relative minimum: (6.4, 0.7) or June

Relative maximum: (0.4, 5.5) or January

(b) $\displaystyle\int_0^{12} R(t)\, dt \approx 37.47$ inches

(c) $\displaystyle\frac{1}{3}\int_9^{12} R(t)\, dt \approx \frac{1}{3}(13) = 4.33$ inches

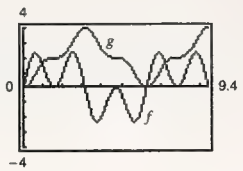
106. (a)



Maximum flow: $R \approx 61.713$ at $t = 9.36$.
[(18.861, 61.178) is a relative maximum.]

(b) Volume $= \displaystyle\int_0^{24} R(t)\, dt \approx 1272$ (5 thousand of gallons)

108. (a)

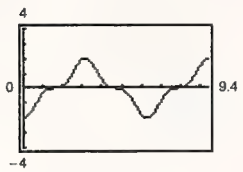


(b) g is nonnegative because the graph of f is positive at the beginning, and generally has more positive sections than negative ones.

(c) The points on g that correspond to the extrema of f are points of inflection of g.

(d) No, some zeros of f, like $x = \pi/2$, do not correspond to an extrema of g. The graph of g continues to increase after $x = \pi/2$ because f remains above the x-axis.

(e)



The graph of h is that of g shifted 2 units downward.

$$g(t) = \int_0^t f(x)\,dx$$
$$= \int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^t f(x)\,dx = 2 + h(t).$$

110. False

$$\int x(x^2 + 1)^2\,dx = \frac{1}{2}\int (x^2 + 1)(2x)\,dx = \frac{1}{4}(x^2 + 1)^2 + C$$

112. True

$$\int_a^b \sin x\,dx = \Big[-\cos x\Big]_a^b = -\cos b + \cos a = -\cos(b + 2\pi) + \cos a = \int_a^{b+2\pi} \sin x\,dx$$

114. False

$$\int \sin^2 2x \cos 2x\,dx = \frac{1}{2}\int (\sin 2x)^2(2\cos 2x)\,dx = \frac{1}{2}\frac{(\sin 2x)^3}{3} + C = \frac{1}{6}\sin^3 2x + C$$

116. Because f is odd, $f(-x) = -f(x)$. Then

$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx$$
$$= -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx.$$

Let $x = -u$, $dx = -du$ in the first integral.

When $x = 0$, $u = 0$. When $x = -a$, $u = a$.

$$\int_{-a}^1 f(x)\,dx = -\int_0^a f(-u)(-du) + \int_0^a f(x)\,dx$$
$$= -\int_0^a f(u)\,du + \int_0^a f(x)\,dx = 0$$

Section 4.6 Numerical Integration

2. Exact: $\displaystyle\int_0^1 \left(\frac{x^2}{2}+1\right)dx = \left[\frac{x^3}{6}+x\right]_0^1 = \frac{7}{6} \approx 1.1667$

Trapezoidal: $\displaystyle\int_0^1 \left(\frac{x^2}{2}+1\right)dx \approx \frac{1}{8}\left[1+2\left(\frac{(1/4)^2}{2}+1\right)+2\left(\frac{(1/2)^2}{2}+1\right)+2\left(\frac{(3/4)^2}{2}+1\right)+\left(\frac{1^2}{2}+1\right)\right] = \frac{75}{64} \approx 1.1719$

Simpson's: $\displaystyle\int_0^1 \left(\frac{x^2}{2}+1\right)dx \approx \frac{1}{12}\left[1+4\left(\frac{(1/4)^2}{2}+1\right)+2\left(\frac{(1/2)^2}{2}+1\right)+4\left(\frac{(3/4)^2}{2}+1\right)+\left(\frac{1^2}{2}+1\right)\right] = \frac{7}{6} \approx 1.1667$

4. Exact: $\displaystyle\int_1^2 \frac{1}{x^2}\,dx = \left[\frac{-1}{x}\right]_1^2 = 0.5000$

Trapezoidal: $\displaystyle\int_1^2 \frac{1}{x^2}\,dx \approx \frac{1}{8}\left[1+2\left(\frac{4}{5}\right)^2+2\left(\frac{4}{6}\right)^2+2\left(\frac{4}{7}\right)^2+\frac{1}{4}\right] \approx 0.5090$

Simpson's: $\displaystyle\int_1^2 \frac{1}{x^2}\,dx \approx \frac{1}{12}\left[1+4\left(\frac{4}{5}\right)^2+2\left(\frac{4}{6}\right)^2+4\left(\frac{4}{7}\right)^2+\frac{1}{4}\right] \approx 0.5004$

6. Exact: $\displaystyle\int_0^8 \sqrt[3]{x}\,dx = \left[\frac{3}{4}x^{4/3}\right]_0^8 = 12.0000$

Trapezoidal: $\displaystyle\int_0^8 \sqrt[3]{x}\,dx \approx \frac{1}{2}[0+2+2\sqrt[3]{2}+2\sqrt[3]{3}+2\sqrt[3]{4}+2\sqrt[3]{5}+2\sqrt[3]{6}+2\sqrt[3]{7}+2] \approx 11.7296$

Simpson's: $\displaystyle\int_0^8 \sqrt[3]{x}\,dx \approx \frac{1}{3}[0+4+2\sqrt[3]{2}+4\sqrt[3]{3}+2\sqrt[3]{4}+4\sqrt[3]{5}+2\sqrt[3]{6}+4\sqrt[3]{7}+2] \approx 11.8632$

8. Exact: $\displaystyle\int_1^3 (4-x^2)\,dx = \left[4x-\frac{x^3}{3}\right]_1^3 = 3-\frac{11}{3} = -\frac{2}{3} \approx -0.6667$

Trapezoidal: $\displaystyle\int_1^3 (4-x^2)\,dx \approx \frac{1}{4}\left\{3+2\left[4-\left(\frac{3}{2}\right)^2\right]+2(0)+2\left[4-\left(\frac{5}{2}\right)^2\right]-5\right\} = -0.7500$

Simpson's: $\displaystyle\int_1^3 (4-x^2)\,dx \approx \frac{1}{6}\left[3+4\left(4-\frac{9}{4}\right)+0+4\left(4-\frac{25}{4}\right)-5\right] \approx -0.6667$

10. Exact: $\displaystyle\int_0^2 x\sqrt{x^2+1}\,dx = \frac{1}{3}\left[(x^2+1)^{3/2}\right]_0^2 = \frac{1}{3}(5^{3/2}-1) \approx 3.393$

Trapezoidal: $\displaystyle\int_0^2 x\sqrt{x^2+1}\,dx \approx \frac{1}{4}\left[0+2\left(\frac{1}{2}\right)\sqrt{(1/2)^2+1}+2(1)\sqrt{1^2+1}+2\left(\frac{3}{2}\right)\sqrt{(3/2)^2+1}+2\sqrt{2^2+1}\right] \approx 3.457$

Simpson's: $\displaystyle\int_0^2 x\sqrt{x^2+1}\,dx \approx \frac{1}{6}\left[0+4\left(\frac{1}{2}\right)\sqrt{(1/2)^2+1}+2(1)\sqrt{1^2+1}+4\left(\frac{3}{2}\right)\sqrt{(3/2)^2+1}+2\sqrt{2^2+1}\right] \approx 3.392$

12. Trapezoidal: $\displaystyle\int_0^2 \frac{1}{\sqrt{1+x^3}}\,dx \approx \frac{1}{4}\left[1+2\left(\frac{1}{\sqrt{1+(1/2)^3}}\right)+2\left(\frac{1}{\sqrt{1+1^3}}\right)+2\left(\frac{1}{\sqrt{1+(3/2)^3}}\right)+\frac{1}{3}\right] \approx 1.397$

Simpson's: $\displaystyle\int_0^2 \frac{1}{\sqrt{1+x^3}}\,dx \approx \frac{1}{6}\left[1+4\left(\frac{1}{\sqrt{1+(1/2)^3}}\right)+2\left(\frac{1}{\sqrt{1+1^3}}\right)+4\left(\frac{1}{\sqrt{1+(3/2)^3}}\right)+\frac{1}{3}\right] \approx 1.405$

Graphing utility: 1.402

14. Trapezoidal: $\displaystyle\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx \approx \frac{\pi}{16}\left[\sqrt{\frac{\pi}{2}}(1) + 2\sqrt{\frac{5\pi}{8}}\sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}}\sin\left(\frac{3\pi}{4}\right) + 2\sqrt{\frac{7\pi}{8}}\sin\left(\frac{7\pi}{8}\right) + 0\right] \approx 1.430$

Simpson's: $\displaystyle\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx \approx \frac{\pi}{24}\left[\sqrt{\frac{\pi}{2}} + 4\sqrt{\frac{5\pi}{8}}\sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}}\sin\left(\frac{3\pi}{4}\right) + 4\sqrt{\frac{7\pi}{8}}\sin\left(\frac{7\pi}{8}\right) + 0\right] \approx 1.458$

Graphing utility: 1.458

16. Trapezoidal: $\displaystyle\int_0^{\sqrt{\pi/4}} \tan(x^2)\,dx \approx \frac{\sqrt{\pi/4}}{8}\left[\tan 0 + 2\tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2\tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 2\tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right]$

≈ 0.271

Simpson's: $\displaystyle\int_0^{\sqrt{\pi/4}} \tan(x^2)\,dx \approx \frac{\sqrt{\pi/4}}{12}\left[\tan 0 + 4\tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2\tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 4\tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right]$

≈ 0.257

Graphing utility: 0.256

18. Trapezoidal: $\displaystyle\int_0^{\pi/2} \sqrt{1+\cos^2 x}\,dx \approx \frac{\pi}{16}\left[\sqrt{2} + 2\sqrt{1+\cos^2(\pi/8)} + 2\sqrt{1+\cos^2(\pi/4)} + 2\sqrt{1+\cos^2(3\pi/8)} + 1\right] \approx 1.910$

Simpson's: $\displaystyle\int_0^{\pi/2} \sqrt{1+\cos^2 x}\,dx \approx \frac{\pi}{24}\left[\sqrt{2} + 4\sqrt{1+\cos^2(\pi/8)} + 2\sqrt{1+\cos^2(\pi/4)} + 4\sqrt{1+\cos^2(3\pi/8)} + 1\right] \approx 1.910$

Graphing utility: 1.910

20. Trapezoidal: $\displaystyle\int_0^{\pi} \frac{\sin x}{x}\,dx \approx \frac{\pi}{8}\left[1 + \frac{2\sin(\pi/4)}{\pi/4} + \frac{2\sin(\pi/2)}{\pi/2} + \frac{2\sin(3\pi/4)}{3\pi/4} + 0\right] \approx 1.836$

Simpson's: $\displaystyle\int_0^{\pi} \frac{\sin x}{x}\,dx \approx \frac{\pi}{12}\left[1 + \frac{4\sin(\pi/4)}{\pi/4} + \frac{2\sin(\pi/2)}{\pi/2} + \frac{4\sin(3\pi/4)}{3\pi/4} + 0\right] \approx 1.852$

Graphing utility: 1.852

22. Trapezoidal: Linear polynomials

Simpson's: Quadratic polynomials

24. $f(x) = \dfrac{1}{x+1}$

$f'(x) = \dfrac{-1}{(x+1)^2}$

$f''(x) = \dfrac{2}{(x+1)^3}$

$f'''(x) = \dfrac{-6}{(x+1)^4}$

$f^{(4)}(x) = \dfrac{24}{(x+1)^5}$

(a) Trapezoidal: Error $\le \dfrac{(1-0)^2}{12(4^2)}(2) = \dfrac{1}{96} \approx 0.01$ since

$f''(x)$ is maximum in $[0, 1]$ when $x = 0$.

(b) Simpson's: Error $\le \dfrac{(1-0)^5}{180(4^4)}(24) = \dfrac{1}{1920} \approx 0.0005$

since $f^{(4)}(x)$ is maximum in $[0, 1]$ when $x = 0$.

26. $f''(x) = \dfrac{2}{(1+x)^3}$ in $[0, 1]$.

(a) $|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = 2$.

Trapezoidal: Error $\le \dfrac{1}{12n^2}(2) < 0.00001$, $n^2 > 16{,}666.67$, $n > 129.10$; let $n = 130$.

$f^{(4)}(x) = \dfrac{24}{(1+x)^5}$ in $[0, 1]$

(b) $|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = 24$.

Simpson's: Error $\le \dfrac{1}{180n^4}(24) < 0.00001$, $n^4 > 13{,}333.33$, $n > 10.75$; let $n = 12$. (In Simpson's Rule n must be even.)

28. $f(x) = (x+1)^{2/3}$

(a) $f''(x) = -\dfrac{2}{9(x+1)^{4/3}}$ in $[0, 2]$.

$|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = \dfrac{2}{9}$.

Trapezoidal: Error $\le \dfrac{8}{12n^4}\left(\dfrac{2}{9}\right) < 0.00001$, $n^2 > 14{,}814.81$, $n > 121.72$; let $n = 122$.

(b) $f^{(4)}(x) = -\dfrac{56}{81(x+1)^{10/3}}$ in $[0, 2]$

$|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = \dfrac{56}{81}$.

Simpson's: Error $\le \dfrac{32}{180n^4}\left(\dfrac{56}{81}\right) < 0.00001$, $n^4 > 12{,}290.81$, $n > 10.53$; let $n = 12$. (In Simpson's Rule n must be even.)

30. $f(x) = \sin(x^2)$

(a) $f''(x) = 2[-2x^2\sin(x^2) + \cos(x^2)]$ in $[0, 1]$.

$|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 2.2853$.

Trapezoidal: Error $\le \dfrac{(1-0)^3}{12n^2}(2.2853) < 0.00001$, $n^2 > 19{,}044.17$, $n > 138.00$; let $n = 139$.

(b) $f^{(4)}(x) = (16x^4 - 12)\sin(x^2) - 48x^2\cos(x^2)$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x \approx 0.852$ and $|f^{(4)}(0.852)| \approx 28.4285$.

Simpson's: Error $\le \dfrac{(1-0)^5}{180n^4}(28.4285) < 0.00001$, $n^4 > 15{,}793.61$, $n > 11.21$; let $n = 12$.

32. The program will vary depending upon the computer or programmable calculator that you use.

34. $f(x) = \sqrt{1-x^2}$ on $[0, 1]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	0.8739	0.7960	0.6239	0.7489	0.7709
8	0.8350	0.7892	0.7100	0.7725	0.7803
10	0.8261	0.7881	0.7261	0.7761	0.7818
12	0.8200	0.7875	0.7367	0.7783	0.7826
16	0.8121	0.7867	0.7496	0.7808	0.7836
20	0.8071	0.7864	0.7571	0.7821	0.7841

36. $f(x) = \dfrac{\sin x}{x}$ on $[1, 2]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	0.7070	0.6597	0.6103	0.6586	0.6593
8	0.6833	0.6594	0.6350	0.6592	0.6593
10	0.6786	0.6594	0.6399	0.6592	0.6593
12	0.6754	0.6594	0.6431	0.6593	0.6593
16	0.6714	0.6594	0.6472	0.6593	0.6593
20	0.6690	0.6593	0.6496	0.6593	0.6593

38. Simpson's Rule: $n = 8$

$$8\sqrt{3}\int_0^{\pi/2} \sqrt{1 - \frac{2}{3}\sin^2\theta}\, d\theta \approx \frac{\sqrt{3}\pi}{6}\left[\sqrt{1 - \frac{2}{3}\sin^2 0} + 4\sqrt{1 - \frac{2}{3}\sin^2\frac{\pi}{16}} + 2\sqrt{1 - \frac{2}{3}\sin^2\frac{\pi}{8}} + \cdots + \sqrt{1 - \frac{2}{3}\sin^2\frac{\pi}{2}}\right]$$
$$\approx 17.476$$

40. (a) Trapezoidal:

$$\int_0^2 f(x)\, dx \approx \frac{2}{2(8)}[4.32 + 2(4.36) + 2(4.58) + 2(5.79) + 2(6.14) + 2(7.25) + 2(7.64) + 2(8.08) + 8.14] \approx 12.518$$

Simpson's:

$$\int_0^2 f(x)\, dx \approx \frac{2}{3(8)}[4.32 + 4(4.36) + 2(4.58) + 4(5.79) + 2(6.14) + 4(7.25) + 2(7.64) + 4(8.08) + 8.14] \approx 12.592$$

(b) Using a graphing utility,

$$y = -1.3727x^3 + 4.0092x^2 - 0.6202x + 4.2844$$

Integrating, $\displaystyle\int_0^2 y\, dx \approx 12.53$

42. Simpson's Rule: $n = 6$

$$\pi = 4\int_0^1 \frac{1}{1 + x^2}\, dx \approx \frac{4}{3(6)}\left[1 + \frac{4}{1 + (1/6)^2} + \frac{2}{1 + (2/6)^2} + \frac{4}{1 + (3/6)^2} + \frac{2}{1 + (4/6)^2} + \frac{4}{1 + (5/6)^2} + \frac{1}{2}\right]$$
$$\approx 3.14159$$

44. Area $\approx \dfrac{120}{2(12)}[75 + 2(81) + 2(84) + 2(76) + 2(67) + 2(68) + 2(69) + 2(72) + 2(68) + 2(56) + 2(42) + 2(23) + 0]$

$= 7435$ sq m

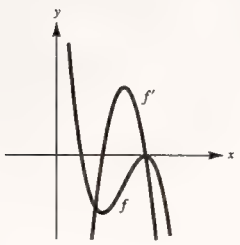
46. The quadratic polynomial

$$p(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}y_1 + \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}y_2 + \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}y_3$$

passes through the three points.

Review Exercises for Chapter 4

2.



4. $u = 3x$

$du = 3\,dx$

$$\int \frac{2}{\sqrt[3]{3x}}\,dx = \frac{2}{3}\int (3x)^{-1/3}(3)\,dx = (3x)^{2/3} + C$$

6. $$\int \frac{x^3 - 2x^2 + 1}{x^2}\,dx = \int (x - 2 + x^{-2})\,dx$$

$$= \frac{1}{2}x^2 - 2x - \frac{1}{x} + C$$

8. $$\int (5\cos x - 2\sec^2 x)\,dx = 5\sin x - 2\tan x + C$$

10. $f''(x) = 6(x - 1)$

$$f'(x) = \int 6(x - 1)\,dx = 3(x - 1)^2 + C_1$$

Since the slope of the tangent line at (2, 1) is 3, it follows that $f'(2) = 3 + C_1 = 3$ when $C_1 = 0$.

$f'(x) = 3(x - 1)^2$

$$f(x) = \int 3(x - 1)^2\,dx = (x - 1)^3 + C_2$$

$f(2) = 1 + C_2 = 1$ when $C_2 = 0$.

$f(x) = (x - 1)^3$

12. 45 mph = 66 ft/sec

30 mph = 44 ft/sec

$a(t) = -a$

$v(t) = -at + 66$ since $v(0) = 66$ ft/sec.

$s(t) = -\frac{a}{2}t^2 + 66t$ since $s(0) = 0$.

Solving the system

$v(t) = -at + 66 = 44$

$s(t) = -\frac{a}{2}t^2 + 66t = 264$

we obtain $t = 24/5$ and $a = 55/12$. We now solve $-(55/12)t + 66 = 0$ and get $t = 72/5$. Thus,

$$s\left(\frac{72}{5}\right) = -\frac{55/12}{2}\left(\frac{72}{5}\right)^2 + 66\left(\frac{72}{5}\right) \approx 475.2 \text{ ft.}$$

Stopping distance from 30 mph to rest is

$475.2 - 264 = 211.2$ ft.

14. $a(t) = -9.8$ m/sec^2

$v(t) = -9.8t + v_0 = -9.8t + 40$

$s(t) = -4.9t^2 + 40t \quad (s(0) = 0)$

(a) $v(t) = -9.8t + 40 = 0$ when $t = \frac{40}{9.8} \approx 4.08$ sec.

(b) $s(4.08) \approx 81.63$ m

(c) $v(t) = -9.8t + 40 = 20$ when $t = \frac{20}{9.8} \approx 2.04$ sec.

(d) $s(2.04) \approx 61.2$ m

16. $x_1 = 2, x_2 = -1, x_3 = 5, x_4 = 3, x_5 = 7$

(a) $\frac{1}{5}\sum_{i=1}^{5} x_i = \frac{1}{5}(2 - 1 + 5 + 3 + 7) = \frac{16}{5}$

(b) $\sum_{i=1}^{5} \frac{1}{x_i} = \frac{1}{2} - 1 + \frac{1}{5} + \frac{1}{3} + \frac{1}{7} = \frac{37}{210}$

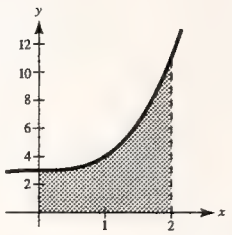
(c) $\sum_{i=1}^{5} (2x_i - x_i^2) = [2(2) - (2)^2] + [2(-1) - (-1)^2] + [2(5) - (5)^2] + [2(3) - (3)^2] + [2(7) - (7)^2] = -56$

(d) $\sum_{i=2}^{5} (x_i - x_{i-1}) = (-1 - 2) + [5 - (-1)] + (3 - 5) + (7 - 3) = 5$

18. $y = 9 - \frac{1}{4}x^2, \Delta x = 1, n = 4$

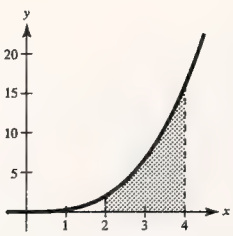
$$S(4) = 1\left[\left(9 - \tfrac{1}{4}(4)\right) + \left(9 - \tfrac{1}{4}(9)\right) + \left(9 - \tfrac{1}{4}(16)\right) + 9 - \tfrac{1}{4}(25)\right]$$
$$\approx 22.5$$
$$s(4) = 1\left[\left(9 - \tfrac{1}{4}(9)\right) + \left(9 - \tfrac{1}{4}(16)\right) + \left(9 - \tfrac{1}{4}(25)\right) + (9 - 9)\right]$$
$$\approx 14.5$$

20. $y = x^2 + 3, \Delta x = \frac{2}{n}$ right endpoints



$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(ci)\,\Delta x$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{2i}{n}\right)^2 + 3\right]\left(\frac{2}{n}\right)$$
$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[\frac{4i^2}{n^2} + 3\right]$$
$$= \lim_{n\to\infty} \frac{2}{n}\left[\frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} + 3n\right]$$
$$= \lim_{n\to\infty} \left[\frac{4}{3}\frac{(n+1)(2n+1)}{n^2} + 6\right] = \frac{8}{3} + 6 = \frac{26}{3}$$

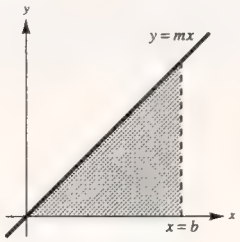
22. $y = \frac{1}{4}x^3, \Delta x = \frac{2}{n}$



$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(ci)\,\Delta x$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{4}\left(2 + \frac{2i}{n}\right)^3\left(\frac{2}{n}\right)$$
$$= \lim_{n\to\infty} \frac{1}{2n} \sum_{i=1}^{n} \left[8 + \frac{24i}{n} + \frac{24i^2}{n^2} + \frac{8i^3}{n^3}\right]$$
$$= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left[1 + \frac{3i}{n} + \frac{3i^2}{n^2} + \frac{i^3}{n^3}\right]$$
$$= \lim_{n\to\infty} \frac{4}{n}\left[n + \frac{3}{n}\frac{n(n+1)}{2} + \frac{3}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{1}{n^3}\frac{n^2(n+1)^2}{4}\right]$$
$$= 4 + 6 + 4 + 1 = 15$$

24. (a) $S = m\left(\frac{b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{2b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{3b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{4b}{4}\right)\left(\frac{b}{4}\right) = \frac{mb^2}{16}(1 + 2 + 3 + 4) = \frac{5mb^2}{8}$

$s = m(0)\left(\frac{b}{4}\right) + m\left(\frac{b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{2b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{3b}{4}\right)\left(\frac{b}{4}\right) = \frac{mb^2}{16}(1 + 2 + 3) = \frac{3mb^2}{8}$



(b) $S(n) = \sum_{i=1}^{n} f\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = \sum_{i=1}^{n}\left(\frac{mbi}{n}\right)\left(\frac{b}{n}\right) = m\left(\frac{b}{n}\right)^2 \sum_{i=1}^{n} i = \frac{mb^2}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{mb^2(n+1)}{2n}$

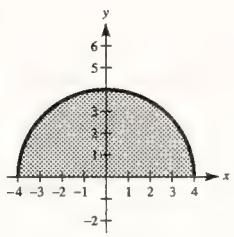
$s(n) = \sum_{i=0}^{n-1} f\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = \sum_{i=0}^{n-1} m\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = m\left(\frac{b}{n}\right)^2 \sum_{i=0}^{n-1} i = \frac{mb^2}{n^2}\left(\frac{(n-1)n}{2}\right) = \frac{mb^2(n-1)}{2n}$

(c) Area $= \lim_{n\to\infty} \frac{mb^2(n+1)}{2n} = \lim_{n\to\infty} \frac{mb^2(n-1)}{2n} = \frac{1}{2}mb^2 = \frac{1}{2}(b)(mb) = \frac{1}{2}$(base)(height)

(d) $\int_0^b mx\,dx = \left[\frac{1}{2}mx^2\right]_0^b = \frac{1}{2}mb^2$

26. $\lim_{\|\Delta\|\to\infty} \sum_{i=1}^{n} 3ci(9 - ci^2)\,\Delta xi = \int_1^3 3x(9 - x^2)\,dx$

28.



$\int_{-4}^{4} \sqrt{16 - x^2}\,dx = \frac{1}{2}\pi(4)^2 = 8\pi$ (semicircle)

30. (a) $\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx = 4 + (-1) = 3$

(b) $\int_6^3 f(x)\,dx = -\int_3^6 f(x)\,dx = -(-1) = 1$

(c) $\int_4^4 f(x)\,dx = 0$

(d) $\int_3^6 -10f(x)\,dx = -10\int_3^6 f(x)\,dx = -10(-1) = 10$

32. $\int_1^3 \frac{12}{x^3}\,dx = \left[\frac{12x^{-2}}{-2}\right]_1^3 = \left[\frac{-6}{x^2}\right]_1^3 = \frac{-6}{9} + 6 = \frac{16}{3}$, (d)

34. $\int_{-1}^{1} (t^2 + 2)\,dt = \left[\frac{t^3}{3} + 2t\right]_{-1}^{1} = \frac{14}{3}$

36. $\int_{-2}^{2} (x^4 + 2x^2 - 5)\,dx = \left[\frac{x^5}{5} + \frac{2x^3}{3} - 5x\right]_{-2}^{2}$

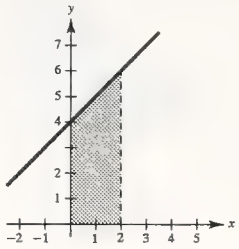
$= \left(\frac{32}{5} + \frac{16}{3} - 10\right) - \left(-\frac{32}{5} - \frac{16}{3} + 10\right)$

$= \frac{52}{15}$

38. $\int_1^2 \left(\frac{1}{x^2} - \frac{1}{x^3}\right)dx = \int_1^2 (x^{-2} - x^{-3})\,dx = \left[-\frac{1}{x} + \frac{1}{2x^2}\right]_1^2 = \left(-\frac{1}{2} + \frac{1}{8}\right) - \left(-1 + \frac{1}{2}\right) = \frac{1}{8}$

40. $\int_{-\pi/4}^{\pi/4} \sec^2 t\,dt = \left[\tan t\right]_{-\pi/4}^{\pi/4} = 1 - (-1) = 2$

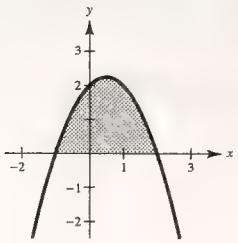
42. $\int_0^2 (x + 4)\,dx = \left[\frac{x^2}{2} + 4x\right]_0^2 = 10$



44. $\int_{-1}^2 (-x^2 + x + 2)\,dx = \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_{-1}^2$

$= \left(-\frac{8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$

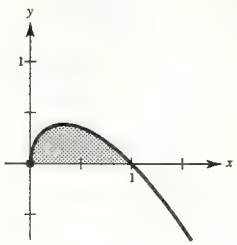
$= \frac{10}{3} + \frac{7}{6} = \frac{9}{2}$



46. $\int_0^1 \sqrt{x}(1 - x)\,dx = (x^{1/2} - x^{3/2})\,dx$

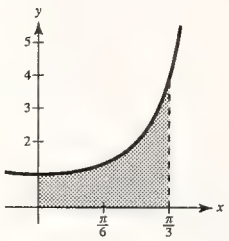
$= \left[\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^1$

$= \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$



48. Area $= \int_0^{\pi/3} \sec^2 x\,dx$

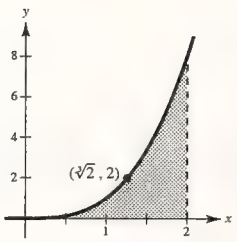
$= \tan x\Big]_0^{\pi/3} = \sqrt{3}$



50. $\frac{1}{2 - 0}\int_0^2 x^3\,dx = \left[\frac{x^4}{8}\right]_0^2 = 2$

$x^3 = 2$

$x = \sqrt[3]{2}$



52. $F'(x) = \frac{1}{x^2}$

54. $F'(x) = \csc^2 x$

56. $\int \left(x + \frac{1}{x}\right)^2 dx = \int (x^2 + 2 + x^{-2})\,dx$

$= \frac{x^3}{3} + 2x - \frac{1}{x} + C$

58. $u = x^3 + 3,\ du = 3x^2\,dx$

$$\int x^2\sqrt{x^3+3}\,dx = \frac{1}{3}\int (x^3+3)^{1/2}\,3x^2\,dx = \frac{2}{9}(x^3+3)^{3/2} + C$$

60. $u = x^2 + 6x - 5,\ du = (2x+6)\,dx$

$$\int \frac{x+3}{(x^2+6x-5)^2} = \frac{1}{2}\int \frac{2x+6}{(x^2+6x-5)^2}\,dx = \frac{-1}{2}(x^2+6x-5)^{-1} + C = \frac{-1}{2(x^2+6x-5)} + C$$

62. $$\int x\sin 3x^2\,dx = \frac{1}{6}\int (\sin 3x^2)(6x)\,dx = -\frac{1}{6}\cos 3x^2 + C$$

64. $$\int \frac{\cos x}{\sqrt{\sin x}}\,dx = \int (\sin x)^{-1/2}\cos x\,dx = 2(\sin x)^{1/2} + C = 2\sqrt{\sin x} + C$$

66. $$\int \sec 2x\tan 2x\,dx = \frac{1}{2}\int (\sec 2x\tan 2x)(2)\,dx = \frac{1}{2}\sec 2x + C$$

68. $$\int \cot^4\alpha\csc^2\alpha\,d\alpha = -\int (\cot\alpha)^4(-\csc^2\alpha)\,d\alpha = -\frac{1}{5}\cot^5\alpha + C$$

70. $$\int_0^1 x^2(x^3+1)^3\,dx = \frac{1}{3}\int_0^1 (x^3+1)^3(3x^2)\,dx = \frac{1}{12}\Big[(x^3+1)^4\Big]_0^1 = \frac{1}{12}(16-1) = \frac{5}{4}$$

72. $$\int_3^6 \frac{x}{3\sqrt{x^2-8}}\,dx = \frac{1}{6}\int_3^6 (x^2-8)^{-1/2}(2x)\,dx = \left[\frac{1}{3}(x^2-8)^{1/2}\right]_3^6 = \frac{1}{3}(2\sqrt{7}-1)$$

74. $u = x + 1,\ x = u - 1,\ dx = du$

When $x = -1$, $u = 0$. When $x = 0$, $u = 1$.

$$2\pi\int_{-1}^0 x^2\sqrt{x+1}\,dx = 2\pi\int_0^1 (u-1)^2\sqrt{u}\,du$$

$$= 2\pi\int_0^1 (u^{5/2} - 2u^{3/2} + u^{1/2})\,du = 2\pi\left[\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right]_0^1 = \frac{32\pi}{105}$$

76. $\displaystyle\int_{-\pi/4}^{\pi/4} \sin 2x\,dx = 0$ since $\sin 2x$ is an odd function.

78. $u = 1 - x,\ x = 1 - u,\ dx = -du$

When $x = a$, $u = 1 - a$. When $x = b$, $u = 1 - b$.

$$P_{a,b} = \int_a^b \frac{1155}{32}x^3(1-x)^{3/2}\,dx = \frac{1155}{32}\int_{1-a}^{1-b} -(1-u)^3u^{3/2}\,du$$

$$= \frac{1155}{32}\int_{1-a}^{1-b} (u^{9/2} - 3u^{7/2} + 3u^{5/2} - u^{3/2})\,du = \frac{1155}{32}\left[\frac{2}{11}u^{11/2} - \frac{2}{3}u^{9/2} + \frac{6}{7}u^{7/2} - \frac{2}{5}u^{5/2}\right]_{1-a}^{1-b}$$

$$= \frac{1155}{32}\left[\frac{2u^{5/2}}{1155}(105u^3 - 385u^2 + 495u - 231)\right]_{1-a}^{1-b} = \left[\frac{u^{5/2}}{16}(105u^3 - 385u^2 + 495u - 231)\right]_{1-a}^{1-b}$$

(a) $$P_{0,0.25} = \left[\frac{u^{5/2}}{16}(105u^3 - 385u^2 + 495u - 231)\right]_1^{0.75} \approx 0.025 = 2.5\%$$

(b) $$P_{0.5,1} = \left[\frac{u^{5/2}}{16}(105u^3 - 385u^2 + 495u - 231)\right]_{0.5}^0 \approx 0.736 = 73.6\%$$

80. $\displaystyle\int_0^2 1.75 \sin\frac{\pi t}{2}\,dt = -\frac{2}{\pi}\left[1.75\cos\frac{\pi t}{2}\right]_0^2 = -\frac{2}{\pi}(1.75)(-1-1) = \frac{7}{\pi} \approx 2.2282$ liters

Increase is

$$\frac{7}{\pi} - \frac{5.1}{\pi} = \frac{1.9}{\pi} \approx 0.6048 \text{ liters.}$$

82. Trapezoidal Rule ($n = 4$): $\displaystyle\int_0^1 \frac{x^{3/2}}{3-x^2}\,dx \approx \frac{1}{8}\left[0 + \frac{2(1/4)^{3/2}}{3-(1/4)^2} + \frac{2(1/2)^{3/2}}{3-(1/2)^2} + \frac{2(3/4)^{3/2}}{3-(3/4)^2} + \frac{1}{2}\right] \approx 0.172$

Simpson's Rule ($n = 4$): $\displaystyle\int_0^1 \frac{x^{3/2}}{3-x^2}\,dx \approx \frac{1}{12}\left[0 + \frac{4(1/4)^{3/2}}{3-(1/4)^2} + \frac{2(1/2)^{3/2}}{3-(1/2)^2} + \frac{4(3/4)^{3/2}}{3-(3/4)^2} + \frac{1}{2}\right] \approx 0.166$

Graphing utility: 0.166

84. Trapezoidal Rule ($n = 4$): $\displaystyle\int_0^\pi \sqrt{1+\sin^2 x}\,dx \approx 3.820$

Simpson's Rule ($n = 4$): 3.820

Graphing utility: 3.820

Problem Solving for Chapter 4

2. (a) $\displaystyle F(x) = \int_2^x \sin t^2\,dt$

x	0	1.0	1.5	1.9	2.0	2.1	2.5	3.0	4.0	5.0
$F(x)$	−0.8048	−0.4945	−0.0265	0.0611	0	−0.0867	−0.3743	−0.0312	−0.0576	−0.2769

(b) $\displaystyle G(x) = \frac{1}{x-2}\int_2^x \sin t^2\,dt$

x	1.9	1.95	1.99	2.01	2.05	2.1
$G(x)$	−0.6106	−0.6873	−0.7436	−0.7697	−0.8174	−0.8671

$\displaystyle\lim_{x\to 2} G(x) \approx -0.75$

(c) $\displaystyle F'(2) = \lim_{x\to 2}\frac{F(x)-F(2)}{x-2}$

$\displaystyle = \lim_{x\to 2}\frac{1}{x-2}\int_2^x \sin t^2\,dt$

$\displaystyle = \lim_{x\to 2} G(x)$

Since $F'(x) = \sin x^2$, $F'(2) = \sin 4 = \displaystyle\lim_{x\to 2} G(x)$.

(**Note:** $\sin 4 \approx -0.7568$)

4. Let d be the distance traversed and a be the uniform acceleration.
We can assume that $v(0) = 0$ and $s(0) = 0$. Then

$$a(t) = a$$

$$v(t) = at$$

$$s(t) = \frac{1}{2}at^2.$$

$s(t) = d$ when $t = \sqrt{\dfrac{2d}{a}}$.

The highest speed is $v = a\sqrt{\dfrac{2d}{a}} = \sqrt{2ad}$.

The lowest speed is $v = 0$.

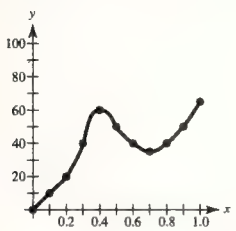
The mean speed is $\dfrac{1}{2}\left(\sqrt{2ad} + 0\right) = \sqrt{\dfrac{ad}{2}}$.

The time necessary to traverse the distance d at the mean speed is

$$t = \frac{d}{\sqrt{ad/2}} = \sqrt{\frac{2d}{a}}$$

which is the same as the time calculated above.

6. (a)



(b) v is increasing (positive acceleration) on (0, 0.4) and (0.7, 1.0).

(c) Average acceleration $= \dfrac{v(0.4) - v(0)}{0.4 - 0} = \dfrac{60 - 0}{0.4} = 150$ mi/hr^2

(d) This integral is the total distance traveled in miles.

$$\int_0^1 v(t)\,dt \approx \frac{1}{10}[0 + 2(20) + 2(60) + 2(40) + 2(40) + 65] = \frac{385}{10} = 38.5 \text{ miles}$$

(e) One approximation is

$$a(0.8) \approx \frac{v(0.9) - v(0.8)}{0.9 - 0.8} = \frac{50 - 40}{0.1} = 100 \text{ mi/hr}^2$$

(other answers possible)

8. $\displaystyle\int_0^x f(t)(x-t)\,dt = \int_0^x xf(t)\,dt - \int_0^x tf(t)\,dt = x\int_0^x f(t)\,dt - \int_0^x tf(t)\,dt$

Thus, $\displaystyle\frac{d}{dx}\int_0^x f(t)(x-t)\,dt = xf(x) + \int_0^x f(t)\,dt - xf(x) = \int_0^x f(t)\,dt$

Differentiating the other integral,

$$\frac{d}{dx}\int_0^x\left(\int_0^x f(v)\,dv\right)dt = \int_0^x f(v)\,dv.$$

Thus, the two original integrals have equal derivatives,

$$\int_0^x f(t)(x-t)\,dt = \int_0^x\left(\int_0^t f(v)\,dv\right)dt + C$$

Letting $x = 0$, we see that $C = 0$.

10. Consider $\displaystyle\int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big]_0^1 = \frac{2}{3}$. The corresponding

Riemann Sum using right-hand endpoints is

$$S(n) = \frac{1}{n}\left[\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right]$$

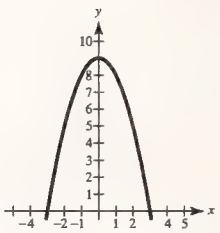
$$= \frac{1}{n^{3/2}}\left[\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}\right]$$

Thus, $\displaystyle\lim_{n\to\infty}\frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^{3/2}} = \frac{2}{3}$.

12. (a) Area $\displaystyle= \int_{-3}^3 (9 - x^2)\,dx = 2\int_0^3 (9 - x^2)\,dx$

$$= 2\left[9x - \frac{x^3}{3}\right]_0^3$$

$$= 2[27 - 9] = 36$$



(b) Base = 6, height = 9. Area $\displaystyle= \frac{2}{3}bh = \frac{2}{3}(6)(9) = 36.$

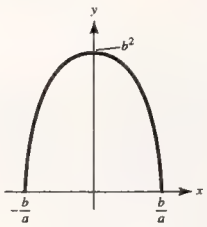
(c) Let the parabola be given by $y = b^2 - a^2x^2$, $a, b > 0$.

$$\text{Area} = 2\int_0^{b/a} (b^2 - a^2x^2)\,dx$$

$$= 2\left[b^2x - a^2\frac{x^3}{3}\right]_0^{b/a}$$

$$= 2\left[b^2\left(\frac{b}{a}\right) - \frac{a^2}{3}\left(\frac{b}{a}\right)^3\right]$$

$$= 2\left[\frac{b^3}{a} - \frac{1}{3}\frac{b^3}{a}\right] = \frac{4}{3}\frac{b^3}{a}$$



Base $\displaystyle= \frac{2b}{a}$, height $= b^2$

Archimedes' Formula: Area $\displaystyle= \frac{2}{3}\left(\frac{2b}{a}\right)(b^2) = \frac{4}{3}\frac{b^3}{a}$

14. (a) $(1+i)^3 = 1 + 3i + 3i^2 + i^3 \Rightarrow (1+i)^3 - i^3 = 3i^2 + 3i + 1$

(b) $3i^2 + 3i + 1 = (i+1)^3 - i^3$

$$\sum_{i=1}^{n}(3i^2 + 3i + 1) = \sum_{i=1}^{n}[(i+1)^3 - i^3]$$
$$= (2^3 - 1^3) + (3^3 - 2^3) + \cdots + [((n+1)^3 - n^3)]$$
$$= (n+1)^3 - 1$$

Hence, $(n+1)^3 = \sum_{i=1}^{n}(3i^2 + 3i + 1) + 1$

(c) $(n+1)^3 - 1 = \sum_{i=1}^{n}(3i^2 + 3i + 1) = \sum_{i=1}^{n} 3i^2 + \frac{3(n)(n+1)}{2} + n$

$$\Rightarrow \sum_{i=1}^{n} 3i^2 = n^3 + 3n^2 + 3n - \frac{3n(n+1)}{2} - n$$
$$= \frac{2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n}{2}$$
$$= \frac{2n^3 + 3n^2 + n}{2}$$
$$= \frac{n(n+1)(2n+1)}{2}$$
$$\Rightarrow \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

16. (a) $C = 0.1\int_8^{20}\left[12\sin\frac{\pi(t-8)}{12}\right]dt = \left[-\frac{14.4}{\pi}\cos\frac{\pi(t-8)}{12}\right]_8^{20} = \frac{-14.4}{\pi}(-1-1) \approx \$9.17$

(b) $C = 0.1\int_{10}^{18}\left[12\sin\frac{\pi(t-8)}{12} - 6\right]dt = \left[-\frac{14.4}{\pi}\cos\frac{\pi(t-8)}{12} - 0.6t\right]_{10}^{18}$

$$= \left[\frac{-14.4}{\pi}\left(\frac{-\sqrt{3}}{2}\right) - 10.8\right] - \left[\frac{-14.4}{\pi}\left(\frac{\sqrt{3}}{2}\right) - 6\right] \approx \$3.14$$

Savings $\approx 9.17 - 3.14 = \$6.03$.

18. (a) Let $A = \int_0^b \frac{f(x)}{f(x) + f(b-x)}\,dx$.

Let $u = b - x$, $du = -dx$.

$$A = \int_b^0 \frac{f(b-u)}{f(b-u) + f(u)}(-du)$$
$$= \int_0^b \frac{f(b-u)}{f(b-u) + f(u)}\,du$$
$$= \int_0^b \frac{f(b-x)}{f(b-x) + f(x)}\,dx$$

Then,

$$2A = \int_0^b \frac{f(x)}{f(x) + f(b-x)}\,dx + \int_0^b \frac{f(b-x)}{f(b-x) + f(x)}\,dx$$
$$= \int_0^b 1\,dx = b.$$

Thus, $A = \frac{b}{2}$.

(b) $b = 1 \Rightarrow \int_0^1 \frac{\sin x}{\sin(1-x) + \sin x}\,dx = \frac{1}{2}$

CHAPTER 5
Logarithmic, Exponential, and Other Transcendental Functions

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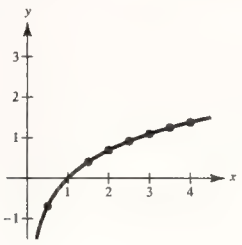
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CHAPTER 5
Logarithmic, Exponential, and Other Transcendental Functions

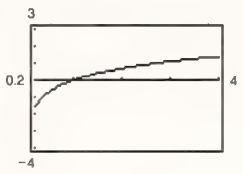
Section 5.1 The Natural Logarithmic Function: Differentiation

Solutions to Even-Numbered Exercises

2. (a)



(b)



The graphs are identical.

4. (a) $\ln 8.3 \approx 2.1163$

(b) $\displaystyle\int_1^{8.3} \frac{1}{t}\,dt \approx 2.1163$

6. (a) $\ln 0.6 \approx -0.5108$

(b) $\displaystyle\int_1^{0.6} \frac{1}{t}\,dt \approx -0.5108$

8. $f(x) = -\ln x$

Reflection in the x-axis

Matches (d)

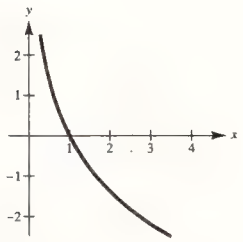
10. $f(x) = -\ln(-x)$

Reflection in the y-axis and the x-axis

Matches (c)

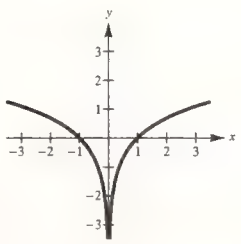
12. $f(x) = -2 \ln x$

Domain: $x > 0$



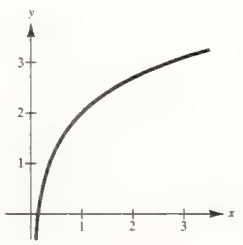
14. $f(x) = \ln|x|$

Domain: $x \neq 0$



16. $g(x) = 2 + \ln x$

Domain: $x > 0$



18. (a) $\ln 0.25 = \ln \frac{1}{4} = \ln 1 - 2 \ln 2 \approx -1.3862$

(b) $\ln 24 = 3 \ln 2 + \ln 3 \approx 3.1779$

(c) $\ln \sqrt[3]{12} = \frac{1}{3}(2 \ln 2 + \ln 3) \approx 0.8283$

(d) $\ln \frac{1}{72} = \ln 1 - (3 \ln 2 + 2 \ln 3) \approx -4.2765$

20. $\ln \sqrt{2^3} = \ln 2^{3/2} = \frac{3}{2} \ln 2$

22. $\ln xyz = \ln x + \ln y + \ln z$

24. $\ln \sqrt{a-1} = \ln(a-1)^{1/2} = \left(\frac{1}{2}\right) \ln(a-1)$

26. $\ln 3e^2 = \ln 3 + 2 \ln e = 2 + \ln 3$

28. $\ln \dfrac{1}{e} = \ln 1 - \ln e = -1$

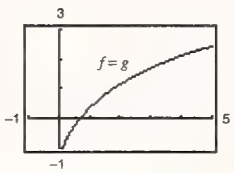
30. $3 \ln x + 2 \ln y - 4 \ln z = \ln x^3 + \ln y^2 - \ln z^4$

$$= \ln \frac{x^3 y^2}{z^4}$$

32. $2[\ln x - \ln(x+1) - \ln(x-1)] = 2 \ln \dfrac{x}{(x+1)(x-1)} = \ln\left(\dfrac{x}{x^2-1}\right)^2$

34. $\dfrac{3}{2}[\ln(x^2+1) - \ln(x+1) - \ln(x-1)] = \dfrac{3}{2} \ln \dfrac{x^2+1}{(x+1)(x-1)} = \ln \sqrt{\left(\dfrac{x^2+1}{x^2-1}\right)^3}$

36.



38. $\lim\limits_{x \to 6^-} \ln(6 - x) = -\infty$

40. $\lim\limits_{x \to 5^+} \ln \dfrac{x}{\sqrt{x-4}} = \ln 5 \approx 1.6094$

42. $y = \ln x^{3/2} = \dfrac{3}{2} \ln x$

$y' = \dfrac{3}{2x}$

At $(1, 0)$, $y' = \frac{3}{2}$.

44. $y = \ln x^{1/2} = \dfrac{1}{2} \ln x$

$y' = \dfrac{1}{2x}$

At $(1, 0)$, $y' = \frac{1}{2}$.

46. $h(x) = \ln(2x^2 + 1)$

$h'(x) = \dfrac{1}{2x^2+1}(4x) = \dfrac{4x}{2x^2+1}$

48. $y = x \ln x$

$\dfrac{dy}{dx} = x\left(\dfrac{1}{x}\right) + \ln x = 1 + \ln x$

50. $y = \ln\sqrt{x^2-4} = \dfrac{1}{2}\ln(x^2-4)$

$\dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{2x}{x^2-4}\right) = \dfrac{x}{x^2-4}$

52. $f(x) = \ln\left(\dfrac{2x}{x+3}\right) = \ln 2x - \ln(x+3)$

$f'(x) = \dfrac{1}{x} - \dfrac{1}{x+3} = \dfrac{3}{x(x+3)}$

54. $h(t) = \dfrac{\ln t}{t}$

$h'(t) = \dfrac{t(1/t) - \ln t}{t^2} = \dfrac{1 - \ln t}{t^2}$

56. $y = \ln(\ln x)$

$\dfrac{dy}{dx} = \dfrac{1/x}{\ln x} = \dfrac{1}{x \ln x}$

58. $y = \ln \sqrt[3]{\dfrac{x-1}{x+1}} = \dfrac{1}{3}[\ln(x-1) - \ln(x+1)]$

$y' = \dfrac{1}{3}\left[\dfrac{1}{x-1} - \dfrac{1}{x+1}\right] = \dfrac{1}{3}\dfrac{2}{x^2-1} = \dfrac{2}{3(x^2-1)}$

60. $f(x) = \ln\left(x + \sqrt{4+x^2}\right)$

$f'(x) = \dfrac{1}{x + \sqrt{4+x^2}}\left(1 + \dfrac{x}{\sqrt{4+x^2}}\right)$

$= \dfrac{1}{\sqrt{4+x^2}}$

62. $y = \dfrac{-\sqrt{x^2+4}}{2x^2} - \dfrac{1}{4}\ln\left(\dfrac{2+\sqrt{x^2+4}}{x}\right) = \dfrac{-\sqrt{x^2+4}}{2x^2} - \dfrac{1}{4}\ln\left(2+\sqrt{x^2+4}\right) + \dfrac{1}{4}\ln x$

$$\frac{dy}{dx} = \frac{-2x^2\left(x/\sqrt{x^2+4}\right) + 4x\sqrt{x^2+4}}{4x^4} - \frac{1}{4}\left(\frac{1}{2+\sqrt{x^2+4}}\right)\left(\frac{x}{\sqrt{x^2+4}}\right) + \frac{1}{4x}$$

Note that:

$$\frac{1}{2+\sqrt{x^2+4}} = \frac{1}{2+\sqrt{x^2+4}} \cdot \frac{2-\sqrt{x^2+4}}{2-\sqrt{x^2+4}} = \frac{2-\sqrt{x^2+4}}{-x^2}$$

$$\text{Hence, } \frac{dy}{dx} = \frac{-1}{2x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} - \frac{1}{4}\frac{\left(2-\sqrt{x^2+4}\right)}{-x^2}\left(\frac{x}{\sqrt{x^2+4}}\right) + \frac{1}{4x}$$

$$= \frac{-1+(1/2)\left(2-\sqrt{x^2+4}\right)}{2x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x}$$

$$= \frac{-\sqrt{x^2+4}}{4x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x} = \frac{\sqrt{x^2+4}}{x^3}$$

64. $y = \ln|\csc x|$

$$y' = \frac{-\csc x \cdot \cot x}{\csc x} = -\cot x$$

66. $y = \ln|\sec x + \tan x|$

$$\frac{dy}{dx} = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x$$

68. $y = \ln\sqrt{1+\sin^2 x} = \dfrac{1}{2}\ln(1+\sin^2 x)$

$$\frac{dy}{dx} = \left(\frac{1}{2}\right)\frac{2\sin x\cos x}{1+\sin^2 x} = \frac{\sin x\cos x}{1+\sin^2 x}$$

70. $g(x) = \displaystyle\int_1^{\ln x} (t^2+3)\,dt$

$$g'(x) = [(\ln x)^2 + 3]\frac{d}{dx}(\ln x) = \frac{(\ln x)^2+3}{x}$$

(Second Fundamental Theorem of Calculus)

72. (a) $y = 4 - x^2 - \ln\left(\dfrac{1}{2}x+1\right)$, $(0, 4)$

$$\frac{dy}{dx} = -2x - \frac{1}{(1/2)x+1}\left(\frac{1}{2}\right)$$

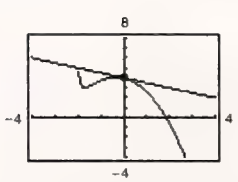
$$= -2x - \frac{1}{x+2}$$

When $x = 0$, $\dfrac{dy}{dx} = -\dfrac{1}{2}$.

Tangent line: $y - 4 = -\dfrac{1}{2}(x-0)$

$$y = -\frac{1}{2}x + 4$$

(b)



74. $\ln(xy) + 5x = 30$

$$\ln x + \ln y + 5x = 30$$

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} + 5 = 0$$

$$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x} - 5$$

$$\frac{dy}{dx} = -\frac{y}{x} - 5y = -\left(\frac{y+5xy}{x}\right)$$

76. $y = x(\ln x) - 4x$

$$y' = x\left(\frac{1}{x}\right) + \ln x - 4 = -3 + \ln x$$

$$(x + y) - xy' = x + x\ln x - 4x - x(-3 + \ln x) = 0$$

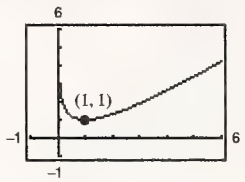
78. $y = x - \ln x$

Domain: $x > 0$

$y' = 1 - \dfrac{1}{x} = 0$ when $x = 1$.

$y'' = \dfrac{1}{x^2} > 0$

Relative minimum: $(1, 1)$



80. $y = \dfrac{\ln x}{x}$

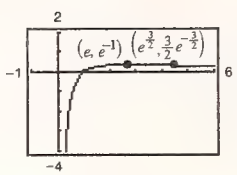
Domain: $x > 0$

$$y' = \frac{x(1/x) - \ln x}{x^2} = \frac{1 - \ln x}{x^2} = 0 \text{ when } x = e.$$

$$y'' = \frac{x^2(-1/x) - (1 - \ln x)(2x)}{x^4} = \frac{2(\ln x) - 3}{x^3} = 0 \text{ when } x = e^{3/2}.$$

Relative maximum: (e, e^{-1})

Point of inflection: $\left(e^{3/2}, \frac{3}{2}e^{-3/2}\right)$



82. $y = x^2 \ln \dfrac{x}{4}$. Domain $x > 0$

$$y' = x^2\left(\frac{1}{x}\right) + 2x \ln \frac{x}{4} = x\left(1 + 2\ln \frac{x}{4}\right) = 0 \text{ when}$$

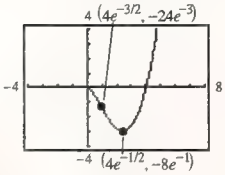
$$-1 = 2\ln \frac{x}{4} \Rightarrow \ln \frac{x}{4} = -\frac{1}{2} \Rightarrow x = 4e^{-1/2}$$

$$y'' = 1 + 2\ln \frac{x}{4} + 2x\left(\frac{1}{x}\right) = 3 + 2\ln \frac{x}{4}$$

$y'' = 0$ when $x = 4e^{-3/2}$

Relative minimum: $(4e^{-1/2}, -8e^{-1})$

Point of inflection: $(4e^{-3/2}, -24e^{-3})$



84. $f(x) = x\ln x, \quad f(1) = 0$

$f'(x) = 1 + \ln x, \quad f(1) = 1$

$f''(x) = \dfrac{1}{x}, \quad f''(1) = 1$

$P_1(x) = f(1) + f'(1)(x - 1) = x - 1, \quad P_1(1) = 0$

$P_2(x) = f(1) + f'(1)(x - 1) + \dfrac{1}{2}f''(1)(x - 1)^2$

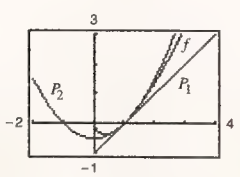
$= (x - 1) + \dfrac{1}{2}(x - 1)^2, \quad P_2(1) = 0$

$P_1'(x) = 1, \quad P_1'(1) = 1$

$P_2'(x) = 1 + (x - 1) = x, \quad P_2'(1) = 1$

$P_2''(x) = x, \quad P_2''(1) = 1$

The values of f, P_1, P_2, and their first derivatives agree at $x = 1$. The values of the second derivatives of f and P_2 agree at $x = 1$.



86. Find x such that $\ln x = 3 - x$.

$$f(x) = x + (\ln x) - 3 = 0$$

$$f'(x) = 1 + \frac{1}{x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n\left[\frac{4 - \ln x_n}{1 + x_n}\right]$$

n	1	2	3
x_n	2	2.2046	2.2079
$f(x_n)$	−0.3069	−0.0049	0.0000

Approximate root: $x = 2.208$

88. $y = \sqrt{(x-1)(x-2)(x-3)}$

$$\ln y = \frac{1}{2}[\ln(x-1) + \ln(x-2) + \ln(x-3)]$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}\right]$$

$$= \frac{1}{2}\left[\frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)}\right]$$

$$\frac{dy}{dx} = \frac{3x^2 - 12x + 11}{2y}$$

$$= \frac{3x^2 - 12x + 11}{2\sqrt{(x-1)(x-2)(x-3)}}$$

90. $y = \sqrt{\dfrac{x^2-1}{x^2+1}}$

$$\ln y = \frac{1}{2}[\ln(x^2-1) - \ln(x^2+1)]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{2x}{x^2-1} - \frac{2x}{x^2+1}\right]$$

$$\frac{dy}{dx} = \sqrt{\frac{x^2-1}{x^2+1}}\left[\frac{2x}{x^4-1}\right]$$

$$= \frac{(x^2-1)^{1/2}2x}{(x^2+1)^{1/2}(x^2-1)(x^2+1)}$$

$$= \frac{2x}{(x^2+1)^{3/2}(x^2-1)^{1/2}}$$

92. $y = \dfrac{(x+1)(x+2)}{(x-1)(x-2)}$

$$\ln y = \ln(x+1) + \ln(x+2) - \ln(x-1) - \ln(x-2)$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x-1} - \frac{1}{x-2}$$

$$\frac{dy}{dx} = y\left[\frac{-2}{x^2-1} + \frac{-4}{x^2-4}\right] = y\left[\frac{-6x^2+12}{(x^2-1)(x^2-4)}\right]$$

$$= \frac{(x+1)(x+2)}{(x-1)(x-2)} \cdot \frac{-6(x^2-2)}{(x+1)(x-1)(x+2)(x-2)}$$

$$= -\frac{6(x^2-2)}{(x-1)^2(x-2)^2}$$

94. The base of the natural logarithmic function is e.

96. $g(x) = \ln f(x), f(x) > 0$

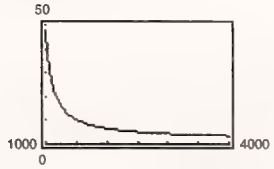
$$g'(x) = \frac{f'(x)}{f(x)}$$

(a) Yes. If the graph of g is increasing, then $g'(x) > 0$. Since $f(x) > 0$, you know that $f'(x) = g'(x)f(x)$ and thus, $f'(x) > 0$. Therefore, the graph of f is increasing.

(b) No. Let $f(x) = x^2 + 1$ (positive and concave up). $g(x) = \ln(x^2+1)$ is not concave up.

98. $t = \dfrac{5.315}{-6.7968 + \ln x}, 1000 < x$

(a)



(b) $t(1167.41) \approx 20$ years

$$T = (1167.41)(20)(12) = \$280{,}178.40$$

(c) $t(1068.45) \approx 30$ years

$$T = (1068.45)(30)(12) = \$384{,}642.00$$

(d) $\dfrac{dt}{dx} = -5.315(-6.7968 + \ln x)^{-2}\left(\dfrac{1}{x}\right)$

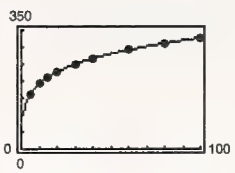
$$= -\frac{5.315}{x(-6.7968 + \ln x)^2}$$

When $x = 1167.41$, $dt/dx \approx -0.0645$. When $x = 1068.45$, $dt/dx \approx -0.1585$.

(e) There are two obvious benefits to paying a higher monthly payment:

1. The term is lower
2. The total amount paid is lower.

100. (a)

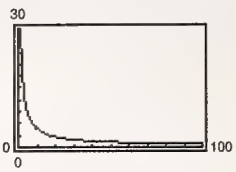


(b) $T'(p) = \dfrac{34.96}{p} + \dfrac{3.955}{\sqrt{p}}$

$T'(10) \approx 4.75$ deg/lb/in^2

$T'(70) \approx 0.97$ deg/lb/in^2

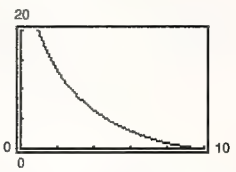
(c)



$\lim_{p\to\infty} T'(p) = 0$

102. $y = 10\ln\left(\dfrac{10 + \sqrt{100 - x^2}}{x}\right) - \sqrt{100 - x^2} = 10\left[\ln\left(10 + \sqrt{100 - x^2}\right) - \ln x\right] - \sqrt{100 - x^2}$

(a)



(c) $\lim_{x\to 10^-} \dfrac{dy}{dx} = 0$

(b) $\dfrac{dy}{dx} = 10\left[\dfrac{-x}{\sqrt{100 - x^2}\left(10 + \sqrt{100 - x^2}\right)} - \dfrac{1}{x}\right] + \dfrac{x}{\sqrt{100 - x^2}}$

$= \dfrac{x}{\sqrt{100 - x^2}}\left[\dfrac{-10}{10 + \sqrt{100 - x^2}}\right] - \dfrac{10}{x} + \dfrac{x}{\sqrt{100 - x^2}}$

$= \dfrac{x}{\sqrt{100 - x^2}}\left[\dfrac{-10}{10 + \sqrt{100 - x^2}} + 1\right] - \dfrac{10}{x}$

$= \dfrac{x}{\sqrt{100 - x^2}}\left[\dfrac{\sqrt{100 - x^2}}{10 + \sqrt{100 - x^2}}\right] - \dfrac{10}{x}$

$= \dfrac{x}{10 + \sqrt{100 - x^2}} - \dfrac{10}{x}$

$= \dfrac{x\left(10 - \sqrt{100 - x^2}\right)}{x^2} - \dfrac{10}{x} = -\dfrac{\sqrt{100 - x^2}}{x}$

When $x = 5$, $dy/dx = -\sqrt{3}$. When $x = 9$, $dy/dx = -\sqrt{19}/9$.

104. $y = \ln x$

$y' = \dfrac{1}{x} > 0$ for $x > 0$.

Since $\ln x$ is increasing on its entire domain $(0, \infty)$, it is a strictly monotonic function and therefore, is one-to-one.

106. False

π is a constant.

$\dfrac{d}{dx}[\ln \pi] = 0$

Section 5.2 The Natural Logarithmic Function: Integration

2. $\displaystyle\int \frac{10}{x}\,dx = 10\int \frac{1}{x}\,dx = 10\ln|x| + C$

4. $u = x - 5$, $du = dx$

$\displaystyle\int \frac{1}{x - 5}\,dx = \ln|x - 5| + C$

6. $\displaystyle\int \frac{1}{3x + 2}\,dx = \frac{1}{3}\int \frac{1}{3x + 2}(3)\,dx$

$= \dfrac{1}{3}\ln|3x + 2| + C$

8. $u = 3 - x^3$, $du = -3x^2\,dx$

$\displaystyle\int \frac{x^2}{3 - x^3}\,dx = -\frac{1}{3}\int \frac{1}{3 - x^3}(-3x^2)\,dx$

$= -\dfrac{1}{3}\ln|3 - x^3| + C$

10. $u = 9 - x^2$, $du = -2x\,dx$

$$\int \frac{x}{\sqrt{9-x^2}}\,dx = -\frac{1}{2}\int (9-x^2)^{-1/2}(-2x)\,dx = -\sqrt{9-x^2} + C$$

12. $$\int \frac{x(x+2)}{x^3+3x^2-4}\,dx = \frac{1}{3}\int \frac{3x^2+6x}{x^3+3x^2-4}\,dx \quad (u = x^3+3x^2-4)$$

$$= \frac{1}{3}\ln|x^3+3x^2-4| + C$$

14. $$\int \frac{2x^2+7x-3}{x-2}\,dx = \int \left(2x+11+\frac{19}{x-2}\right)dx$$

$$= x^2 + 11x + 19\ln|x-2| + C$$

16. $$\int \frac{x^3-6x-20}{x+5}\,dx = \int \left(x^2-5x+19-\frac{115}{x+5}\right)dx$$

$$= \frac{x^3}{3} - \frac{5x^2}{2} + 19x - 115\ln|x+5| + C$$

18. $$\int \frac{x^3-3x^2+4x-9}{x^2+3}\,dx = \int \left(-3+x+\frac{x}{x^2+3}\right)dx$$

$$= -3x + \frac{x^2}{2} + \frac{1}{2}\ln(x^2+3) + C$$

20. $$\int \frac{1}{x\ln(x^3)}\,dx = \frac{1}{3}\int \frac{1}{\ln x}\cdot\frac{1}{x}\,dx$$

$$= \frac{1}{3}\ln|\ln|x|| + C$$

22. $u = 1 + x^{1/3}$, $du = \dfrac{1}{3x^{2/3}}\,dx$

$$\int \frac{1}{x^{2/3}(1+x^{1/3})}\,dx = 3\int \frac{1}{1+x^{1/3}}\left(\frac{1}{3x^{2/3}}\right)dx$$

$$= 3\ln|1+x^{1/3}| + C$$

24. $$\int \frac{x(x-2)}{(x-1)^3}\,dx = \int \frac{x^2-2x+1-1}{(x-1)^3}\,dx$$

$$= \int \frac{(x-1)^2}{(x-1)^3}\,dx - \int \frac{1}{(x-1)^3}\,dx$$

$$= \int \frac{1}{x-1}\,dx - \int \frac{1}{(x-1)^3}\,dx$$

$$= \ln|x-1| + \frac{1}{2(x-1)^2} + C$$

26. $u = 1 + \sqrt{3x}$, $du = \dfrac{3}{2\sqrt{3x}}dx \Rightarrow dx = \dfrac{2}{3}(u-1)\,du$

$$\int \frac{1}{1+\sqrt{3x}}\,dx = \int \frac{1}{u}\frac{2}{3}(u-1)\,du$$

$$= \frac{2}{3}\int \left(1-\frac{1}{u}\right)du$$

$$= \frac{2}{3}[u - \ln|u|] + C$$

$$= \frac{2}{3}\left[1+\sqrt{3x} - \ln\left(1+\sqrt{3x}\right)\right] + C$$

$$= \frac{2}{3}\sqrt{3x} - \frac{2}{3}\ln\left(1+\sqrt{3x}\right) + C_1$$

28. $u = x^{1/3} - 1,\ du = \dfrac{1}{3x^{2/3}}\,dx \implies dx = 3(u+1)^2\,du$

$$\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\,dx = \int \frac{u+1}{u}3(u+1)^2\,du$$

$$= 3\int \frac{u+1}{u}(u^2+2u+1)\,du$$

$$= 3\int \left(u^2+3u+3+\frac{1}{u}\right)du$$

$$= 3\left[\frac{u^3}{3}+\frac{3u^2}{2}+3u+\ln|u|\right]+C$$

$$= 3\left[\frac{(x^{1/3}-1)^3}{3}+\frac{3(x^{1/3}-1)^2}{2}+3(x^{1/3}-1)+\ln|x^{1/3}-1|\right]+C$$

$$= 3\ln|x^{1/3}-1| + \frac{3x^{2/3}}{2} + 3x^{1/3} + x + C_1$$

30. $\displaystyle\int \tan 5\theta\,d\theta = \frac{1}{5}\int \frac{5\sin 5\theta}{\cos 5\theta}\,d\theta$

$$= -\frac{1}{5}\ln|\cos 5\theta| + C$$

32. $\displaystyle\int \sec\frac{x}{2}\,dx = 2\int \sec\frac{x}{2}\left(\frac{1}{2}\right)dx$

$$= 2\ln\left|\sec\frac{x}{2}+\tan\frac{x}{2}\right| + C$$

34. $u = \cot t,\ du = -\csc^2 t\,dt$

$$\int \frac{\csc^2 t}{\cot t}\,dt = -\ln|\cot t| + C$$

36. $\displaystyle\int (\sec t + \tan t)\,dt = \ln|\sec t + \tan t| - \ln|\cos t| + C$

$$= \ln\left|\frac{\sec t+\tan t}{\cos t}\right| + C$$

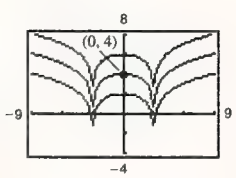
$$= \ln|\sec t(\sec t + \tan t)| + C$$

38. $\displaystyle y = \int \frac{2x}{x^2-9}\,dx$

$$= \ln|x^2-9| + C$$

$(0, 4)$: $4 = \ln|0-9| + C \implies C = 4 - \ln 9$

$y = \ln|x^2-9| + 4 - \ln 9$

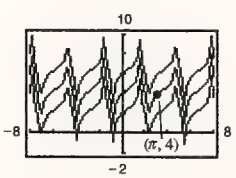


40. $\displaystyle r = \int \frac{\sec^2 t}{\tan t + 1}\,dt$

$$= \ln|\tan t + 1| + C$$

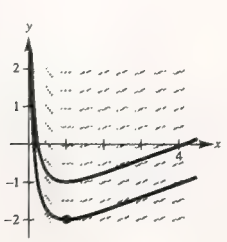
$(\pi, 4)$: $4 = \ln|0+1| + C \implies C = 4$

$r = \ln|\tan t + 1| + 4$



42. $\dfrac{dy}{dx} = \dfrac{\ln x}{x}$, $(1, -2)$

(a)



(b) $\displaystyle y = \int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2} + C$

$y(1) = -2 \implies -2 = \dfrac{(\ln 1)^2}{2} + C \implies C = -2$

Hence, $\displaystyle y = \frac{(\ln x)^2}{2} - 2$.

44. $\displaystyle\int_{-1}^{1}\frac{1}{x+2}\,dx = \Big[\ln|x+2|\Big]_{-1}^{1}$

$= \ln 3 - \ln 1 = \ln 3$

46. $u = \ln x,\ du = \dfrac{1}{x}\,dx$

$\displaystyle\int_{e}^{e^2}\frac{1}{x\ln x}dx = \int_{e}^{e^2}\left(\frac{1}{\ln x}\right)\frac{1}{x}\,dx = \Big[\ln|\ln|x||\Big]_{e}^{e^2} = \ln 2$

48. $\displaystyle\int_0^1\frac{x-1}{x+1}\,dx = \int_0^1 1\,dx + \int_0^1\frac{-2}{x+1}\,dx$

$= \Big[x - 2\ln|x+1|\Big]_0^1 = 1 - 2\ln 2$

50. $\displaystyle\int_{0.1}^{0.2}(\csc 2\theta - \cot 2\theta)^2\,d\theta = \int_{0.1}^{0.2}(\csc^2 2\theta - 2\csc 2\theta\cot 2\theta + \cot^2 2\theta)\,d\theta$

$\displaystyle= \int_{0.1}^{0.2}(2\csc^2 2\theta - 2\csc 2\theta\cot 2\theta - 1)\,d\theta$

$\displaystyle= \Big[-\cot 2\theta + \csc 2\theta - \theta\Big]_{0.1}^{0.2} \approx 0.0024$

52. $\ln|\sin x| + C = \ln\left|\dfrac{1}{\csc x}\right| + C = -\ln|\csc x| + C$

54. $-\ln|\csc x + \cot x| + C = -\ln\left|\dfrac{(\csc x + \cot x)(\csc x - \cot x)}{(\csc x - \cot x)}\right| + C = -\ln\left|\dfrac{\csc^2 x - \cot^2 x}{\csc x - \cot x}\right| + C$

$= -\ln\left|\dfrac{1}{\csc x - \cot x}\right| + C = \ln|\csc x - \cot x| + C$

56. $\displaystyle\int\frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx = -\left(1+\sqrt{x}\right)^2 + 6\left(1+\sqrt{x}\right) - 4\ln\left(1+\sqrt{x}\right) + C_1$

$= 4\sqrt{x} - x - 4\ln\left(1+\sqrt{x}\right) + C$ where $C = C_1 + 5$.

58. $\displaystyle\int\frac{\tan^2 2x}{\sec 2x}\,dx = \frac{1}{2}[\ln|\sec 2x + \tan 2x| - \sin 2x] + C$

60. $\displaystyle\int_{-\pi/4}^{\pi/4}\frac{\sin^2 x - \cos^2 x}{\cos x}\,dx = \Big[\ln|\sec x + \tan x| - 2\sin x\Big]_{-\pi/4}^{\pi/4}$

$\displaystyle= \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) - 2\sqrt{2} \approx -1.066$

Note: In Exercises 62 and 64, you can use the Second Fundamental Theorem of Calculus or integrate the function.

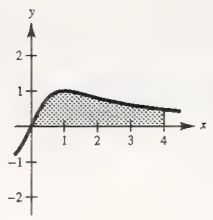
62. $\displaystyle F(x) = \int_0^x \tan t\,dt$

$F'(x) = \tan x$

64. $\displaystyle F(x) = \int_1^{x^2}\frac{1}{t}\,dt$

$F'(x) = \dfrac{2x}{x^2} = \dfrac{2}{x}$

66.



$A \approx 3$

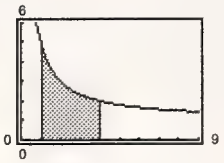
Matches (a)

68. $A = \displaystyle\int_1^4 \frac{x+4}{x}\,dx = \int_1^4 \left(1 + \frac{4}{x}\right) dx$

$= \left[x + 4\ln x\right]_1^4$

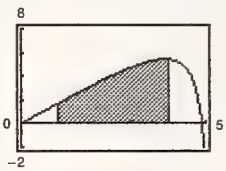
$= 4 + 4\ln 4 - 1$

$= 3 + 4\ln 4 \approx 8.5452$



70. $\displaystyle\int_1^4 (2x - \tan(0.3x))\,dx = \left[x^2 + \frac{10}{3}\ln|\cos(0.3x)|\right]_1^4$

$= \left[16 + \frac{10}{3}\ln\cos(1.2)\right] - \left[1 + \frac{10}{3}\ln\cos(0.3)\right] \approx 11.7686$



72. Substitution: $(u = x^2 + 4)$ and Power Rule

74. Substitution: $(u = \tan x)$ and Log Rule

76. Answers will vary.

78. Average value $= \dfrac{1}{4-2}\displaystyle\int_2^4 \frac{4(x+1)}{x^2}\,dx$

$= 2\displaystyle\int_2^4 \left(\frac{1}{x} + \frac{1}{x^2}\right) dx$

$= 2\left[\ln x - \frac{1}{x}\right]_2^4$

$= 2\left[\ln 4 - \frac{1}{4} - \ln 2 + \frac{1}{2}\right]$

$= 2\left[\ln 2 + \frac{1}{4}\right] = \ln 4 + \frac{1}{2} \approx 1.8863$

80. Average value $= \dfrac{1}{2-0}\displaystyle\int_0^2 \sec\frac{\pi x}{6}\,dx$

$= \left[\frac{1}{2}\left(\frac{6}{\pi}\right)\ln\left|\sec\frac{\pi x}{6} + \tan\frac{\pi x}{6}\right|\right]_0^2$

$= \frac{3}{\pi}\left[\ln\left(2 + \sqrt{3}\right) - \ln(1+0)\right]$

$= \frac{3}{\pi}\ln\left(2 + \sqrt{3}\right)$

82. $t = \dfrac{10}{\ln 2}\displaystyle\int_{250}^{300} \frac{1}{T - 100}\,dT$

$= \dfrac{10}{\ln 2}\left[\ln(T-100)\right]_{250}^{300} = \dfrac{10}{\ln 2}[\ln 200 - \ln 150] = \dfrac{10}{\ln 2}\left[\ln\left(\frac{4}{3}\right)\right] \approx 4.1504$ units of time

84. $\dfrac{dS}{dt} = \dfrac{k}{t}$

$S(t) = \displaystyle\int \frac{k}{t}\,dt = k\ln|t| + C = k\ln t + C$ since $t > 1$.

$S(2) = k\ln 2 + C = 200$

$S(4) = k\ln 4 + C = 300$

Solving this system yields $k = 100/\ln 2$ and $C = 100$. Thus,

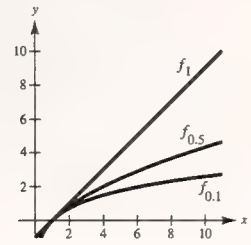
$S(t) = \dfrac{100\ln t}{\ln 2} + 100 = 100\left[\dfrac{\ln t}{\ln 2} + 1\right]$.

86. $k = 1$: $f_1(x) = x - 1$

$k = 0.5$: $f_{0.5}(x) = \dfrac{\sqrt{x} - 1}{0.5} = 2\left(\sqrt{x} - 1\right)$

$k = 0.1$: $f_{0.1}(x) = \dfrac{\sqrt[10]{x} - 1}{0.1} = 10\left(\sqrt[10]{x} - 1\right)$

$\lim\limits_{k \to 0^+} f_k(x) = \ln x$



88. False

$\dfrac{d}{dx}[\ln x] = \dfrac{1}{x}$

90. False; the integrand has a nonremovable discontinuity at $x = 0$.

Section 5.3 Inverse Functions

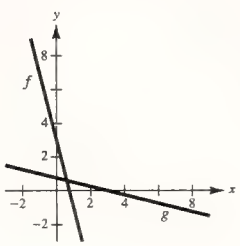
2. (a) $f(x) = 3 - 4x$

$g(x) = \dfrac{3 - x}{4}$

$f(g(x)) = f\left(\dfrac{3 - x}{4}\right) = 3 - 4\left(\dfrac{3 - x}{4}\right) = x$

$g(f(x)) = g(3 - 4x) = \dfrac{3 - (3 - 4x)}{4} = x$

(b)



4. (a) $f(x) = 1 - x^3$

$g(x) = \sqrt[3]{1 - x}$

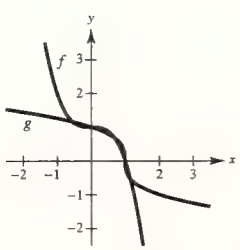
$f(g(x)) = f\left(\sqrt[3]{1 - x}\right) = 1 - \left(\sqrt[3]{1 - x}\right)^3$

$= 1 - (1 - x) = x$

$g(f(x)) = g(1 - x^3)$

$= \sqrt[3]{1 - (1 - x^3)} = \sqrt[3]{x^3} = x$

(b)



6. (a) $f(x) = 16 - x^2,\ x \geq 0$

$g(x) = \sqrt{16 - x}$

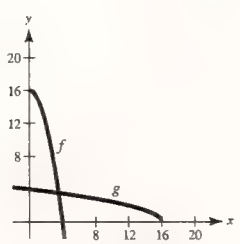
$f(g(x)) = f\left(\sqrt{16 - x}\right) = 16 - \left(\sqrt{16 - x}\right)^2$

$= 16 - (16 - x) = x$

$g(f(x)) = g(16 - x^2) = \sqrt{16 - (16 - x^2)}$

$= \sqrt{x^2} = x$

(b)



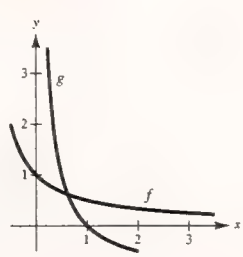
8. (a) $f(x) = \dfrac{1}{1 + x},\ x \geq 0$

$g(x) = \dfrac{1 - x}{x},\ 0 < x \leq 1$

$f(g(x)) = f\left(\dfrac{1 - x}{x}\right) = \dfrac{1}{1 + \dfrac{1 - x}{x}} = \dfrac{1}{\dfrac{1}{x}} = x$

$g(f(x)) = g\left(\dfrac{1}{1 + x}\right) = \dfrac{1 - \dfrac{1}{1 + x}}{\dfrac{1}{1 + x}} = \dfrac{x}{1 + x} \cdot \dfrac{1 + x}{1} = x$

(b)

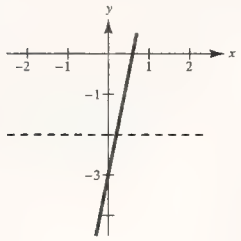


10. Matches (b)

12. Matches (d)

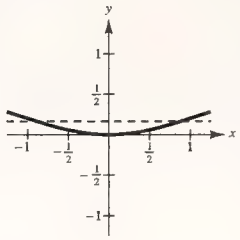
14. $f(x) = 5x - 3$

One-to-one; has an inverse



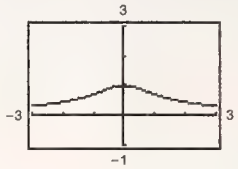
16. $F(x) = \dfrac{x^2}{x^2 + 4}$

Not one-to-one; does not have an inverse



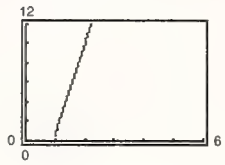
18. $g(t) = \dfrac{1}{\sqrt{t^2 + 1}}$

Not one-to-one; does not have an inverse



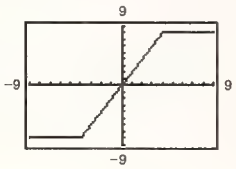
20. $f(x) = 5x\sqrt{x - 1}$

One-to-one; has an inverse



22. $h(x) = |x + 4| - |x - 4|$

Not one-to-one; does not have an inverse



24. $f(x) = \cos\dfrac{3x}{2}$

$f'(x) = -\dfrac{3}{2}\sin\dfrac{3x}{2} = 0$ when $x = 0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \ldots$

f is not strictly monotonic on $(-\infty, \infty)$. Therefore, f does not have an inverse.

26. $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12 = 3(x - 2)^2 \geq 0$ for all x.

f is increasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

28. $f(x) = \ln(x - 3), x > 3$

$f'(x) = \dfrac{1}{x - 3} > 0$ for $x > 3$.

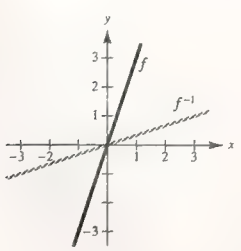
f is increasing on $(3, \infty)$. Therefore, f is strictly monotonic and has an inverse.

30. $f(x) = 3x = y$

$x = \dfrac{y}{3}$

$y = \dfrac{x}{3}$

$f^{-1}(x) = \dfrac{x}{3}$

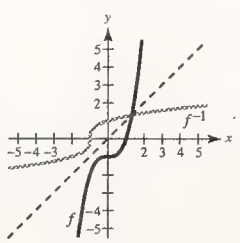


32. $f(x) = x^3 - 1 = y$

$x = \sqrt[3]{y + 1}$

$y = \sqrt[3]{x + 1}$

$f^{-1}(x) = \sqrt[3]{x + 1}$

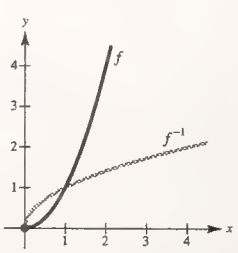


34. $f(x) = x^2 = y, \ 0 \leq x$

$x = \sqrt{y}$

$y = \sqrt{x}$

$f^{-1}(x) = \sqrt{x}$

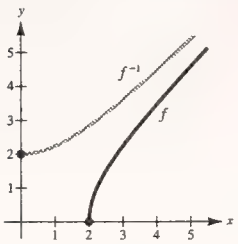


36. $f(x) = \sqrt{x^2 - 4} = y,\ x \geq 2$

$x = \sqrt{y^2 + 4}$

$y = \sqrt{x^2 + 4}$

$f^{-1}(x) = \sqrt{x^2 + 4},\ x \geq 0$

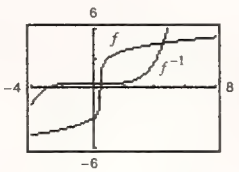


38. $f(x) = 3\sqrt[5]{2x - 1} = y$

$x = \dfrac{y^5 + 243}{486}$

$y = \dfrac{x^5 + 243}{486}$

$f^{-1}(x) = \dfrac{x^5 + 243}{486}$



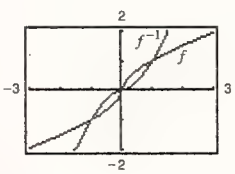
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

40. $f(x) = x^{3/5} = y$

$x = y^{5/3}$

$y = x^{5/3}$

$f^{-1}(x) = x^{5/3}$



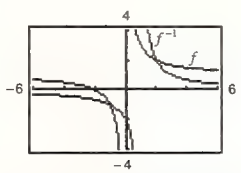
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

42. $f(x) = \dfrac{x + 2}{x} = y$

$x = \dfrac{2}{y - 1}$

$y = \dfrac{2}{x - 1}$

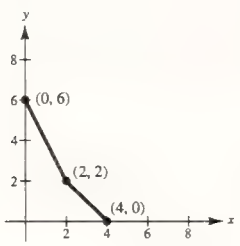
$f^{-1}(x) = \dfrac{2}{x - 1}$



The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

44.

x	0	2	4
$f^{-1}(x)$	6	2	0



46. $f(x) = k(2 - x - x^3)$ is one-to-one for all $k \neq 0$. Since $f^{-1}(3) = -2, f(-2) = 3 = k(2 - (-2) - (-2)^3) = 12k \implies k = \frac{1}{4}$.

48. $f(x) = |x + 2|$ on $[-2, \infty)$

$f'(x) = \dfrac{|x + 2|}{x + 2}(1) = 1 > 0$ on $(-2, \infty)$

f is increasing on $[-2, \infty)$. Therefore, f is strictly monotonic and has an inverse.

50. $f(x) = \cot x$ on $(0, \pi)$

$f'(x) = -\csc^2 x < 0$ on $(0, \pi)$

f is decreasing on $(0, \pi)$. Therefore, f is strictly monotonic and has an inverse.

52. $f(x) = \sec x$ on $\left[0, \dfrac{\pi}{2}\right)$

$f'(x) = \sec x \tan x > 0$ on $\left(0, \dfrac{\pi}{2}\right)$

f is increasing on $[0, \pi/2)$. Therefore, f is strictly monotonic and has an inverse.

54. $f(x) = 2 - \dfrac{3}{x^2} = y$ on $(0, 10)$

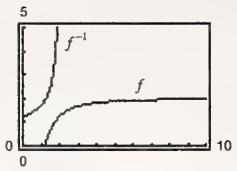
$2x^2 - 3 = x^2y$

$x^2(2 - y) = 3$

$x = \pm\sqrt{\dfrac{3}{2 - y}}$

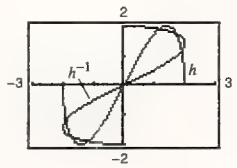
$y = \pm\sqrt{\dfrac{3}{2 - x}}$

$f^{-1}(x) = \sqrt{\dfrac{3}{2 - x}},\ x < 2$



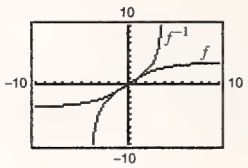
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

56. (a), (b)



(c) h is not one-to-one and does not have an inverse. The inverse relation is not an inverse function.

58. (a), (b)



(c) Yes, f is one-to-one and has an inverse. The inverse relation is an inverse function.

60. $f(x) = -3$

Not one-to-one; does not have an inverse

62. $f(x) = ax + b$

f is one-to-one; has an inverse

$ax + b = y$

$x = \dfrac{y - b}{a}$

$y = \dfrac{x - b}{a}$

$f^{-1}(x) = \dfrac{x - b}{a},\ a \neq 0$

64. $f(x) = 16 - x^4$ is one-to-one for $x \geq 0$.

$16 - x^4 = y$

$16 - y = x^4$

$\sqrt[4]{16 - y} = x$

$\sqrt[4]{16 - x} = y$

$f^{-1}(x) = \sqrt[4]{16 - x},\ x \leq 16$

66. $f(x) = |x - 3|$ is one-to-one for $x \geq 3$.

$x - 3 = y$

$x = y + 3$

$y = x + 3$

$f^{-1}(x) = x + 3,\ x \geq 0$

68. No, there could be two times $t_1 \neq t_2$ for which $h(t_1) = h(t_2)$.

70. Yes, the area function is increasing and hence one-to-one. The inverse function gives the radius r corresponding to the area A.

72. $f(x) = \dfrac{1}{27}(x^5 + 2x^3); f(-3) = \dfrac{1}{27}(-243 - 54) = -11 = a.$

$f'(x) = \dfrac{1}{27}(5x^4 + 6x^2)$

$(f^{-1})'(-11) = \dfrac{1}{f'(f^{-1}(-11))} = \dfrac{1}{f'(-3)} = \dfrac{27}{5(-3)^4 + 6(-3)^2} = \dfrac{1}{17}$

74. $f(x) = \cos 2x, f(0) = 1 = a$

$f'(x) = -2 \sin 2x$

$(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(0)} = \dfrac{1}{-2 \sin 0} = \dfrac{1}{0}$ which is undefined.

76. $f(x) = \sqrt{x-4}, f(8) = 2 = a$

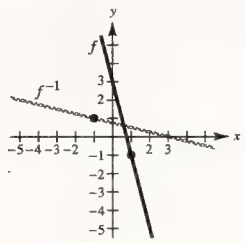
$f'(x) = \dfrac{1}{2\sqrt{x-4}}$

$(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(8)} = \dfrac{1}{1/(2\sqrt{8-4})} = \dfrac{1}{1/4} = 4$

78. (a) Domain f = Domain $f^{-1} = (-\infty, \infty)$

(b) Range f = Range $f^{-1} = (-\infty, \infty)$

(c)



(d) $f(x) = 3 - 4x, \ (1, -1)$

$f'(x) = -4$

$f'(1) = -4$

$f^{-1}(x) = \dfrac{3-x}{4}, \ (-1, 1)$

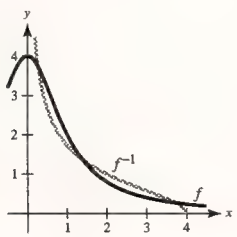
$(f^{-1})'(x) = -\dfrac{1}{4}$

$(f^{-1})'(-1) = -\dfrac{1}{4}$

80. (a) Domain $f = [0, \infty)$, Domain $f^{-1} = (0, 4]$

(b) Range $f = (0, 4]$, Range $f^{-1} = [0, \infty)$

(c)



(d) $f(x) = \dfrac{4}{1+x^2}$

$f'(x) = \dfrac{-8x}{(x^2+1)^2}, \ f'(1) = -2$

$f^{-1}(x) = \sqrt{\dfrac{4-x}{x}}$

$(f^{-1})'(x) = \dfrac{-2}{x^2\sqrt{\dfrac{4-x}{x}}}, \ (f^{-1})'(2) = -\dfrac{1}{2}$

82. $x = 2 \ln(y^2 - 3)$

$1 = 2\dfrac{1}{y^2-3} 2y \dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{y^2-3}{4y}$. At $(0, 4)$, $\dfrac{dy}{dx} = \dfrac{16-3}{16} = \dfrac{13}{16}$.

In Exercises 84 and 86, use the following.

$f(x) = \frac{1}{8}x - 3$ **and** $g(x) = x^3$

$f^{-1}(x) = 8(x+3)$ **and** $g^{-1}(x) = \sqrt[3]{x}$

84. $(g^{-1} \circ f^{-1})(-3) = g^{-1}(f^{-1}(-3)) = g^{-1}(0) = 0$

86. $(g^{-1} \circ g^{-1})(-4) = g^{-1}(g^{-1}(-4)) = g^{-1}(\sqrt[3]{-4})$

$= \sqrt[3]{\sqrt[3]{-4}} = -\sqrt[9]{4}$

In Exercises 88 and 90, use the following.

$$f(x) = x + 4 \text{ and } g(x) = 2x - 5$$

$$f^{-1}(x) = x - 4 \text{ and } g^{-1}(x) = \frac{x + 5}{2}$$

88. $(f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x))$

$$= f^{-1}\left(\frac{x + 5}{2}\right)$$

$$= \frac{x + 5}{2} - 4$$

$$= \frac{x - 3}{2}$$

90. $(g \circ f)(x) = g(f(x))$

$$= g(x + 4)$$

$$= 2(x + 4) - 5$$

$$= 2x + 3$$

Hence, $(g \circ f)^{-1}(x) = \frac{x - 3}{2}$

(Note: $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$)

92. The graphs of f and f^{-1} are mirror images with respect to the line $y = x$.

94. Theorem 5.9: Let f be differentiable on an interval I. If f has an inverse g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \ f'(g(x)) \neq 0$$

96. f is not one-to-one because different x-values yield the same y-value.

Example: $f(3) = f\left(-\frac{4}{3}\right) = \frac{3}{5}$

Not continuous at ± 2.

98. If f has an inverse, then f and f^{-1} are both one-to-one. Let $(f^{-1})^{-1}(x) = y$ then $x = f^{-1}(y)$ and $f(x) = y$. Thus, $(f^{-1})^{-1} = f$.

100. If f has an inverse and $f(x_1) = f(x_2)$, then $f^{-1}(f(x_1)) = f^{-1}(f(x_2)) \Rightarrow x_1 = x_2$. Therefore, f is one-to-one. If $f(x)$ is one-to-one, then for every value b in the range, there corresponds exactly one value a in the domain. Define $g(x)$ such that the domain of g equals the range of f and $g(b) = a$. By the reflexive property of inverses, $g = f^{-1}$.

102. True; if f has a y-intercept.

104. False

Let $f(x) = x$ or $g(x) = 1/x$.

106. From Theorem 5.9, we have:

$$g'(x) = \frac{1}{f'(g(x))}$$

$$g''(x) = \frac{f'(g(x))(0) - f''(g(x))g'(x)}{[f'(g(x))]^2}$$

$$= -\frac{f''(g(x)) \cdot [1/(f'(g(x)))]}{[f'(g(x))]^2}$$

$$= -\frac{f''(g(x))}{[f'(g(x))]^3}$$

If f is increasing and concave down, then $f' > 0$ and $f'' < 0$ which implies that g is increasing and concave up.

Section 5.4 Exponential Functions: Differentiation and Integration

2. $e^{-2} = 0.1353\ldots$

$\ln 0.1353\ldots = -2$

4. $\ln 0.5 = -0.6931\ldots$

$e^{-0.6931\ldots} = \frac{1}{2}$

6. $e^{\ln 2x} = 12$

$2x = 12$

$x = 6$

8. $4e^x = 83$

$e^x = \frac{83}{4}$

$x = \ln\left(\frac{83}{4}\right) \approx 3.033$

10. $-6 + 3e^x = 8$

$3e^x = 14$

$e^x = \frac{14}{3}$

$x = \ln\left(\frac{14}{3}\right) \approx 1.540$

12. $200e^{-4x} = 15$

$e^{-4x} = \frac{15}{200} = \frac{3}{40}$

$-4x = \ln\left(\frac{3}{40}\right)$

$x = \frac{1}{4}\ln\left(\frac{40}{3}\right) \approx 0.648$

14. $\ln x^2 = 10$

$x^2 = e^{10}$

$x = \pm e^5 \approx \pm 148.4132$

16. $\ln 4x = 1$

$4x = e^1 = e$

$x = \frac{e}{4} \approx 0.680$

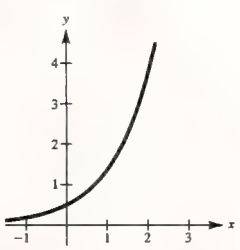
18. $\ln(x-2)^2 = 12$

$(x-2)^2 = e^{12}$

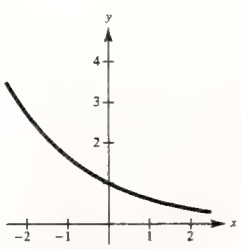
$x - 2 = e^6$

$x = 2 + e^6 \approx 405.429$

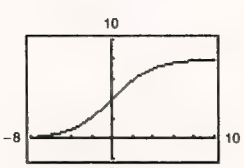
20. $y = \frac{1}{2}e^x$



22. $y = e^{-x/2}$

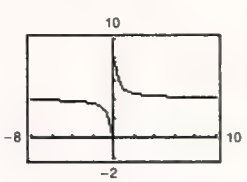


24. (a)



Horizontal asymptotes: $y = 0$ and $y = 8$

(b)



Horizontal asymptote: $y = 4$

26. $y = Ce^{-ax}$

Horizontal asymptote: $y = 0$

Reflection in the y-axis

Matches (d)

28. $y = \dfrac{C}{1 + e^{-ax}}$

$$\lim_{x \to \infty} \frac{C}{1 + e^{-ax}} = C$$

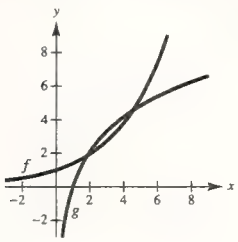
$$\lim_{x \to -\infty} \frac{C}{1 + e^{-ax}} = 0$$

Horizontal asymptotes: $y = C$ and $y = 0$

Matches (b)

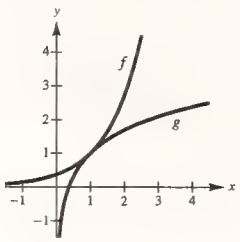
30. $f(x) = e^{x/3}$

$g(x) = \ln x^3 = 3 \ln x$



32. $f(x) = e^{x-1}$

$g(x) = 1 + \ln x$



34. In the same way,

$$\lim_{x \to \infty} \left(1 + \frac{r}{x}\right)^x = e^r \text{ for } r > 0.$$

36. $1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \dfrac{1}{120} + \dfrac{1}{720} + \dfrac{1}{5040} = 2.71\overline{825396}$

$$e \approx 2.718281828$$

$$e > 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040}$$

38. (a) $y = e^{2x}$

$y' = 2e^{2x}$

At $(0, 1)$, $y' = 2$.

(b) $y = e^{-2x}$

$y' = -2e^{-2x}$

At $(0, 1)$, $y' = -2$.

40. $f(x) = e^{1-x}$

$f'(x) = -e^{1-x}$

42. $y = e^{-x^2}$

$$\frac{dy}{dx} = -2xe^{-x^2}$$

44. $y = x^2e^{-x}$

$$\frac{dy}{dx} = -x^2e^{-x} + 2xe^{-x}$$

$$= xe^{-x}(2 - x)$$

46. $g(t) = e^{-3/t^2}$

$$g'(t) = e^{-3/t^2}(6t^{-3}) = \frac{6}{t^3e^{3/t^2}}$$

48. $y = \ln\left(\dfrac{1 + e^x}{1 - e^x}\right)$

$$= \ln(1 + e^x) - \ln(1 - e^x)$$

$$\frac{dy}{dx} = \frac{e^x}{1 + e^x} + \frac{e^x}{1 - e^x}$$

$$= \frac{2e^x}{1 - e^{2x}}$$

50. $y = \ln\left(\dfrac{e^x + e^{-x}}{2}\right)$

$$= \ln(e^x + e^{-x}) - \ln 2$$

$$\frac{dy}{dx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$= \frac{e^{2x} - 1}{e^{2x} + 1}$$

52. $y = \dfrac{e^x - e^{-x}}{2}$

$$\frac{dy}{dx} = \frac{e^x + e^{-x}}{2}$$

54. $y = xe^x - e^x = e^x(x - 1)$

$$\frac{dy}{dx} = e^x + e^x(x - 1) = xe^x$$

56. $f(x) = e^3 \ln x$

$$f'(x) = \frac{e^3}{x}$$

58. $y = \ln e^x = x$

$$\frac{dy}{dx} = 1$$

60. $e^{xy} + x^2 - y^2 = 10$

$$\left(x\frac{dy}{dx} + y\right)e^{xy} + 2x - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(xe^{xy} - 2y) = -ye^{xy} - 2x$$

$$\frac{dy}{dx} = -\frac{ye^{xy} + 2x}{xe^{xy} - 2y}$$

62. $g(x) = \sqrt{x} + e^x \ln x$

$$g'(x) = \frac{1}{2\sqrt{x}} + \frac{e^x}{x} + e^x \ln x$$

$$g''(x) = -\frac{1}{4x^{3/2}} + \frac{xe^x - e^x}{x^2} + \frac{e^x}{x} + e^x \ln x$$

$$= -\frac{1}{4x\sqrt{x}} + \frac{e^x(2x - 1)}{x^2} + e^x \ln x$$

64. $y = e^x(3\cos 2x - 4\sin 2x)$

$$y' = e^x(-6\sin 2x - 8\cos 2x) + e^x(3\cos 2x - 4\sin 2x)$$
$$= e^x(-10\sin 2x - 5\cos 2x) = -5e^x(2\sin 2x + \cos 2x)$$

$$y'' = -5e^x(4\cos 2x - 2\sin 2x) - 5e^x(2\sin 2x + \cos 2x) = -5e^x(5\cos 2x) = -25e^x\cos 2x$$

$$y'' - 2y' = -25e^x\cos 2x - 2(-5e^x)(2\sin 2x + \cos 2x) = -5e^x(3\cos 2x - 4\sin 2x) = -5y$$

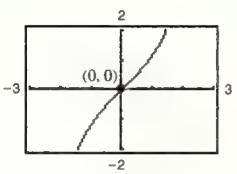
Therefore, $y'' - 2y' = -5y \Rightarrow y'' - 2y' + 5y = 0$.

66. $f(x) = \dfrac{e^x - e^{-x}}{2}$

$$f'(x) = \frac{e^x + e^{-x}}{2} > 0$$

$$f''(x) = \frac{e^x - e^{-x}}{2} = 0 \text{ when } x = 0.$$

Point of inflection: $(0, 0)$



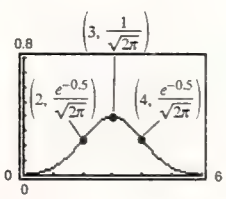
68. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-3)^2/2}$

$$g'(x) = \frac{-1}{\sqrt{2\pi}}(x - 3)e^{-(x-3)^2/2}$$

$$g''(x) = \frac{1}{\sqrt{2\pi}}(x - 2)(x - 4)e^{-(x-3)^2/2}$$

Relative maximum: $\left(3, \dfrac{1}{\sqrt{2\pi}}\right) \approx (3, 0.399)$

Points of inflection: $\left(2, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right), \left(4, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right) \approx (2, 0.242), (4, 0.242)$



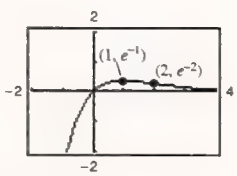
70. $f(x) = xe^{-x}$

$f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1 - x) = 0$ when $x = 1$.

$f''(x) = -e^{-x} + (-e^{-x})(1 - x) = e^{-x}(x - 2) = 0$ when $x = 2$.

Relative maximum: $(1, e^{-1})$

Point of inflection: $(2, 2e^{-2})$



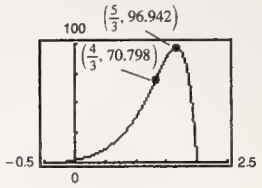
72. $f(x) = -2 + e^{3x}(4 - 2x)$

$f'(x) = e^{3x}(-2) + 3e^{3x}(4 - 2x) = e^{3x}(10 - 6x) = 0$ when $x = \frac{5}{3}$.

$f''(x) = e^{3x}(-6) + 3e^{3x}(10 - 6x) = e^{3x}(24 - 18x) = 0$ when $x = \frac{4}{3}$.

Relative maximum: $\left(\frac{5}{3}, 96.942\right)$

Point of inflection: $\left(\frac{4}{3}, 70.798\right)$



74. (a) $f(c) = f(c + x)$

$$10ce^{-c} = 10(c + x)e^{-(c+x)}$$

$$\frac{c}{e^c} = \frac{c + x}{e^{c+x}}$$

$$ce^{c+x} = (c + x)e^c$$

$$ce^x = c + x$$

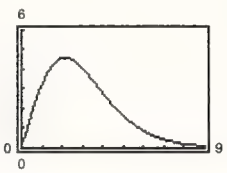
$$ce^x - c = x$$

$$c = \frac{x}{e^x - 1}$$

(b) $A(x) = xf(c) = x\left[10\left(\frac{x}{e^x - 1}\right)e^{-(x/(e^x-1))}\right]$

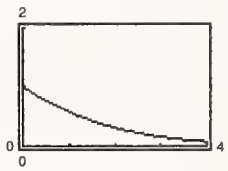
$$= \frac{10x^2}{e^x - 1}e^{x/(1-e^x)}$$

(c) $A(x) = \frac{10x^2}{e^x - 1}e^{x/(1-e^x)}$



The maximum area is 4.591 for $x = 2.118$ and $f(x) = 2.547$.

(d) $c = \frac{x}{e^x - 1}$



$$\lim_{x \to 0^+} c = 1$$

$$\lim_{x \to \infty} c = 0$$

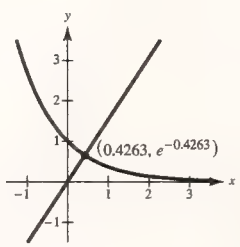
76. Let (x_0, y_0) be the desired point on $y = e^{-x}$.

$$y = e^{-x}$$

$$y' = -e^{-x} \quad \text{(Slope of tangent line)}$$

$$-\frac{1}{y'} = e^x \quad \text{(Slope of normal line)}$$

$$y - e^{-x_0} = e^{x_0}(x - x_0)$$



We want (0, 0) to satisfy the equation:

$$-e^{-x_0} = -x_0e^{x_0}$$

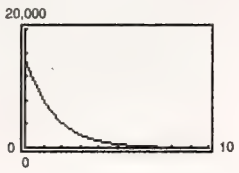
$$1 = x_0e^{2x_0}$$

$$x_0e^{2x_0} - 1 = 0$$

Solving by Newton's Method or using a computer, the solution is $x_0 \approx 0.4263$.

$(0.4263, e^{-0.4263})$

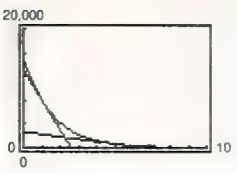
78. $V = 15{,}000e^{-0.6286t}, 0 \le t \le 10$

(a) 

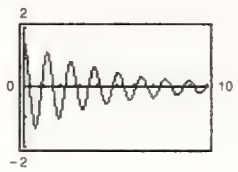
(b) $\dfrac{dV}{dt} = -9429e^{-0.6286t}$

When $t = 1, \dfrac{dV}{dt} \approx -5028.84.$

When $t = 5, \dfrac{dV}{dt} \approx -406.89.$

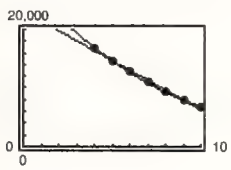
(c) 

80. $1.56e^{-0.22t} \cos 4.9t \le 0.25$ (3 inches equals one-fourth foot.) Using a graphing utility or Newton's Method, we have $t \ge 7.79$ seconds.



82. (a) $V_1 = -1686.79t + 23{,}181.79$

$V_2 = 109.52t^2 - 3220.12t + 28{,}110.36$



(b) The slope represents the rate of decrease in value of the car.

(c) $V_3 = 31{,}450.77(0.8592)^t = 31{,}450.77e^{-0.1518t}$

(d) Horizontal asymptote: $\lim_{t \to \infty} V_3(t) = 0$

As $t \to \infty$, the value of the car approaches 0.

(e) $\dfrac{dV_3}{dt} = -4774.2e^{-0.1518t}$

For $t = 5, \dfrac{dV_3}{dt} \approx -2235$ dollars/year.

For $t = 9, \dfrac{dV_3}{dt} \approx -1218$ dollars/year.

84. $f(x) = e^{-x^2/2}, f(0) = 1$

$f'(x) = -xe^{-x^2/2}, f'(0) = 0$

$f''(x) = x^2e^{-x^2/2} - e^{-x^2/2} = e^{-x^2/2}(x^2 - 1), f''(0) = -1$

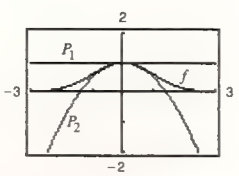
$P_1(x) = 1 + 0(x - 0) = 1, P_1(0) = 1$

$P_1'(x) = 0, \ P_1'(0) = 0$

$P_2(x) = 1 + 0(x - 0) - \dfrac{1}{2}(x - 0)^2 = 1 - \dfrac{x^2}{2}, P_2(0) = 1$

$P_2'(x) = -x, P_2'(0) = 0$

$P_2''(x) = -1, P_2''(0) = -1$



The values of f, P_1, P_2 and their first derivatives agree at $x = 0$. The values of the second derivatives of f and P_2 agree at $x = 0$.

86. n^{th} term is $x^n/n!$ in polynomial:

$$y_4 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

Conjecture: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

88. Let $u = -x^4$, $du = -4x^3\,dx$.

$$\int e^{-x^4}(-4x^3)\,dx = e^{-x^4} + C$$

90. $\displaystyle\int_3^4 e^{3-x}\,dx = \left[-e^{3-x}\right]_3^4 = -e^{-1} + 1 = 1 - \frac{1}{e}$

92. $\displaystyle\int x^2 e^{x^3/2}\,dx = \frac{2}{3}\int e^{x^3/2}\left(\frac{3x^2}{2}\right)dx = \frac{2}{3}e^{x^3/2} + C$

94. $\displaystyle\int \frac{e^{1/x^2}}{x^3}\,dx = -\frac{1}{2}\int e^{1/x^2}\left(\frac{-2}{x^3}\right)dx = -\frac{1}{2}e^{1/x^2} + C$

96. Let $u = 1 + e^{2x}$, $du = 2e^{2x}\,dx$.

$$\int \frac{e^{2x}}{1 + e^{2x}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{1 + e^{2x}}\,dx = \frac{1}{2}\ln(1 + e^{2x}) + C$$

98. Let $u = \dfrac{-x^2}{2}$, $du = -x\,dx$.

$$\int_0^{\sqrt{2}} xe^{-x^2/2}\,dx = -\int_0^{\sqrt{2}} e^{-x^2/2}(-x)\,dx = \left[-e^{-x^2/2}\right]_0^{\sqrt{2}} = 1 - e^{-1} = \frac{e-1}{e}$$

100. Let $u = e^x + e^{-x}$, $du = (e^x - e^{-x})dx$.

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \ln(e^x + e^{-x}) + C$$

102. Let $u = e^x + e^{-x}$, $du = (e^x - e^{-x})\,dx$.

$$\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2}\,dx = 2\int (e^x + e^{-x})^{-2}(e^x - e^{-x})\,dx$$

$$= \frac{-2}{e^x + e^{-x}} + C$$

104. $\displaystyle\int \frac{e^{2x} + 2e^x + 1}{e^x}\,dx = \int (e^x + 2 + e^{-x})\,dx$

$$= e^x + 2x - e^{-x} + C$$

106. $\displaystyle\int e^{\sec 2x} \sec 2x \tan 2x\,dx = \frac{1}{2}e^{\sec 2x} + C$

$(u = \sec 2x,\ du = 2 \sec 2x \tan 2x)$

108. $\displaystyle\int \ln(e^{2x-1})\,dx = \int (2x - 1)\,dx$

$$= x^2 - x + C$$

110. $\displaystyle y = \int (e^x - e^{-x})^2\,dx$

$$= \int (e^{2x} - 2 + e^{-2x})\,dx$$

$$= \frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x} + C$$

112. $\displaystyle f'(x) = \int (\sin x + e^{2x})\,dx = -\cos x + \frac{1}{2}e^{2x} + C_1$

$$f'(0) = -1 + \frac{1}{2} + C_1 = \frac{1}{2} \Rightarrow C_1 = 1$$

$$f'(x) = -\cos x + \frac{1}{2}e^{2x} + 1$$

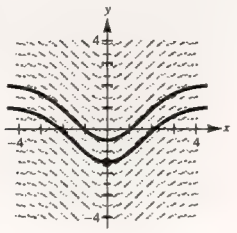
$$f(x) = \int \left(-\cos x + \frac{1}{2}e^{2x} + 1\right)dx$$

$$= -\sin x + \frac{1}{4}e^{2x} + x + C_2$$

$$f(0) = \frac{1}{4} + C_2 = \frac{1}{4} \Rightarrow C_2 = 0$$

$$f(x) = x - \sin x + \frac{1}{4}e^{2x}$$

114. (a)



(b) $\dfrac{dy}{dx} = xe^{-0.2x^2}, \quad \left(0, -\dfrac{3}{2}\right)$

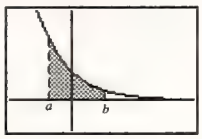
$$y = \int xe^{-0.2x^2}\,dx = \frac{1}{-0.4}\int e^{-0.2x^2}(-0.4x)\,dx$$

$$= -\frac{1}{0.4}e^{-0.2x^2} + C = -2.5e^{-0.2x^2} + C$$

$$\left(0, -\frac{3}{2}\right): -\frac{3}{2} = -2.5e^0 + C = -2.5 + C \Rightarrow C = 1$$

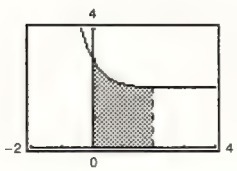
$$y = -2.5e^{-0.2x^2} + 1$$

116. $\displaystyle\int_a^b e^{-x}\,dx = \left[-e^{-x}\right]_a^b = e^{-a} - e^{-b}$



118. $\displaystyle\int_0^2 (e^{-2x} + 2)\,dx = \left[-\frac{1}{2}e^{-2x} + 2x\right]_0^2$

$$= -\frac{1}{2}e^{-4} + 4 + \frac{1}{2} \approx 4.491$$



120. (a) $\displaystyle\int_0^4 \sqrt{x}e^x\,dx,\ n = 12$

Midpoint Rule: 92.1898

Trapezoidal Rule: 93.8371

Simpson's Rule: 92.7385

Graphing Utility: 92.7437

(b) $\displaystyle\int_0^2 2xe^{-x}\,dx,\ n = 12$

Midpoint Rule: 1.1906

Trapezoidal Rule: 1.1827

Simpson's Rule: 1.1880

Graphing Utility: 1.18799

122. $\displaystyle\int_0^x 0.3^{-0.3t}\,dt = \frac{1}{2}$

$$\left[-e^{-0.3t}\right]_0^x = \frac{1}{2}$$

$$-e^{-0.3x} + 1 = \frac{1}{2}$$

$$e^{-0.3x} = \frac{1}{2}$$

$$-0.3x = \ln\frac{1}{2} = -\ln 2$$

$$x = \frac{\ln 2}{0.3} \approx 2.31 \text{ minutes}$$

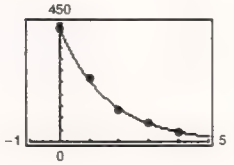
124.

t	0	1	2	3	4
R	425	240	118	71	36
$\ln R$	6.052	5.481	4.771	4.263	3.584

(a) $\ln R = -0.6155t + 6.0609$

$$R = e^{-0.6155t + 6.0609} = 428.78e^{-0.6155t}$$

(b)



(c) $\displaystyle\int_0^4 R(t)\,dt = \int_0^4 428.78e^{-0.6155t}\,dt \approx 637.2$ liters

126. The graphs of $f(x) = \ln x$ and $g(x) = e^x$ are mirror images across the line $y = x$.

128. (a) Log Rule: $(u = e^x + 1)$

(b) Substitution: $(u = x^2)$

130. $\ln \dfrac{e^a}{e^b} = \ln e^a - \ln e^b = a - b$

$\ln e^{a-b} = a - b$

Therefore, $\ln \dfrac{e^a}{e^b} = \ln e^{a-b}$ and since $y = \ln x$ is one-to-one, we have $\dfrac{e^a}{e^b} = e^{a-b}$.

Section 5.5 Bases Other than e and Applications

2. $y = \left(\dfrac{1}{2}\right)^{t/8}$

At $t_0 = 16$, $y = \left(\dfrac{1}{2}\right)^{16/8} = \dfrac{1}{4}$

4. $y = \left(\dfrac{1}{2}\right)^{t/5}$

At $t_0 = 2$, $y = \left(\dfrac{1}{2}\right)^{2/5} \approx 0.7579$

6. $\log_{27} 9 = \log_{27} 27^{2/3} = \frac{2}{3}$

8. $\log_a \dfrac{1}{a} = \log_a 1 - \log_a a = -1$

10. (a) $27^{2/3} = 9$

$\log_{27} 9 = \dfrac{2}{3}$

(b) $16^{3/4} = 8$

$\log_{16} 8 = \dfrac{3}{4}$

12. (a) $\log_3 \frac{1}{9} = -2$

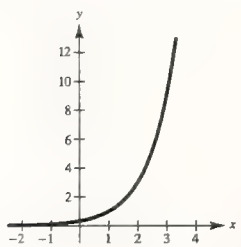
$3^{-2} = \frac{1}{9}$

(b) $49^{1/2} = 7$

$\log_{49} 7 = \frac{1}{2}$

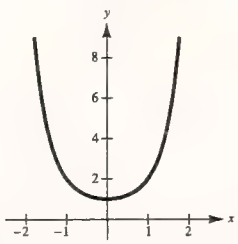
14. $y = 3^{x-1}$

x	-1	0	1	2	3
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9



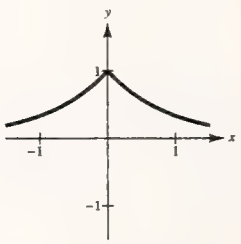
16. $y = 2^{x^2}$

x	-2	-1	0	1	2
y	16	2	1	2	16



18. $y = 3^{-|x|}$

x	0	± 1	± 2
y	1	$\frac{1}{3}$	$\frac{1}{9}$



20. (a) $\log_3 \dfrac{1}{81} = x$

$3^x = \dfrac{1}{81}$

$x = -4$

(b) $\log_6 36 = x$

$6^x = 36$

$x = 2$

22. (a) $\log_b 27 = 3$

$b^3 = 27$

$b = 3$

(b) $\log_b 125 = 3$

$b^3 = 125$

$b = 5$

24. (a) $\log_3 x + \log_3(x-2) = 1$

$$\log_3[x(x-2)] = 1$$
$$x(x-2) = 3^1$$
$$x^2 - 2x - 3 = 0$$
$$(x+1)(x-3) = 0$$
$$x = -1 \text{ OR } x = 3$$

$x = 3$ is the only solution since the domain of the logarithmic function is the set of all *positive* real numbers.

(b) $\log_{10}(x+3) - \log_{10} x = 1$

$$\log_{10}\frac{x+3}{x} = 1$$
$$\frac{x+3}{x} = 10^1$$
$$x + 3 = 10x$$
$$3 = 9x$$
$$x = \frac{1}{3}$$

26. $5^{6x} = 8320$

$$6x \ln 5 = \ln 8320$$
$$x = \frac{\ln 8320}{6 \ln 5} \approx 0.935$$

28. $3(5^{x-1}) = 86$

$$5^{x-1} = \frac{86}{3}$$
$$(x-1)\ln 5 = \ln\left(\frac{86}{3}\right)$$
$$x - 1 = \frac{\ln\left(\frac{86}{3}\right)}{\ln 5}$$
$$x = 1 + \frac{\ln\left(\frac{86}{3}\right)}{\ln 5} \approx 3.085$$

30. $\left(1 + \frac{0.10}{365}\right)^{365t} = 2$

$$365t \ln\left(1 + \frac{0.10}{365}\right) = \ln 2$$
$$t = \frac{1}{365} \cdot \frac{\ln 2}{\ln\left(1 + \frac{0.10}{365}\right)} \approx 6.932$$

32. $\log_{10}(t-3) = 2.6$

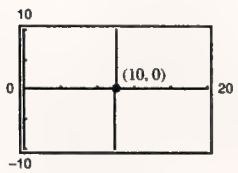
$$t - 3 = 10^{2.6}$$
$$t = 3 + 10^{2.6} \approx 401.107$$

34. $\log_5\sqrt{x-4} = 3.2$

$$\sqrt{x-4} = 5^{3.2}$$
$$x - 4 = (5^{3.2})^2 = 5^{6.4}$$
$$x = 4 + 5^{6.4} \approx 29{,}748.593$$

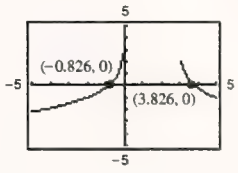
36. $f(t) = 300(1.0075^{12t}) - 735.41$

Zero: $t \approx 10$



38. $g(x) = 1 - 2\log_{10}[x(x-3)]$

Zeros: $x \approx -0.826, 3.826$

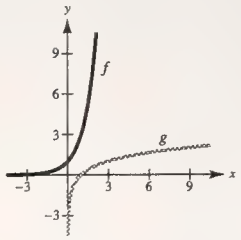


40. $f(x) = 3^x$

$g(x) = \log_3 x$

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$g(x)$	-2	-1	0	1	2



42. $g(x) = 2^{-x}$

$g'(x) = -(\ln 2)\, 2^{-x}$

44. $y = x(6^{-2x})$

$$\frac{dy}{dx} = x[-2(\ln 6)6^{-2x}) + 6^{-2x}$$
$$= 6^{-2x}[-2x(\ln 6) + 1]$$
$$= 6^{-2x}(1 - 2x \ln 6)$$

46. $f(t) = \dfrac{3^{2t}}{t}$

$$f'(t) = \frac{t(2 \ln 3)\, 3^{2t} - 3^{2t}}{t^2}$$
$$= \frac{3^{2t}(2t \ln 3 - 1)}{t^2}$$

48. $g(\alpha) = 5^{-\alpha/2} \sin 2\alpha$

$g'(\alpha) = 5^{-\alpha/2}\, 2 \cos 2\alpha - \frac{1}{2}(\ln 5)\, 5^{-\alpha/2} \sin 2\alpha$

50. $y = \log_{10}(2x) = \log_{10} 2 + \log_{10} x$

$$\frac{dy}{dx} = 0 + \frac{1}{x \ln 10} = \frac{1}{x \ln 10}$$

52. $h(x) = \log_3 \dfrac{x\sqrt{x-1}}{2}$

$$= \log_3 x + \frac{1}{2}\log_3 (x-1) - \log_3 2$$

$$h'(x) = \frac{1}{x \ln 3} + \frac{1}{2} \cdot \frac{1}{(x-1)\ln 3} - 0$$
$$= \frac{1}{\ln 3}\left[\frac{1}{x} + \frac{1}{2(x-1)}\right]$$
$$= \frac{1}{\ln 3}\left[\frac{3x-2}{2x(x-1)}\right]$$

54. $y = \log_{10} \dfrac{x^2 - 1}{x}$

$$= \log_{10}(x^2 - 1) - \log_{10} x$$

$$\frac{dy}{dx} = \frac{2x}{(x^2-1)\ln 10} - \frac{1}{x \ln 10}$$
$$= \frac{1}{\ln 10}\left[\frac{2x}{x^2-1} - \frac{1}{x}\right]$$
$$= \frac{1}{\ln 10}\left[\frac{x^2+1}{x(x^2-1)}\right]$$

56. $f(t) = t^{3/2} \log_2 \sqrt{t+1} = t^{3/2}\, \frac{1}{2}\, \frac{\ln(t+1)}{\ln 2}$

$$f'(t) = \frac{1}{2 \ln 2}\left[t^{3/2}\frac{1}{t+1} + \frac{3}{2}t^{1/2}\ln(t+1)\right]$$

58. $y = x^{x-1}$

$$\ln y = (x-1)(\ln x)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = (x-1)\left(\frac{1}{x}\right) + \ln x$$
$$\frac{dy}{dx} = y\left[\frac{x-1}{x} + \ln x\right]$$
$$= x^{x-2}(x - 1 + x \ln x)$$

60. $y = (1+x)^{1/x}$

$$\ln y = \frac{1}{x}\ln(1+x)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x}\left(\frac{1}{1+x}\right) + \ln(1+x)\left(-\frac{1}{x^2}\right)$$
$$\frac{dy}{dx} = \frac{y}{x}\left[\frac{1}{x+1} - \frac{\ln(x+1)}{x}\right]$$
$$= \frac{(1+x)^{1/x}}{x}\left[\frac{1}{x+1} - \frac{\ln(x+1)}{x}\right]$$

62. $\displaystyle\int 5^{-x}\, dx = \frac{-5^{-x}}{\ln 5} + C$

64. $\displaystyle\int_{-2}^{0} (3^3 - 5^2)\, dx = \int_{-2}^{0} (27 - 25)\, dx$

$$= \int_{-2}^{0} 2\, dx$$
$$= \Big[2x\Big]_{-2}^{0} = 4$$

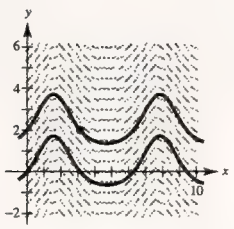
66. $\int (3-x)\,7^{(3-x)^2}\,dx = -\frac{1}{2}\int -2(3-x)\,7^{(3-x)^2}\,dx$

$= -\frac{1}{2\ln 7}[7^{(3-x)^2}] + C$

68. $\int 2^{\sin x}\cos x\,dx,\ u = \sin x,\ du = \cos x\,dx$

$\frac{1}{\ln 2}\,2^{\sin x} + C$

70. (a)



(b) $\frac{dy}{dx} = e^{\sin x}\cos x \qquad (\pi, 2)$

$y = \int e^{\sin x}\cos x\,dx = e^{\sin x} + C$

$(\pi, 2)$: $2 = e^{\sin \pi} + C = 1 + C \Rightarrow C = 1$

$y = e^{\sin x} + 1$

72. $\log_b x = \frac{\ln x}{\ln b} = \frac{\log_{10} x}{\log_{10} b}$

74. $f(x) = \log_{10} x$

(a) Domain: $x > 0$

(b) $y = \log_{10} x$

$10^y = x$

$f^{-1}(x) = 10^x$

(c) $\log_{10} 1000 = \log_{10} 10^3 = 3$

$\log_{10} 10{,}000 = \log_{10} 10^4 = 4$

If $1000 \le x \le 10{,}000$, then $3 \le f(x) \le 4$.

(d) If $f(x) < 0$, then $0 < x < 1$.

(e) $f(x) + 1 = \log_{10} x + \log_{10} 10$

$= \log_{10}(10x)$

x must have been increased by a factor of 10.

(f) $\log_{10}\left(\frac{x_1}{x_2}\right) = \log_{10} x_1 - \log_{10} x_2$

$= 3n - n = 2n$

Thus, $x_1/x_2 = 10^{2n} = 100^n$.

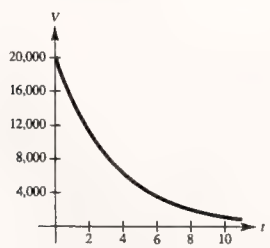
76. $f(x) = a^x$

(a) $f(u+v) = a^{u+v} = a^u a^v = f(u)f(v)$

(b) $f(2x) = a^{2x} = (a^x)^2 = [f(x)]^2$

78. $V(t) = 20{,}000\left(\frac{3}{4}\right)^t$

(a)



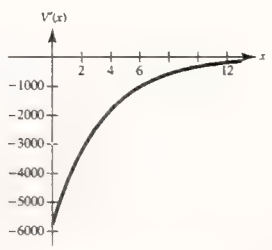
$V(2) = 20{,}000\left(\frac{3}{4}\right)^2 = \$11{,}250$

(b) $\frac{dV}{dt} = 20{,}000\left(\ln\frac{3}{4}\right)\left(\frac{3}{4}\right)^t$

When $t = 1$: $\frac{dV}{dt} \approx -4315.23$

When $t = 4$: $\frac{dV}{dt} \approx -1820.49$

(c)



Horizontal asymptote: $v' = 0$

As the car ages, it is worth less each year and depreciates less each year, but the value of the car will never reach \$0.

80. $P = \$2500,\ r = 6\% = 0.06,\ t = 20$

$$A = 2500\left(1 + \frac{0.06}{n}\right)^{20n}$$

$$A = 2500e^{(0.06)(20)} = 8300.29$$

n	1	2	4	12	365	Continuous
A	8017.84	8155.09	8226.66	8275.51	8299.47	8300.29

82. $P = \$5000,\ r = 7\% = 0.07,\ t = 25$

$$A = 5000\left(1 + \frac{0.07}{n}\right)^{25n}$$

$$A = 5000e^{0.07(25)}$$

n	1	2	4	12	365	Continuous
A	27,137.16	27,924.63	28,340.78	28,627.09	28,768.19	28,773.01

84. $100{,}000 = Pe^{0.06t} \Rightarrow P = 100{,}000e^{-0.06t}$

t	1	10	20	30	40	50
P	94,176.45	54,881.16	30,119.42	16,529.89	9071.80	4978.71

86. $100{,}000 = P\left(1 + \frac{0.07}{365}\right)^{365t} \Rightarrow P = 100{,}000\left(1 + \frac{0.07}{365}\right)^{-365t}$

t	1	10	20	30	40	50
P	93,240.01	49,661.86	24,663.01	12,248.11	6082.64	3020.75

88. Let $P = \$100,\ 0 \le t \le 20$.

(a) $A = 100e^{0.03t}$

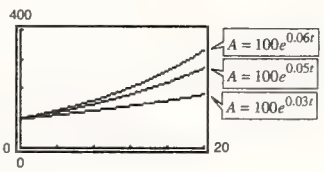
$A(20) \approx 182.21$

(b) $A = 100e^{0.05t}$

$A(20) \approx 271.83$

(c) $A = 100e^{0.06t}$

$A(20) \approx 332.01$



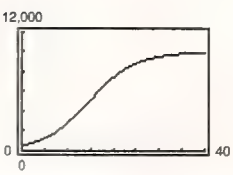
90. (a) $\lim_{n\to\infty} \frac{0.86}{1 + e^{-0.25n}} = 0.86$ or 86%

(b) $P' = \frac{-0.86(-0.25)(e^{-0.25n})}{(1 + e^{-0.25n})} = \frac{0.215e^{-0.25n}}{(1 + e^{-0.25n})}$

$P'(3) \approx 0.069$

$P'(10) \approx 0.016$

92. (a)



(b) Limiting size: 10,000 fish

(c) $p(t) = \frac{10{,}000}{1 + 19e^{-t/5}}$

$$p'(t) = \frac{e^{-t/5}}{(1 + 19e^{t/5})^2}\left(\frac{19}{5}\right)(10{,}000)$$

$$= \frac{38{,}000e^{-t/5}}{(1 + 19e^{-t/5})^2}$$

$p'(1) \approx 113.5$ fish/month

$p'(10) \approx 403.2$ fish/month

(d) $p''(t) = -\frac{38{,}000}{5}(e^{-t/5})\left[\frac{1 - 19e^{-t/5}}{(1 + 19e^{-t/5})^3}\right] = 0$

$19e^{-t/5} = 1$

$\frac{t}{5} = \ln 19$

$t = 5 \ln 19 \approx 14.72$

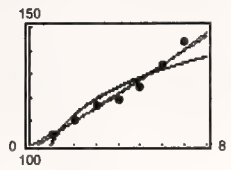
94. (a) $y_1 = 6.0536x + 97.5571$

$y_2 = 100.0751 + 17.8148 \ln x$

$y_3 = 99.4557(1.0506)^x$

$y_4 = 101.2875x^{0.1471}$

(b)



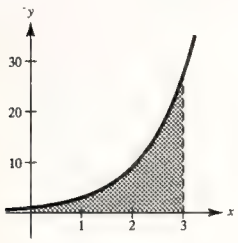
y_3 seems best.

(c) The slope of 6.0536 is the annual rate of change in the amount given to philanthropy.

(d) For 1996, $x = 6$ and $y_1' = 6.0536$, $y_2' \approx 2.9691$, $y_3' \approx 6.6015$, $y_4' \approx 3.2321$.

y_3 is increasing at the greatest rate in 1996.

96. $A = \int_0^3 3^x\,dx = \left[\frac{3^x}{\ln 3}\right]_0^3 = \frac{26}{\ln 3} \approx 23.666$



98.

x	1	10^{-1}	10^{-2}	10^{-4}	10^{-6}
$(1 + x)^{1/x}$	2	2.594	2.705	2.718	2.718

100.

t	0	1	2	3	4
y	600	630	661.50	694.58	729.30

$y = C(k^t)$

When $t = 0$, $y = 600 \Rightarrow C = 600$.

$y = 600(k^t)$

$\frac{630}{600} = 1.05$, $\frac{661.50}{630} = 1.05$, $\frac{694.58}{661.50} \approx 1.05$, $\frac{729.30}{694.58} \approx 1.05$

Let $k = 1.05$.

$y = 600(1.05)^t$

102. True.

$f(e^{n+1}) - f(e^n) = \ln e^{n+1} - \ln e^n$

$= n + 1 - n$

$= 1$

104. True.

$\frac{d^n y}{dx^n} = Ce^x$

$= y$ for $n = 1, 2, 3, \ldots$

106. True.

$f(x) = g(x)e^x = 0 \Rightarrow$

$g(x) = 0$ since $e^x > 0$ for all x.

108. $y = x^{\sin x}$

$$\ln y = \ln x^{\sin x} = \sin x \cdot \ln x$$

$$\frac{y'}{y} = \sin x\left(\frac{1}{x}\right) + \cos x \cdot \ln x$$

$$y' = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \cdot \ln x\right]$$

At $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$y' = \left(\frac{\pi}{2}\right)^{\sin(\pi/2)}\left[\frac{\sin(\pi/2)}{\pi/2} + \cos\left(\frac{\pi}{2}\right)\ln\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{\pi}{2}\left[\frac{2}{\pi} + 0\right] = 1$$

Tangent line: $y - \frac{\pi}{2} = 1\left(x - \frac{\pi}{2}\right)$

$$y = x$$

Section 5.6 Differential Equations: Growth and Decay

2. $\frac{dy}{dx} = 4 - x$

$$y = \int (4 - x)\,dx = 4x - \frac{x^2}{2} + C$$

4. $\frac{dy}{dx} = 4 - y$

$$\frac{dy}{4 - y} = dx$$

$$\int \frac{-1}{4 - y}\,dy = \int -dx$$

$$\ln|4 - y|\,dy = -x + C_1$$

$$4 - y = e^{-x + C_1} = Ce^{-x}$$

$$y = 4 - Ce^{-x}$$

6. $y' = \frac{\sqrt{x}}{3y}$

$$3yy' = \sqrt{x}$$

$$\int 3yy'\,dx = \int \sqrt{x}\,dx$$

$$\frac{3y^2}{2} = \frac{2}{3}x^{3/2} + C_1$$

$$9y^2 - 4x^{3/2} = C$$

8. $y' = x(1 + y)$

$$\frac{y'}{1 + y} = x$$

$$\int \frac{y'}{1 + y}\,dx = \int x\,dx$$

$$\int \frac{dy}{1 + y} = \int x\,dx$$

$$\ln(1 + y) = \frac{x^2}{2} + C_1$$

$$1 + y = e^{(x^2/2) + C_1}$$

$$y = e^{C_1}e^{x^2/2} - 1$$

$$= Ce^{x^2/2} - 1$$

10. $xy + y' = 100x$

$$y' = 100x + xy = x(100 - y)$$

$$\frac{y'}{100 - y} = x$$

$$\int \frac{y'}{100 - y}\,dx = \int x\,dx$$

$$\int \frac{1}{100 - y}\,dy = \int x\,dx$$

$$-\ln(100 - y) = \frac{x^2}{2} + C_1$$

$$\ln(100 - y) = -\frac{x^2}{2} - C_1$$

$$100 - y = e^{-(x^2/2) - C_1}$$

$$-y = e^{-C_1}e^{-x^2/2} - 100$$

$$y = 100 - Ce^{-x^2/2}$$

12. $\dfrac{dP}{dt} = k(10 - t)$

$$\int \frac{dP}{dt}\,dt = \int k(10 - t)\,dt$$

$$\int dP = -\frac{k}{2}(10 - t)^2 + C$$

$$P = -\frac{k}{2}(10 - t)^2 + C$$

14. $\dfrac{dy}{dx} = kx(L - y)$

$$\frac{1}{L - y}\frac{dy}{dx} = kx$$

$$\int \frac{1}{L - y}\frac{dy}{dx}\,dx = \int kx\,dx$$

$$\int \frac{1}{L - y}\,dy = \frac{kx^2}{2} + C_1$$

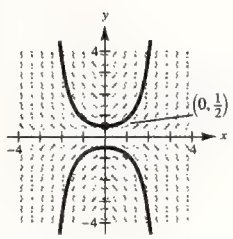
$$-\ln(L - y) = \frac{kx^2}{2} + C_1$$

$$L - y = e^{-(kx^2/2) - C_1}$$

$$-y = -L + e^{-C_1}e^{-kx^2/2}$$

$$y = L - Ce^{-kx^2/2}$$

16. (a)



(b) $\dfrac{dy}{dx} = xy, \left(0, \dfrac{1}{2}\right)$

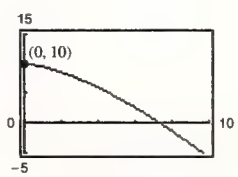
$$\frac{dy}{y} = x\,dx$$

$$\ln|y| = \frac{x^2}{2} + C$$

$$y = e^{x^2/2 + C} = C_1e^{x^2/2}$$

$$\left(0, \frac{1}{2}\right): \frac{1}{2} = C_1e^0 \Rightarrow C_1 = \frac{1}{2} \Rightarrow y = \frac{1}{2}e^{x^2/2}$$

18. $\dfrac{dy}{dt} = -\dfrac{3}{4}\sqrt{t}, \quad (0, 10)$



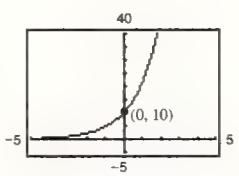
$$\int dy = \int -\frac{3}{4}\sqrt{t}\,dt$$

$$y = -\frac{1}{2}t^{3/2} + C$$

$$10 = -\frac{1}{2}(0)^{3/2} + C \Rightarrow C = 10$$

$$y = -\frac{1}{2}t^{3/2} + 10$$

20. $\dfrac{dy}{dt} = \dfrac{3}{4}y, \quad (0, 10)$



$$\int \frac{dy}{y} = \int \frac{3}{4}\,dt$$

$$\ln y = \frac{3}{4}t + C_1$$

$$y = e^{(3/4)t + C_1}$$

$$= e^{C_1}e^{(3/4)t} = Ce^{3t/4}$$

$$10 = Ce^0 \Rightarrow C = 10$$

$$y = 10e^{3t/4}$$

22. $\dfrac{dN}{dt} = kN$

$N = Ce^{kt}$ (Theorem 5.16)

$(0, 250)$: $C = 250$

$(1, 400)$: $400 = 250e^k \Rightarrow k = \ln\dfrac{400}{250} = \ln\dfrac{8}{5}$

When $t = 4$, $N = 250e^{4\ln(8/5)} = 250e^{\ln(8/5)^4}$

$$= 250\left(\frac{8}{5}\right)^4 = \frac{8192}{5}$$

24. $\dfrac{dP}{dt} = kP$

$P = Ce^{kt}$ (Theorem 5.16)

$(0, 5000)$: $C = 5000$

$(1, 4750)$: $4750 = 5000e^k \Rightarrow k = \ln\left(\dfrac{19}{20}\right)$

When $t = 5$, $P = 5000e^{\ln(19/20)(5)}$

$$= 5000\left(\frac{19}{20}\right)^5 \approx 3868.905$$

26. $y = Ce^{kt}$, $(0, 4)$, $\left(5, \frac{1}{2}\right)$

$C = 4$

$y = 4e^{kt}$

$\frac{1}{2} = 4e^{5k}$

$k = \frac{\ln(1/8)}{5} \approx -0.4159$

$y = 4e^{-0.4159t}$

28. $y = Ce^{kt}$, $\left(3, \frac{1}{2}\right)$, $(4, 5)$

$\frac{1}{2} = Ce^{3k}$

$5 = Ce^{4k}$

$2Ce^{3k} = \frac{1}{5}Ce^{4k}$

$10e^{3k} = e^{4k}$

$10 = e^{k}$

$k = \ln 10 \approx 2.3026$

$y = Ce^{2.3026t}$

$5 = Ce^{2.3026(4)}$

$C \approx 0.0005$

$y = 0.0005e^{2.3026t}$

30. $y' = \frac{dy}{dt} = ky$

32. $\frac{dy}{dx} = \frac{1}{2}x^2y$

$\frac{dy}{dx} > 0$ when $y > 0$. Quadrants I and II.

34. Since $y = Ce^{[\ln(1/2)/1620]t}$, we have $1.5 = Ce^{[\ln(1/2)/1620](1000)} \Rightarrow C \approx 2.30$ which implies that the initial quantity is 2.30 grams. When $t = 10{,}000$, we have $y = 2.30e^{[\ln(1/2)/1620](10{,}000)} \approx 0.03$ gram.

36. Since $y = Ce^{[\ln(1/2)/5730]t}$, we have $2.0 = Ce^{[\ln(1/2)/5730](10{,}000)} \Rightarrow C \approx 6.70$ which implies that the initial quantity is 6.70 grams. When $t = 1000$, we have $y = 6.70e^{[\ln(1/2)/5730](1000)} \approx 5.94$ grams.

38. Since $y = Ce^{[\ln(1/2)/5730]t}$, we have $3.2 = Ce^{[\ln(1/2)/5730]1000} \Rightarrow C \approx 3.61$.

Initial quantity: 3.61 grams.

When $t = 10{,}000$, we have $y \approx 1.08$ grams.

40. Since $y = Ce^{[\ln(1/2)/24{,}360]t}$, we have $0.4 = Ce^{[\ln(1/2)/24{,}360](10{,}000)} \Rightarrow C \approx 0.53$ which implies that the initial quantity is 0.53 gram. When $t = 1000$, we have $y = 0.53e^{[\ln(1/2)/24{,}360](1000)} \approx 0.52$ gram.

42. Since $\frac{dy}{dx} = ky$, $y = Ce^{kt}$ or $y = y_0e^{kt}$.

$\frac{1}{2}y_0 = y_0e^{5730k}$

$k = -\frac{\ln 2}{5730}$

$0.15y_0 = y_0e^{(-\ln 2/5730)t}$

$\ln 0.15 = -\frac{(\ln 2)t}{5730}$

$t = -\frac{5730 \ln 0.15}{\ln 2} \approx 15{,}682.8$ years.

44. Since $A = 20{,}000e^{0.055t}$, the time to double is given by $40{,}000 = 20{,}000e^{0.055t}$ and we have

$2 = e^{0.055t}$

$\ln 2 = 0.055t$

$t = \frac{\ln 2}{0.055} \approx 12.6$ years.

Amount after 10 years:
$A = 20{,}000e^{(0.055)(10)} \approx \$34{,}665.06$

46. Since $A = 10{,}000e^{rt}$ and $A = 20{,}000$ when $t = 5$, we have the following.

$$20{,}000 = 10{,}000e^{5r}$$

$$r = \frac{\ln 2}{5} \approx 0.1386 = 13.86\%$$

Amount after 10 years: $A = 10{,}000e^{[(\ln 2)/5](10)} = \$40{,}000$

48. Since $A = 2000e^{rt}$ and $A = 5436.56$ when $t = 10$, we have the following.

$$5436.56 = 2000e^{10r}$$

$$r = \frac{\ln(5436.56/2000)}{10} \approx 0.10 = 10\%$$

The time to double is given by

$$4000 = 2000e^{0.10t}$$

$$t = \frac{\ln 2}{0.10} \approx 6.93 \text{ years.}$$

50. $500{,}000 = P\left(1 + \dfrac{0.06}{12}\right)^{(12)(40)}$

$$P = 500{,}000(1.005)^{-480} \approx \$45{,}631.04$$

52. $500{,}000 = P\left(1 + \dfrac{0.09}{12}\right)^{(12)(25)}$

$$P = 500{,}000\left(1 + \frac{0.09}{12}\right)^{-300}$$

$$\approx \$53{,}143.92$$

54. (a) $2000 = 1000(1 + 0.6)^t$

$$2 = 1.06^t$$

$$\ln 2 = t \ln 1.06$$

$$t = \frac{\ln 2}{\ln 1.06} \approx 11.90 \text{ years}$$

(b) $2000 = 1000\left(1 + \dfrac{0.06}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.06}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.06}{12}\right)$$

$$t = \frac{1}{12} \frac{\ln 2}{\ln\left(1 + \dfrac{0.06}{12}\right)} \approx 11.58 \text{ years}$$

(c) $2000 = 1000\left(1 + \dfrac{0.06}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.06}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.06}{365}\right)$$

$$t = \frac{1}{365} \frac{\ln 2}{\ln\left(1 + \dfrac{0.06}{365}\right)} \approx 11.55 \text{ years}$$

(d) $2000 = 1000e^{0.06t}$

$$2 = e^{0.06t}$$

$$\ln 2 = 0.06t$$

$$t = \frac{\ln 2}{0.06} \approx 11.55 \text{ years}$$

56. (a) $2000 = 1000(1 + 0.055)^t$

$$2 = 1.055^t$$

$$\ln 2 = t \ln 1.055$$

$$t = \frac{\ln 2}{\ln 1.055} \approx 12.95 \text{ years}$$

(b) $2000 = 1000\left(1 + \dfrac{0.055}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.055}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.055}{12}\right)$$

$$t = \frac{1}{12} \frac{\ln 2}{\ln\left(1 + \dfrac{0.055}{12}\right)} \approx 12.63 \text{ years}$$

(c) $2000 = 1000\left(1 + \dfrac{0.055}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.055}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.055}{365}\right)$$

$$t = \frac{1}{365} \frac{\ln 2}{\ln\left(1 + \dfrac{0.055}{365}\right)} \approx 12.60 \text{ years}$$

(d) $2000 = 1000e^{0.055t}$

$$2 = e^{0.055t}$$

$$\ln 2 = 0.055t$$

$$t = \frac{\ln 2}{0.055} \approx 12.60 \text{ years}$$

58. $P = Ce^{kt} = Ce^{0.031t}$

$P(-1) = 11.6 = Ce^{0.031(-1)} \Rightarrow C = 11.9652$

$P = 11.9652e^{0.031t}$

$P(10) \approx 16.31$ or 16,310,000 people in 2010

60. $P = Ce^{kt} = Ce^{-0.004t}$

$P(-1) = 3.6 = Ce^{-0.004(-1)} \Rightarrow C = 3.5856$

$P = 3.5856e^{-0.004t}$

$P(10) \approx 3.45$ or 3,450,000 people in 2010

62. (a) $N = 100.1596(1.2455)^t$

(b) $N = 400$ when $t = 6.3$ hours (graphing utility)

Analytically,

$$400 = 100.1596(1.2455)^t$$

$$1.2455^t = \frac{400}{100.1596} = 3.9936$$

$$t \ln 1.2455 = \ln 3.9936$$

$$t = \frac{\ln 3.9936}{\ln 1.2455} \approx 6.3 \text{ hours.}$$

64. $y = Ce^{kt}$, (0, 742,000), (2, 632,000)

$C = 742{,}000$

$632{,}000 = 742{,}000e^{2k}$

$$k = \frac{\ln(632/742)}{2} \approx -0.0802$$

$y \approx 742{,}000e^{-0.0802t}$

When $t = 4$, $y \approx \$538{,}372$.

66. (a) $20 = 30(1 - e^{30k})$

$30e^{30k} = 10$

$$k = \frac{\ln(1/3)}{30} = \frac{-\ln 3}{30} \approx -0.0366$$

$N \approx 30(1 - e^{-0.0366t})$

(b) $25 = 30(1 - e^{-0.0366t})$

$$e^{-0.0366t} = \frac{1}{6}$$

$$t = \frac{-\ln 6}{-0.0366} \approx 49 \text{ days}$$

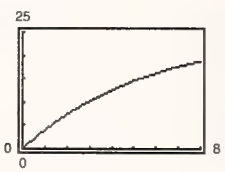
68. $S = 25(1 - e^{kt})$

(a) $4 = 25(1 - e^{k(1)}) \Rightarrow 1 - e^k = \frac{4}{25} \Rightarrow e^k = \frac{21}{25} \Rightarrow k = \ln\left(\frac{21}{25}\right) \approx -0.1744$

(b) 25,000 units $\left(\lim_{t\to\infty} S = 25\right)$

(c) When $t = 5$, $S \approx 14.545$ which is 14,545 units.

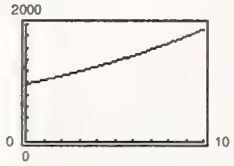
(d)



70. (a) $R = 979.3993(1.0694)^t = 979.3993e^{0.0671t}$

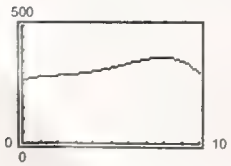
$I = -0.1385t^4 + 2.1770t^3 - 9.9755t^2 + 23.8513t + 266.4923$

(b)

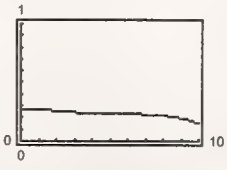


Rate of growth $= R'(t) = 65.7e^{0.0671t}$

(c)



(d) $P(t) = \frac{I}{R}$



72. $93 = 10 \log_{10} \dfrac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-6.7 = \log_{10} I \Rightarrow I = 10^{-6.7}$

$80 = 10 \log_{10} \dfrac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-8 = \log_{10} I \Rightarrow I = 10^{-8}$

Percentage decrease: $\left(\dfrac{10^{-6.7} - 10^{-8}}{10^{-6.7}}\right)(100) \approx 95\%$

74. Since $\dfrac{dy}{dt} = k(y - 80)$

$$\int \frac{1}{y - 80}\, dy = \int k\, dt$$

$$\ln(y - 80) = kt + C.$$

When $t = 0$, $y = 1500$. Thus, $C = \ln 1420$.

When $t = 1$, $y = 1120$. Thus,

$$k(1) + \ln 1420 = \ln(1120 - 80)$$

$$k = \ln 1040 - \ln 1420 = \ln \frac{104}{142}.$$

Thus, $y = 1420e^{[\ln(104/142)]t} + 80$.

When $t = 5$, $y \approx 379.2°$.

76. True

78. True

Section 5.7 Differential Equations: Separation of Variables

2. Differential equation: $y' = \dfrac{2xy}{x^2 - y^2}$

Solution: $x^2 + y^2 = Cy$

Check: $2x + 2yy' = Cy'$

$$y' = \frac{-2x}{(2y - C)}$$

$$y' = \frac{-2xy}{2y^2 - Cy} = \frac{-2xy}{2y^2 - (x^2 + y^2)} = \frac{-2xy}{y^2 - x^2} = \frac{2xy}{x^2 - y^2}$$

4. Differential equation: $y'' + 2y' + 2y = 0$

Solution: $y = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x$

Check:

$$y' = -(C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x$$

$$y'' = 2C_1 e^{-x} \sin x - 2C_2 e^{-x} \cos x$$

$$y'' + 2y' + 2y = 2C_1 e^{-x} \sin x - 2C_2 e^{-x} \cos x +$$

$$2(-(C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x) + 2(C_1 e^{-x} \cos x + C_2 e^{-x} \sin x)$$

$$= (2C_1 - 2C_1 - 2C_2 + 2C_2)e^{-x} \sin x + (-2C_2 - 2C_1 + 2C_2 + 2C_1)e^{-x} \cos x = 0$$

6. $y = \frac{2}{3}(e^{-2x} + e^x)$

$y' = \frac{2}{3}(-2e^{-2x} + e^x)$

$y'' = \frac{2}{3}(4e^{-2x} + e^x)$

Substituting, $y'' + 2y' = \frac{2}{3}(4e^{-2x} + e^x) + 2\left(\frac{2}{3}\right)(-2e^{-2x} + e^x) = 2e^x$.

In Exercises 8–12, the differential equation is $y^{(4)} - 16y = 0$.

8. $y = 3\cos 2x$

$y^{(4)} = 48\cos 2x$

$y^{(4)} - 16y = 48\cos 2x - 48\cos 2x = 0$, Yes.

10. $y = 5\ln x$

$y^{(4)} = -\dfrac{30}{x^4}$

$y^{(4)} - 16y = -\dfrac{30}{x^4} - 80\ln x \neq 0$, No.

12. $y = 3e^{2x} - 4\sin 2x$

$y^{(4)} = 48e^{2x} - 64\sin 2x$

$y^{(4)} - 16y = (24e^{2x} - 64\sin 2x) - 16(3e^{2x} - 4\sin 2x) = 0$, Yes

In 14–18, the differential equation is $xy' - 2y = x^3e^x$.

14. $y = x^2e^x$, $y' = x^2e^x + 2xe^x = e^x(x^2 + 2x)$

$xy' - 2y = x(e^x(x^2 + 2x)) - 2(x^2e^x) = x^3e^x$, Yes.

16. $y = \sin x$, $y' = \cos x$

$xy' - 2y = x(\cos x) - 2(\sin x) \neq x^3e^x$, No.

18. $y = x^2e^x - 5x^2$, $y' = x^2e^x + 2xe^x - 10x$

$xy' - 2y = x[x^2e^x + 2xe^x - 10x] - 2[x^2e^x - 5x^2] = x^3e^x$, Yes.

20. $y = A\sin \omega t$

$\dfrac{d^2y}{dt^2} = -A\omega^2 \sin \omega t$

Since $(d^2y/dt^2) + 16y = 0$, we have

$-A\omega^2 \sin \omega t + 16A \sin \omega t = 0$.

Thus, $\omega^2 = 16$ and $\omega = \pm 4$.

22. $2x^2 - y^2 = C$ passes through $(3, 4)$

$2(9) - 16 = C \Rightarrow C = 2$

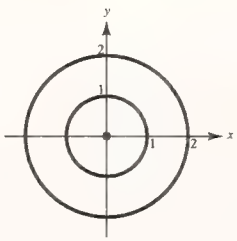
Particular solution: $2x^2 - y^2 = 2$

24. Differential equation: $yy' + x = 0$

General solution: $x^2 + y^2 = C$

Particular solutions: $C = 0$, Point

$C = 1$, $C = 4$, Circles



26. Differential equation: $3x + 2yy' = 0$

General solution: $3x^2 + 2y^2 = C$

$6x + 4yy' = 0$

$2(3x + 2yy') = 0$

$3x + 2yy' = 0$

Initial condition:
$y(1) = 3$: $3(1)^2 + 2(3)^2 = 3 + 18 = 21 = C$

Particular solution: $3x^2 + 2y^2 = 21$

28. Differential equation: $xy'' + y' = 0$

General solution: $y = C_1 + C_2 \ln x$

$y' = C_2\left(\frac{1}{x}\right), y'' = -C_2\left(\frac{1}{x^2}\right)$

$xy'' + y' = x\left(-C_2\frac{1}{x^2}\right) + C_2\frac{1}{x} = 0$

Initial conditions: $y(2) = 0,\ y'(2) = \frac{1}{2}$

$0 = C_1 + C_2 \ln 2$

$y' = \frac{C_2}{x}$

$\frac{1}{2} = \frac{C_2}{2} \Rightarrow C_2 = 1,\ C_1 = -\ln 2$

Particular solution: $y = -\ln 2 + \ln x = \ln\frac{x}{2}$

30. Differential equation: $9y'' - 12y' + 4y = 0$

General solution: $y = e^{2x/3}(C_1 + C_2x)$

$y' = \frac{2}{3}e^{2x/3}(C_1 + C_2x) + C_2e^{2x/3} = e^{2x/3}\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right)$

$y'' = \frac{2}{3}e^{2x/3}\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right) + e^{2x/3}\frac{2}{3}C_2 = \frac{2}{3}e^{2x/3}\left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2x\right)$

$9y'' - 12y' + 4y = 9\left(\frac{2}{3}e^{2x/3}\right)\left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2x\right) - 12(e^{2x/3})\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right) + 4(e^{2x/3})(C_1 + C_2x) = 0$

Initial conditions: $y(0) = 4,\ y(3) = 0$

$0 = e^2(C_1 + 3C_2)$

$4 = (1)(C_1 + 0) \Rightarrow C_1 = 4$

$0 = e^2(4 + 3C_2) \Rightarrow C_2 = -\frac{4}{3}$

Particular solution: $y = e^{2x/3}\left(4 - \frac{4}{3}x\right)$

32. $\frac{dy}{dx} = x^3 - 4x$

$y = \int (x^3 - 4x)\,dx = \frac{x^4}{4} - 2x^2 + C$

34. $\frac{dy}{dx} = \frac{e^x}{1 + e^x}$

$y = \int \frac{e^x}{1 + e^x}\,dx = \ln(1 + e^x) + C$

36. $\frac{dy}{dx} = x\cos x^2$

$y = \int x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2) + C$

$(u = x^2,\ du = 2x\,dx)$

38. $\frac{dy}{dx} = \tan^2 x = \sec^2 x - 1$

$y = \int (\sec^2 x - 1)\,dx = \tan x - x + C$

40. $\frac{dy}{dx} = x\sqrt{5 - x}$. Let $u = \sqrt{5 - x},\ u^2 = 5 - x,\ dx = -2u\,du$

$$y = \int x\sqrt{5 - x}\,dx = \int (5 - u^2)u(-2u)\,du$$

$$= \int (-10u^2 + 2u^4)\,du$$

$$= \frac{-10u^3}{3} + \frac{2u^5}{5} + C$$

$$= -\frac{10}{3}(5 - x)^{3/2} + \frac{2}{5}(5 - x)^{5/2} + C$$

42. $\dfrac{dy}{dx} = 5e^{-x/2}$

$$y = \int 5e^{-x/2}\,dx = 5(-2)\int e^{-x/2}\left(-\frac{1}{2}\right)dx$$
$$= -10e^{-x/2} + C$$

44. $\dfrac{dy}{dx} = \dfrac{x^2+2}{3y^2}$

$$\int 3y^2\,dy = \int (x^2+2)\,dx$$
$$y^3 = \frac{x^3}{3} + 2x + C$$

46. $\dfrac{dr}{ds} = 0.05s$

$$\int dr = \int 0.05s\,ds$$
$$r = 0.025s^2 + C$$

48. $xy' = y$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$
$$\ln y = \ln x + \ln C = \ln Cx$$
$$y = Cx$$

50. $y\dfrac{dy}{dx} = 6\cos \pi x$

$$\int y\,dy = \int 6\cos \pi x\,dx$$
$$\frac{y^2}{2} = \frac{6}{\pi}\sin \pi x + C_1$$
$$y^2 = \frac{12}{\pi}\sin \pi x + C$$

52. $\sqrt{x^2-9}\dfrac{dy}{dx} = 5x$

$$\int dy = \int \frac{5x}{\sqrt{x^2-9}}\,dx$$
$$y = 5(x^2-9)^{1/2} + C$$

54. $4y\dfrac{dy}{dx} = 3e^x$

$$\int 4y\,dy = \int 3e^x\,dx$$
$$2y^2 = 3e^x + C$$

56. $\sqrt{x} + \sqrt{y}\,y' = 0$

$$\int y^{1/2}\,dy = -\int x^{1/2}\,dx$$
$$\frac{2}{3}y^{3/2} = -\frac{2}{3}x^{3/2} + C_1$$
$$y^{3/2} + x^{3/2} = C$$

Initial condition: $y(1) = 4$,
$(4)^{3/2} + (1)^{3/2} = 8 + 1 = 9 = C$

Particular solution: $y^{3/2} + x^{3/2} = 9$

58. $2xy' - \ln x^2 = 0$

$$2x\frac{dy}{dx} = 2\ln x$$
$$\int dy = \int \frac{\ln x}{x}\,dx$$
$$y = \frac{(\ln x)^2}{2} + C$$

$y(1) = 2$: $2 = C$

$$y = \frac{1}{2}(\ln x)^2 + 2$$

60. $y\sqrt{1-x^2}\dfrac{dy}{dx} = x\sqrt{1-y^2}$

$$\int (1-y^2)^{-1/2}y\,dy = \int (1-x^2)^{-1/2}x\,dx$$
$$-(1-y^2)^{1/2} = -(1-x^2)^{1/2} + C$$

$y(0) = 1$: $0 = -1 + C \Rightarrow C = 1$

$$\sqrt{1-y^2} = \sqrt{1-x^2} - 1$$

62. $\dfrac{dr}{ds} = e^{r-2s}$

$$\int e^{-r}dr = \int e^{-2s}\,ds$$
$$-e^{-r} = -\frac{1}{2}e^{-2s} + C$$

$r(0) = 0$: $-1 = -\dfrac{1}{2} + C \Rightarrow C = -\dfrac{1}{2}$

$$-e^{-r} = -\frac{1}{2}e^{-2s} - \frac{1}{2}$$
$$e^{-r} = \frac{1}{2}e^{-2s} + \frac{1}{2}$$
$$-r = \ln\left(\frac{1}{2}e^{-2s} + \frac{1}{2}\right) = \ln\left(\frac{1+e^{-2s}}{2}\right)$$
$$r = \ln\left(\frac{2}{1+e^{-2s}}\right)$$

64. $dT + k(T - 70)\,dt = 0$

$$\int \frac{dT}{T-70} = -k\int dt$$

$$\ln(T-70) = -kt + C_1$$

$$T - 70 = Ce^{-kt}$$

Initial condition: $T(0) = 140$;
$140 - 70 = 70 = Ce^0 = C$

Particular solution: $T - 70 = 70e^{-kt}$, $T = 70(1 + e^{-kt})$

66. $\dfrac{dy}{dx} = \dfrac{2y}{3x}$

$$\int \frac{3}{y}\,dy = \int \frac{2}{x}\,dx$$

$$\ln y^3 = \ln x^2 + \ln C$$

$$y^3 = Cx^2$$

Initial condition: $y(8) = 2$, $2^3 = C(8^2)$, $C = \dfrac{1}{8}$

Particular solution: $8y^3 = x^2$, $y = \dfrac{1}{2}x^{2/3}$

68. $m = \dfrac{dy}{dx} = \dfrac{y-0}{x-0} = \dfrac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + C_1 = \ln x + \ln C = \ln Cx$$

$$y = Cx$$

70. $f(x, y) = x^3 + 3x^2y^2 - 2y^2$

$f(tx, ty) = t^3x^3 + 3t^4x^2y^2 - 2t^2y^2$

Not homogeneous

72. $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{tx\,ty}{\sqrt{t^2x^2 + t^2y^2}}$$

$$= \frac{t^2xy}{t\sqrt{x^2 + y^2}} = t\frac{xy}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 1

74. $f(x, y) = \tan(x + y)$

$f(tx, ty) = \tan(tx + ty) = \tan[t(x + y)]$

Not homogeneous

76. $f(x, y) = \tan\dfrac{y}{x}$

$$f(tx, ty) = \tan\frac{ty}{tx} = \tan\frac{y}{x}$$

Homogeneous of degree 0

78. $y' = \dfrac{(x^3 + y^3)}{xy^2}$

$xy^2\,dy = (x^3 + y^3)\,dx$

$y = vx$, $dy = x\,dv + v\,dx$

$x(vx)^2(x\,dv + v\,dx) = (x^3 + (vx)^3)\,dx$

$x^4v^2\,dv + x^3v^3\,dx = x^3\,dx + v^3x^3\,dx$

$xv^2\,dv = dx$

$$\int v^2\,dv = \int \frac{1}{x}\,dx$$

$$\frac{v^3}{3} = \ln|x| + C$$

$$\left(\frac{y}{x}\right)^3 = 3\ln|x| + C$$

$$y^3 = 3x^3\ln|x| + Cx^3$$

80. $y' = \dfrac{x^2 + y^2}{2xy}$, $y = vx$

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x^2v}$$

$$2v\,dx + 2x\,dv = \frac{1 + v^2}{v}\,dx$$

$$\int \frac{2v}{v^2 - 1}\,dv = -\int \frac{dx}{x}$$

$$\ln(v^2 - 1) = -\ln x + \ln C = \ln\frac{C}{x}$$

$$v^2 - 1 = \frac{C}{x}$$

$$\frac{y^2}{x^2} - 1 = \frac{C}{x}$$

$$y^2 - x^2 = Cx$$

82. $y' = \dfrac{2x + 3y}{x}, y = vx$

$$v + x\frac{dv}{dx} = \frac{2x + 3vx}{x} = 2 + 3v$$

$$x\frac{dv}{dx} = 2 + 2v \Rightarrow \int \frac{dv}{1 + v} = 2\int \frac{dx}{x}$$

$$\ln|1 + v| = \ln x^2 + \ln C = \ln x^2 C$$

$$1 + v = x^2 C$$

$$1 + \frac{y}{x} = x^2 C$$

$$\frac{y}{x} = x^2 C - 1$$

$$y = Cx^3 - x$$

84. $-y^2\,dx + x(x + y)\,dy = 0, y = vx$

$$-x^2 v^2\,dx + (x^2 + x^2 v)(v\,dx + x\,dv) = 0$$

$$\int \frac{1 + v}{v}\,dv = -\int \frac{dx}{x}$$

$$v + \ln v = -\ln x + \ln C_1 = \ln \frac{C_1}{x}$$

$$v = \ln \frac{C_1}{xv}$$

$$\frac{C_1}{vx} = e^v$$

$$\frac{C_1}{y} = e^{y/x}$$

$$y = Ce^{-y/x}$$

Initial condition: $y(1) = 1, 1 = Ce^{-1} \Rightarrow C = e$

Particular solution: $y = e^{1-y/x}$

86. $(2x^2 + y^2)\,dx + xy\,dy = 0$

Let $y = vx, dy = x\,dv + v\,dx$.

$$(2x^2 + v^2x^2)\,dx + x(vx)(x\,dv + v\,dx) = 0$$

$$(2x^2 + 2x^2v^2)\,dx + x^3v\,dv = 0$$

$$(2 + 2v^2)\,dx = -xv\,dv$$

$$\frac{-2}{x}\,dx = \frac{v}{1 + v^2}\,dv$$

$$-2\ln x = \frac{1}{2}\ln(1 + v^2) + C_1$$

$$\ln x^{-2} = \ln(1 + v^2)^{1/2} + \ln C$$

$$x^{-2} = C(1 + v^2)^{1/2}$$

$$\frac{1}{x^2} = C\left(1 + \frac{y^2}{x^2}\right)^{1/2} = \frac{C}{x}(x^2 + y^2)^{1/2}$$

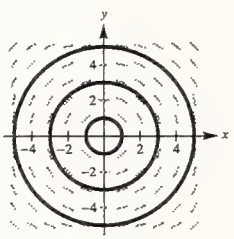
$$\frac{1}{x} = C(x^2 + y^2)^{1/2}$$

$y(1) = 0$: $1 = C(1 + 0) \Rightarrow C = 1$

$$\frac{1}{x} = \sqrt{x^2 + y^2}$$

$$1 = x\sqrt{x^2 + y^2}$$

88. $\dfrac{dy}{dx} = -\dfrac{x}{y}$

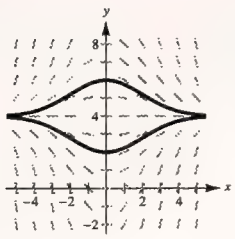


$$y\,dy = -x\,dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C_1$$

$$y^2 + x^2 = C$$

90. $\dfrac{dy}{dx} = 0.25x(4 - y)$



$$\frac{dy}{4 - y} = 0.25x\,dx$$

$$\int \frac{dy}{y - 4} = \int -0.25x\,dx$$

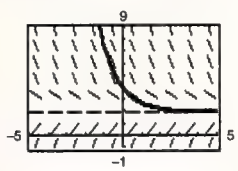
$$= = -\frac{1}{4}\int x\,dx$$

$$\ln|y - 4| = -\frac{1}{8}x^2 + C_1$$

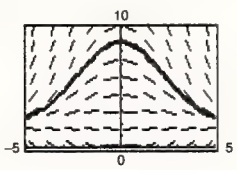
$$y - 4 = e^{C_1 - (1/8)x^2} = Ce^{-(1/8)x^2}$$

$$y = 4 + Ce^{-(1/8)x^2}$$

92. $\dfrac{dy}{dx} = 2 - y, y(0) = 4$



94. $\dfrac{dy}{dx} = 0.2x(2 - y), y(0) = 9$



96. $\dfrac{dy}{dt} = ky, y = Ce^{kt}$

Initial conditions: $y(0) = 20, y(1) = 16$

$$20 = Ce^0 = C$$

$$16 = 20e^k$$

$$k = \ln\frac{4}{5}$$

Particular solution: $y = 20e^{t\ln(4/5)}$

When 75% has been changed:

$$5 = 20e^{t\ln(4/5)}$$

$$\frac{1}{4} = e^{t\ln(4/5)}$$

$$t = \frac{\ln(1/4)}{\ln(4/5)} \approx 6.2 \text{ hr}$$

98. $\dfrac{dy}{dx} = k(x - 4)$

The direction field satisfies $(dy/dx) = 0$ along $x = 4$: Matches (b).

100. $\dfrac{dy}{dx} = ky^2$

The direction field satisfies $(dy/dx) = 0$ along $y = 0$, and grows more positive as y increases. Matches (d).

102. From Exercise 101,

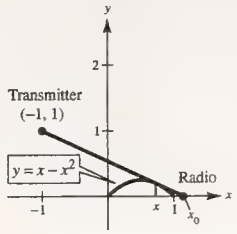
$$w = 1200 - Ce^{-kt}, k = 1$$

$$w = 1200 - Ce^{-t}$$

$$w(0) = w_0 = 1200 - C \Rightarrow C = 1200 - w_0$$

$$w = 1200 - (1200 - w_0)e^{-t}$$

104. Let the radio receiver be located at $(x_0, 0)$. The tangent line to $y = x - x^2$ joins $(-1, 1)$ and $(x_0, 0)$.



(a) If (x, y) is the point of tangency on the $y = x - x^2$, then

$$1 - 2x = \frac{y - 1}{x + 1} = \frac{x - x^2 - 1}{x + 1}$$

$$x - 2x^2 + 1 - 2x = x - x^2 - 1$$

$$x^2 + 2x - 2 = 0$$

$$x = \left(\frac{-2 \pm \sqrt{4 + 8}}{2}\right) = -1 + \sqrt{3}$$

$$y = x - x^2 = 3\sqrt{3} - 5$$

Then $\dfrac{1 - 0}{-1 - x_0} = \dfrac{1 - 3\sqrt{3} + 5}{-1 + 1 - \sqrt{3}} = \dfrac{6 - 3\sqrt{3}}{-\sqrt{3}}$

$$\sqrt{3} = (1 + x_0)(6 - 3\sqrt{3})$$

$$= 6 - 3\sqrt{3} + x_0(6 - 3\sqrt{3})$$

$$x_0 = \frac{4\sqrt{3} - 6}{6 - 3\sqrt{3}} \approx 1.155$$

(b) Now let the transmitter be located at $(-1, h)$.

$$1 - 2x = \frac{y - h}{x + 1} = \frac{x - x^2 - h}{x + 1}$$

$$x - 2x^2 + 1 - 2x = x - x^2 - h$$

$$x^2 + 2x - h - 1 = 0$$

$$x = \frac{\left(-2 \pm \sqrt{4 + 4(h + 1)}\right)}{2}$$

$$= -1 + \sqrt{2 + h}$$

$$y = x - x^2$$

$$= 3\sqrt{2 + h} - h - 4$$

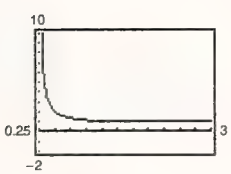
Then, $\dfrac{h - 0}{-1 - x_0} = \dfrac{h - \left(3\sqrt{2 + h} - h - 4\right)}{-1 - \left(-1 + \sqrt{2 + h}\right)}$

$$= \frac{2h + 4 - 3\sqrt{2 + h}}{-\sqrt{2 + h}}$$

$$\frac{x_0 + 1}{h} = \frac{\sqrt{2 + h}}{2h + 4 - 3\sqrt{2 + h}}$$

$$x_0 = \frac{h\sqrt{2 + h}}{2h + 4 - 3\sqrt{2 + h}} - 1$$

(c)



There is a vertical asymptote at $h = \frac{1}{4}$, which is the height of the mountain.

106. Given family (hyperbolas): $x^2 - 2y^2 = C$

$$2x - 4yy' = 0$$

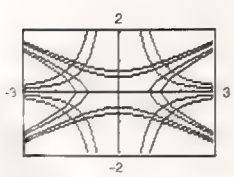
$$y' = \frac{x}{2y}$$

Orthogonal trajectory: $y' = \dfrac{-2y}{x}$

$$\int \frac{dy}{y} = -\int \frac{2}{x}\,dx$$

$$\ln y = -2 \ln x + \ln k$$

$$y = kx^{-2} = \frac{k}{x^2}$$



108. Given family (parabolas): $y^2 = 2Cx$

$$2yy' = 2C$$

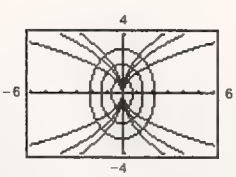
$$y' = \frac{C}{y} = \frac{y^2}{2x}\left(\frac{1}{y}\right) = \frac{y}{2x}$$

Orthogonal trajectory (ellipse): $y' = -\dfrac{2x}{y}$

$$\int y\,dy = -\int 2x\,dx$$

$$\frac{y^2}{2} = -x^2 + K_1$$

$$2x^2 + y^2 = K$$



110. Given family (exponential functions): $y = Ce^x$

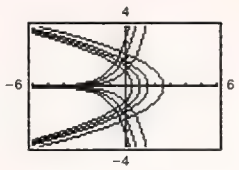
$$y' = Ce^x = y$$

Orthogonal trajectory (parabolas): $y' = -\frac{1}{y}$

$$\int y\,dy = -\int dx$$

$$\frac{y^2}{2} = -x + K_1$$

$$y^2 = -2x + K$$



112. The number of initial conditions matches the number of constants in the general solution.

114. Two families of curves are mutually orthogonal if each curve in the first family intersects each curve in the second family at right angles.

116. True

$$\frac{dy}{dx} = (x - 2)(y + 1)$$

118. True

$$x^2 + y^2 = 2Cy \qquad x^2 + y^2 = 2Kx$$

$$\frac{dy}{dx} = \frac{x}{C - y} \qquad \frac{dy}{dx} = \frac{K - x}{y}$$

$$\frac{x}{C - y} \cdot \frac{K - x}{y} = \frac{Kx - x^2}{Cy - y^2} = \frac{2Kx - 2x^2}{2Cy - 2y^2} = \frac{x^2 + y^2 - 2x^2}{x^2 + y^2 - 2y^2} = \frac{y^2 - x^2}{x^2 - y^2} = -1$$

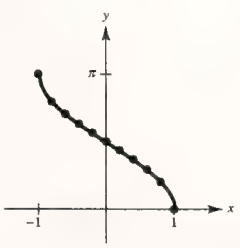
Section 5.8 Inverse Trigonometric Functions: Differentiation

2. $y = \arccos x$

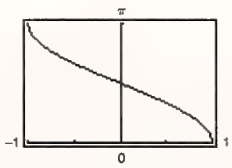
(a)

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y	3.142	2.499	2.214	1.982	1.772	1.571	1.369	1.159	0.927	0.634	0

(b)



(c)



(d) Intercepts: $\left(0, \frac{\pi}{2}\right)$ and $(1, 0)$

No symmetry

4. $\left(\_\_, \frac{\pi}{4}\right) = \left(1, \frac{\pi}{4}\right)$

$\left(\_\_, -\frac{\pi}{6}\right) = \left(-\frac{\sqrt{3}}{3}, -\frac{\pi}{6}\right)$

$\left(-\sqrt{3}, \_\_\right) = \left(-\sqrt{3}, -\frac{\pi}{3}\right)$

6. $\arcsin 0 = 0$

8. $\arccos 0 = \dfrac{\pi}{2}$

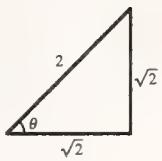
10. $\operatorname{arc\,cot}\left(-\sqrt{3}\right) = \dfrac{5\pi}{6}$

12. $\arccos\left(-\dfrac{\sqrt{3}}{2}\right) = \dfrac{5\pi}{6}$

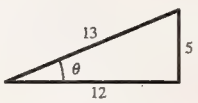
14. $\arcsin(-0.39) \approx -0.40$

16. $\arctan(-3) \approx -1.25$

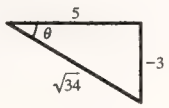
18. (a) $\tan\left(\arccos\dfrac{\sqrt{2}}{2}\right) = \tan\left(\dfrac{\pi}{4}\right) = 1$



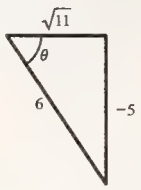
(b) $\cos\left(\arcsin\dfrac{5}{13}\right) = \dfrac{12}{13}$



20. (a) $\sec\left[\arctan\left(-\dfrac{3}{5}\right)\right] = \dfrac{\sqrt{34}}{5}$



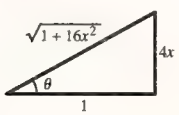
(b) $\tan\left[\arcsin\left(-\dfrac{5}{6}\right)\right] = -\dfrac{5\sqrt{11}}{11}$



22. $y = \sec(\arctan 4x)$

$\theta = \arctan 4x$

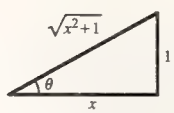
$y = \sec\theta = \sqrt{1 + 16x^2}$



24. $y = \cos(\operatorname{arccot} x)$

$\theta = \operatorname{arccot} x$

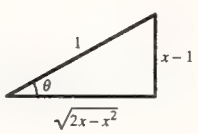
$y = \cos\theta = \dfrac{x}{\sqrt{x^2 + 1}}$



26. $y = \sec[\arcsin(x - 1)]$

$\theta = \arcsin(x - 1)$

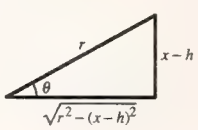
$y = \sec\theta = \dfrac{1}{\sqrt{2x - x^2}}$



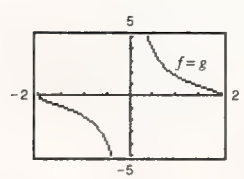
28. $y = \cos\left(\arcsin\dfrac{x - h}{r}\right)$

$\theta = \arcsin\dfrac{x - h}{r}$

$y = \cos\theta = \dfrac{\sqrt{r^2 - (x - h)^2}}{r}$



30.

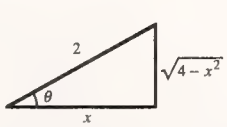


Asymptote: $x = 0$

$\arccos\dfrac{x}{2} = \theta$

$\cos\theta = \dfrac{x}{2}$

$\tan\theta = \dfrac{\sqrt{4 - x^2}}{x}$



32. $\arctan(2x - 5) = -1$

$2x - 5 = \tan(-1)$

$x = \dfrac{1}{2}(\tan(-1) + 5) \approx 1.721$

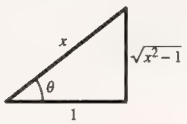
34. $\arccos x = \operatorname{arcsec} x$

$x = \cos(\operatorname{arcsec} x)$

$x = \dfrac{1}{x}$

$x^2 = 1$

$x = \pm 1$



36. (a) $\arcsin(-x) = -\arcsin x$, $|x| \le 1$.

Let $y = \arcsin(-x)$. Then,

$-x = \sin y \implies x = -\sin y \implies x = \sin(-y)$.

Thus, $-y = \arcsin x \implies y = -\arcsin x$. Therefore, $\arcsin(-x) = -\arcsin x$.

(b) $\arccos(-x) = \pi - \arccos x$, $|x| \le 1$.

Let $y = \arccos(-x)$. Then,

$-x = \cos y \implies x = -\cos y \implies x = \cos(\pi - y)$.

Thus, $\pi - y = \arccos x \implies y = \pi - \arccos x$. Therefore, $\arccos(-x) = \pi - \arccos x$.

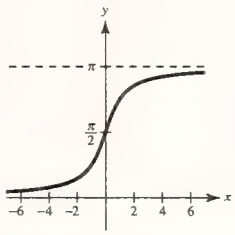
38. $f(x) = \arctan x + \dfrac{\pi}{2}$

$x = \tan\left(y - \dfrac{\pi}{2}\right)$

Domain: $(-\infty, \infty)$

Range: $(0, \pi)$

$f(x)$ is the graph of $\arctan x$ shifted $\pi/2$ units upward.



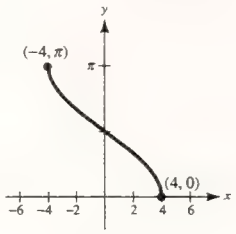
40. $f(x) = \arccos\left(\dfrac{x}{4}\right)$

$\dfrac{x}{4} = \cos y$

$x = 4\cos y$

Domain: $[-4, 4]$

Range: $[0, \pi]$



42. $f(t) = \arcsin t^2$

$f'(t) = \dfrac{2t}{\sqrt{1 - t^4}}$

44. $f(x) = \operatorname{arcsec} 2x$

$f'(x) = \dfrac{2}{|2x|\sqrt{4x^2 - 1}} = \dfrac{1}{|x|\sqrt{4x^2 - 1}}$

46. $f(x) = \arctan \sqrt{x}$

$f'(x) = \left(\dfrac{1}{1 + x}\right)\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{1}{2\sqrt{x}(1 + x)}$

48. $h(x) = x^2 \arctan x$

$h'(x) = 2x \arctan x + \dfrac{x^2}{1 + x^2}$

50. $f(x) = \arcsin x + \arccos x = \dfrac{\pi}{2}$

$f'(x) = 0$

52. $y = \ln(t^2 + 4) - \dfrac{1}{2}\arctan\dfrac{t}{2}$

$y' = \dfrac{2t}{t^2 + 4} - \dfrac{1}{2} \cdot \dfrac{1}{1 + \left(\dfrac{t}{2}\right)^2}\left(\dfrac{1}{2}\right)$

$= \dfrac{2t}{t^2 + 4} - \dfrac{1}{t^2 + 4} = \dfrac{2t - 1}{t^2 + 4}$

54. $y = \dfrac{1}{2}\left[x\sqrt{4 - x^2} + 4\arcsin\left(\dfrac{x}{2}\right)\right]$

$y' = \dfrac{1}{2}\left[x\dfrac{1}{2}(4 - x^2)^{-1/2}(-2x) + \sqrt{4 - x^2} + 2\dfrac{1}{\sqrt{1 - (x/2)^2}}\right]$

$= \dfrac{1}{2}\left[\dfrac{-x^2}{\sqrt{4 - x^2}} + \sqrt{4 - x^2} + \dfrac{4}{\sqrt{4 - x^2}}\right]$

$= \sqrt{4 - x^2}$

56. $y = x\arctan 2x - \dfrac{1}{4}\ln(1 + 4x^2)$

$\dfrac{dy}{dx} = \dfrac{2x}{1 + 4x^2} + \arctan(2x) - \dfrac{1}{4}\left(\dfrac{8x}{1 + 4x^2}\right) = \arctan(2x)$

58. $y = 25 \arcsin\frac{x}{5} - x\sqrt{25 - x^2}$

$$y' = 5\frac{1}{\sqrt{1 - (x/2)^2}} - \sqrt{25 - x^2} - x\frac{1}{2}(25 - x^2)^{-1/2}(-2x)$$

$$= \frac{25}{\sqrt{25 - x^2}} - \frac{(25 - x^2)}{\sqrt{25 - x^2}} + \frac{x^2}{\sqrt{25 - x^2}}$$

$$= \frac{2x^2}{\sqrt{25 - x^2}}$$

60. $y = \arctan\frac{x}{2} - \frac{1}{2(x^2 + 4)}$

$$y' = \frac{1}{2}\frac{1}{1 + (x/2)^2} + \frac{1}{2}(x^2 + 4)^{-2}(2x)$$

$$= \frac{2}{x^2 + 4} + \frac{x}{(x^2 + 4)^2}$$

$$= \frac{2x^2 + 8 + x}{(x^2 + 4)^2}$$

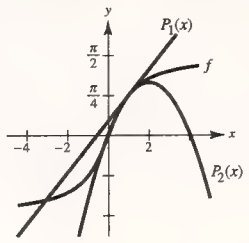
62. $f(x) = \arctan x$, $a = 1$

$$f'(x) = \frac{1}{1 + x^2}$$

$$f''(x) = \frac{-2x}{(1 + x^2)^2}$$

$$P_1(x) = f(1) + f'(1)(x - 1) = \frac{\pi}{4} + \frac{1}{2}(x - 1)$$

$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2}f''(1)(x - 1)^2 = \frac{\pi}{4} + \frac{1}{2}(x - 1) - \frac{1}{4}(x - 1)^2$$



64. $f(x) = \arcsin x - 2x$

$f'(x) = \frac{1}{\sqrt{1 - x^2}} - 2 = 0$ when $\sqrt{1 - x^2} = \frac{1}{2}$ or $x = \pm\frac{\sqrt{3}}{2}$.

$$f''(x) = \frac{x}{(1 - x^2)^{3/2}}$$

$f''\left(\frac{\sqrt{3}}{2}\right) > 0$

Relative minimum: $\left(\frac{\sqrt{3}}{2}, -0.68\right)$

$f''\left(-\frac{\sqrt{3}}{2}\right) < 0$

Relative maximum: $\left(-\frac{\sqrt{3}}{2}, 0.68\right)$

66. $f(x) = \arcsin x - 2\arctan x$

$$f'(x) = \frac{1}{\sqrt{1 - x^2}} - \frac{2}{1 + x^2} = 0$$

$$1 + x^2 = 2\sqrt{1 - x^2}$$

$$1 + 2x^2 + x^4 = 4(1 - x^2)$$

$$x^4 + 6x^2 - 3 = 0$$

$$x = \pm 0.681$$

By the First Derivative Test, $(-0.681, 0.447)$ is a relative maximum and $(0.681, -0.447)$ is a relative minimum.

68. $\arctan 0 = 0$. π is not in the range of $y = \arctan x$.

70. The derivatives are algebraic. See Theorem 5.18.

72. (a) $\cot\theta = \frac{x}{3}$

$$\theta = \operatorname{arccot}\left(\frac{x}{3}\right)$$

(b) $\frac{d\theta}{dt} = \frac{-3}{x^2 + 9}\frac{dx}{dt}$

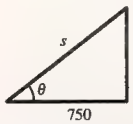
If $x = 10$, $\frac{d\theta}{dt} \approx 11.001$ rad/hr.

If $x = 3$, $\frac{d\theta}{dt} \approx 66.667$ rad/hr.

A lower altitude results in a greater rate of change of θ.

74. $\cos\theta = \frac{750}{s}$

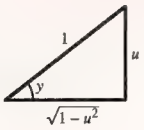
$$\theta = \arccos\left(\frac{750}{s}\right)$$



$$\frac{d\theta}{dt} = \frac{d\theta}{ds} \cdot \frac{ds}{dt} = \frac{-1}{\sqrt{1 - (750/s)^2}}\left(\frac{-750}{s^2}\right)\frac{ds}{dt}$$

$$= \frac{750}{s\sqrt{s^2 - 750^2}}\frac{ds}{dt}$$

76. (a) Let $y = \arcsin u$. Then

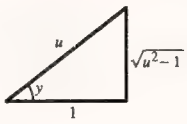


$$\sin y = u$$

$$\cos y \cdot y' = u'$$

$$\frac{dy}{dx} = \frac{u'}{\cos y} = \frac{u'}{\sqrt{1-u^2}}.$$

(c) Let $y = \operatorname{arcsec} u$. Then



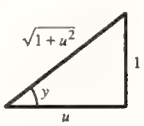
$$\sec y = u$$

$$\sec y \tan y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{\sec y \tan y} = \frac{u'}{|u|\sqrt{u^2-1}}.$$

Note: The absolute value sign in the formula for the derivative of arcsec u is necessary because the inverse secant function has a positive slope at every value in its domain.

(e) Let $y = \operatorname{arccot} u$. Then

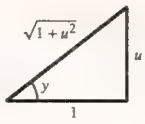


$$\cot y = u$$

$$-\csc^2 y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{-\csc^2 y} = -\frac{u'}{1+u^2}.$$

(b) Let $y = \arctan u$. Then

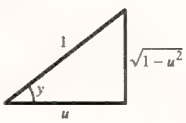


$$\tan y = u$$

$$\sec^2 y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{\sec^2 y} = \frac{u'}{1+u^2}.$$

(d) Let $y = \arccos u$. Then

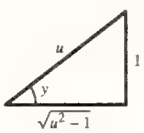


$$\cos y = u$$

$$-\sin y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = -\frac{u'}{\sin y} = -\frac{u'}{\sqrt{1-u^2}}.$$

(f) Let $y = \operatorname{arccsc} u$. Then



$$\csc y = u$$

$$-\csc y \cot y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{-\csc y \cot y} = -\frac{u'}{|u|\sqrt{u^2-1}}$$

Note: The absolute value sign in the formula for the derivative of arccsc u is necessary because the inverse cosecant function has a negative slope at every value in its domain.

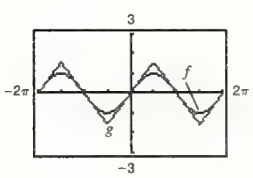
78. $f(x) = \sin x$

$g(x) = \arcsin(\sin x)$

(a) The range of $y = \arcsin x$ is $-\pi/2 \le y \le \pi/2$.

(b) Maximum: $\pi/2$

Minimum: $-\pi/2$



80. False

The range of $y = \arcsin x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

82. False

$$\arcsin^2 0 + \arccos^2 0 = 0 + \left(\frac{\pi}{2}\right)^2 \neq 1$$

Section 5.9 Inverse Trigonometric Functions: Integration

2. $\displaystyle\int \frac{3}{\sqrt{1-4x^2}}\,dx = \frac{3}{2}\int \frac{2}{\sqrt{1-4x^2}}\,dx = \frac{3}{2}\arcsin(2x) + C$

4. $\displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\arcsin\frac{x}{2}\right]_0^1 = \frac{\pi}{6}$

6. $\displaystyle\int \frac{4}{1+9x^2}\,dx = \frac{4}{3}\int \frac{3}{1+9x^2}\,dx = \frac{4}{3}\arctan(3x) + C$

8. $\displaystyle\int_{\sqrt{3}}^3 \frac{1}{9+x^2}\,dx = \left[\frac{1}{3}\arctan\frac{x}{3}\right]_{\sqrt{3}}^3 = \frac{\pi}{36}$

10. $\displaystyle\int \frac{1}{4+(x-1)^2}\,dx = \frac{1}{2}\arctan\left(\frac{x-1}{2}\right) + C$

12. $\displaystyle\int \frac{x^4-1}{x^2+1}\,dx = \int (x^2-1)\,dx = \frac{1}{3}x^3 - x + C$

14. Let $u = t^2$, $du = 2t\,dt$.

$$\int \frac{t}{t^4 + 16}\,dt = \frac{1}{2}\int \frac{1}{(4)^2 + (t^2)^2}(2t)\,dt = \frac{1}{8}\arctan\frac{t^2}{4} + C$$

16. Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{1}{x\sqrt{x^4 - 4}}\,dx = \frac{1}{2}\int \frac{1}{x^2\sqrt{(x^2)^2 - 2^2}}(2x)\,dx$$
$$= \frac{1}{4}\operatorname{arcsec}\frac{x^2}{2} + C$$

18. Let $u = \arccos x$, $du = -\dfrac{1}{\sqrt{1 - x^2}}\,dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1 - x^2}}\,dx = -\int_0^{1/\sqrt{2}} \frac{-\arccos x}{\sqrt{1 - x^2}}\,dx$$
$$= \left[-\frac{1}{2}\arccos^2 x\right]_0^{1/\sqrt{2}} = \frac{3\pi^2}{32} \approx 0.925$$

20. Let $u = 1 + x^2$, $du = 2x\,dx$.

$$\int_{-\sqrt{3}}^{0} \frac{x}{1 + x^2}\,dx = \frac{1}{2}\int_{-\sqrt{3}}^{0} \frac{1}{1 + x^2}(2x)\,dx$$
$$= \left[\frac{1}{2}\ln(1 + x^2)\right]_{-\sqrt{3}}^{0} = -\ln 2$$

22. $$\int_1^2 \frac{1}{3 + (x - 2)^2}\,dx = \int_1^2 \frac{1}{(\sqrt{3})^2 + (x - 2)^2}\,dx = \left[\frac{1}{\sqrt{3}}\arctan\left(\frac{x - 2}{\sqrt{3}}\right)\right]_1^2 = \frac{\sqrt{3}\pi}{18}$$

24. $$\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\,dx = \arctan(\sin x)\Big]_0^{\pi/2} = \frac{\pi}{4}$$

26. $$\int \frac{3}{2\sqrt{x}(1 + x)}\,dx.\ u = \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx,\ dx = 2u\,du$$
$$\frac{3}{2}\int \frac{2u\,du}{u(1 + u^2)} = 3\int \frac{du}{1 + u^2} = 3\arctan u + C$$
$$= 3\arctan\sqrt{x} + C$$

28. $$\int \frac{4x + 3}{\sqrt{1 - x^2}}\,dx = (-2)\int \frac{-2x}{\sqrt{1 - x^2}}\,dx + 3\int \frac{1}{\sqrt{1 - x^2}}\,dx = -4\sqrt{1 - x^2} + 3\arcsin x + C$$

30. $$\int \frac{x - 2}{(x + 1)^2 + 4}\,dx = \frac{1}{2}\int \frac{2x + 2}{(x + 1)^2 + 4}\,dx - \int \frac{3}{(x + 1)^2 + 4}\,dx$$
$$= \frac{1}{2}\ln(x^2 + 2x + 5) - \frac{3}{2}\arctan\left(\frac{x + 1}{2}\right) + C$$

32. $$\int_{-2}^{2} \frac{dx}{x^2 + 4x + 13} = \int_{-2}^{2} \frac{dx}{(x + 2)^2 + 9} = \left[\frac{1}{3}\arctan\left(\frac{x + 2}{3}\right)\right]_{-2}^{2} = \frac{1}{3}\arctan\left(\frac{4}{3}\right)$$

34. $$\int \frac{2x - 5}{x^2 + 2x + 2}\,dx = \int \frac{2x + 2}{x^2 + 2x + 2}\,dx - 7\int \frac{1}{1 + (x + 1)^2}\,dx = \ln|x^2 + 2x + 2| - 7\arctan(x + 1) + C$$

36. $$\int \frac{2}{\sqrt{-x^2 + 4x}}\,dx = \int \frac{2}{\sqrt{4 - (x^2 - 4x + 4)}}\,dx$$
$$= \int \frac{2}{\sqrt{4 - (x - 2)^2}}\,dx$$
$$= 2\arcsin\left(\frac{x - 2}{2}\right) + C$$

38. Let $u = x^2 - 2x$, $du = (2x - 2)\,dx$.

$$\int \frac{x - 1}{\sqrt{x^2 - 2x}}\,dx = \frac{1}{2}\int (x^2 - 2x)^{-1/2}(2x - 2)\,dx$$
$$= \sqrt{x^2 - 2x} + C$$

40. $$\int \frac{1}{(x - 1)\sqrt{x^2 - 2x}}\,dx = \int \frac{1}{(x - 1)\sqrt{(x - 1)^2 - 1}}\,dx = \operatorname{arcsec}|x - 1| + C$$

42. Let $u = x^2 - 4$, $du = 2x\,dx$.

$$\int \frac{x}{\sqrt{9 + 8x^2 - x^4}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{25 - (x^2 - 4)^2}}\,dx = \frac{1}{2}\arcsin\left(\frac{x^2 - 4}{5}\right) + C$$

44. Let $u = \sqrt{x - 2}$, $u^2 + 2 = x$, $2u\,du = dx$

$$\int \frac{\sqrt{x - 2}}{x + 1}\,dx = \int \frac{2u^2}{u^2 + 3}\,du = \int \frac{2u^2 + 6 - 6}{u^2 + 3}\,du = 2\int du - 6\int \frac{1}{u^2 + 3}\,du$$

$$= 2u - \frac{6}{\sqrt{3}}\arctan\frac{u}{\sqrt{3}} + C = 2\sqrt{x - 2} - 2\sqrt{3}\arctan\sqrt{\frac{x - 2}{3}} + C$$

46. The term is $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$: $x^2 + 3x = x^2 + 3x + \frac{9}{4} - \frac{9}{4} = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4}$

48. (a) $\int e^{x^2}\,dx$ cannot be evaluated using the basic integration rules.

(b) $\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$, $u = x^2$

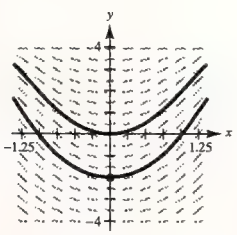
(c) $\int \frac{1}{x^2}e^{1/x}\,dx = -e^{1/x} + C$, $u = \frac{1}{x}$

50. (a) $\int \frac{1}{1 + x^4}\,dx$ cannot be evaluated using the basic integration rules.

(b) $\int \frac{x}{1 + x^4}\,dx = \frac{1}{2}\int \frac{2x}{1 + (x^2)^2}\,dx = \frac{1}{2}\arctan(x^2) + C$, $u = x^2$

(c) $\int \frac{x^3}{1 + x^4}\,dx = \frac{1}{4}\int \frac{4x^3}{1 + x^4}\,dx = \frac{1}{4}\ln(1 + x^4) + C$, $u = 1 + x^4$

52. (a)



(b) $\frac{dy}{dx} = x\sqrt{16 - y^2}$, $(0, -2)$

$$\frac{dy}{\sqrt{16 - y^2}} = x\,dx$$

$$\arcsin\left(\frac{y}{4}\right) = \frac{x^2}{2} + C$$

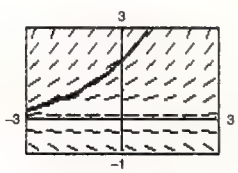
$(0, -2)$: $\arcsin\left(-\frac{2}{4}\right) = C \Rightarrow C = -\frac{\pi}{6}$

$$\arcsin\left(\frac{y}{4}\right) = \frac{x^2}{2} - \frac{\pi}{6}$$

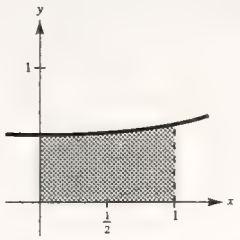
$$\frac{y}{4} = \sin\left(\frac{x^2}{2} - \frac{\pi}{6}\right)$$

$$y = 4\sin\left(\frac{x^2}{2} - \frac{\pi}{6}\right)$$

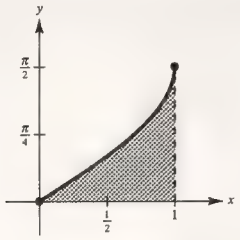
54. $\frac{dy}{dx} = \frac{2y}{\sqrt{16 - x^2}}$, $y(0) = 2$



56. $A = \int_0^1 \frac{1}{\sqrt{4 - x^2}}\,dx = \left[\arcsin\frac{x}{2}\right]_0^1 = \frac{\pi}{6}$



58. $\int_0^1 \arcsin x\,dx \approx 0.571$



60. $F(x) = \frac{1}{2}\int_x^{x+2} \frac{2}{t^2 + 1}\,dt$

(a) $F(x)$ represents the average value of $f(x)$ over the interval $[x, x + 2]$. Maximum at $x = -1$, since the graph is greatest on $[-1, 1]$.

(b) $F(x) = \left[\arctan t\right]_x^{x+2} = \arctan(x + 2) - \arctan x$

$$F'(x) = \frac{1}{1 + (x + 2)^2} - \frac{1}{1 + x^2} = \frac{(1 + x^2) - (x^2 + 4x + 5)}{(x^2 + 1)(x^2 + 4x + 5)} = \frac{-4(x + 1)}{(x^2 + 1)(x^2 + 4x + 5)} = 0 \text{ when } x = -1.$$

62. $\int \frac{1}{\sqrt{6x - x^2}}\,dx$

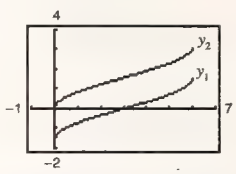
(a) $6x - x^2 = 9 - (x^2 - 6x + 9) = 9 - (x - 3)^2$

$$\int \frac{1}{\sqrt{6x - x^2}}\,dx = \int \frac{dx}{\sqrt{9 - (x - 3)^2}} = \arcsin\left(\frac{x - 3}{3}\right) + C$$

(b) $u = \sqrt{x}, u^2 = x, 2u\,du = dx$

$$\int \frac{1}{\sqrt{6u^2 - u^4}}(2u\,du) = \int \frac{2}{\sqrt{6 - u^2}}\,du = 2\arcsin\left(\frac{u}{\sqrt{6}}\right) + C = 2\arcsin\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$

(c)



The antiderivatives differ by a constant, $\pi/2$.

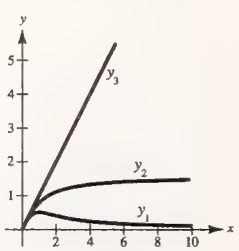
Domain: $[0, 6]$

64. Let $f(x) = \arctan x - \frac{x}{1 + x^2}$

$$f'(x) = \frac{1}{1 + x^2} - \frac{1 - x^2}{(1 + x^2)^2} = \frac{2x^2}{(1 + x^2)} > 0 \text{ for } x > 0.$$

Since $f(0) = 0$ and f is increasing for $x > 0$, $\arctan x - \frac{x}{1 + x^2} > 0$ for $x > 0$. Thus,

$$\arctan x > \frac{x}{1 + x^2}.$$



Let $g(x) = x - \arctan x$

$$g'(x) = 1 - \frac{1}{1 + x^2} = \frac{x^2}{1 + x^2} > 0 \text{ for } x > 0.$$

Since $g(0) = 0$ and g is increasing for $x > 0$, $x - \arctan x > 0$ for $x > 0$. Thus, $x > \arctan x$.

Therefore,

$$\frac{x}{1 + x^2} < \arctan x < x.$$

Section 5.10 Hyperbolic Functions

2. (a) $\cosh(0) = \dfrac{e^0 + e^0}{2} = 1$

(b) $\operatorname{sech}(1) = \dfrac{2}{e + e^{-1}} \approx 0.648$

4. (a) $\sinh^{-1}(0) = 0$

(b) $\tanh^{-1}(0) = 0$

6. (a) $\operatorname{csch}^{-1}(2) = \ln\left(\dfrac{1 + \sqrt{5}}{2}\right) \approx 0.481$

(b) $\coth^{-1}(3) = \dfrac{1}{2}\ln\left(\dfrac{4}{2}\right) \approx 0.347$

8. $\dfrac{1 + \cosh 2x}{2} = \dfrac{1 + (e^{2x} + e^{-2x})/2}{2} = \dfrac{e^{2x} + 2 + e^{-2x}}{4} = \left(\dfrac{e^x + e^{-x}}{2}\right)^2 = \cosh^2 x$

10. $2 \sinh x \cosh x = 2\left(\dfrac{e^x - e^{-x}}{2}\right)\left(\dfrac{e^x + e^{-x}}{2}\right) = \dfrac{e^{2x} - e^{-2x}}{2} = \sinh 2x$

12. $2\cosh\left(\dfrac{x+y}{2}\right)\cosh\left(\dfrac{x-y}{2}\right) = 2\left[\dfrac{e^{(x+y)/2} + e^{-(x+y)/2}}{2}\right]\left[\dfrac{e^{(x-y)/2} + e^{-(x-y)/2}}{2}\right]$

$= 2\left[\dfrac{e^x + e^y + e^{-y} + e^{-x}}{4}\right] = \dfrac{e^x + e^{-x}}{2} + \dfrac{e^y + e^{-y}}{2}$

$= \cosh x + \cosh y$

14. $\tanh x = \dfrac{1}{2}$

$\left(\dfrac{1}{2}\right)^2 + \operatorname{sech}^2 x = 1 \Rightarrow \operatorname{sech}^2 x = \dfrac{3}{4} \Rightarrow \operatorname{sech} x = \dfrac{\sqrt{3}}{2}$

$\cosh x = \dfrac{1}{\sqrt{3/2}} = \dfrac{2\sqrt{3}}{3}$

$\coth x = \dfrac{1}{1/2} = 2$

$\sinh x = \tanh x \cosh x = \left(\dfrac{1}{2}\right)\left(\dfrac{2\sqrt{3}}{3}\right) = \dfrac{\sqrt{3}}{3}$

$\operatorname{csch} x = \dfrac{1}{\sqrt{3}/3} = \sqrt{3}$

Putting these in order:

$\sinh x = \dfrac{\sqrt{3}}{3}$ $\quad \operatorname{csch} x = \sqrt{3}$

$\cosh x = \dfrac{2\sqrt{3}}{3}$ $\quad \operatorname{sech} x = \dfrac{\sqrt{3}}{2}$

$\tanh x = \dfrac{1}{2}$ $\quad \coth x = 2$

16. $y = \coth(3x)$

$y' = -3\operatorname{csch}^2(3x)$

18. $g(x) = \ln(\cosh x)$

$g'(x) = \dfrac{1}{\cosh x}(\sinh x) = \tanh x$

20. $y = x\cosh x - \sinh x$

$y' = x\sinh x + \cosh x - \cosh x = x\sinh x$

22. $h(t) = t - \coth t$

$h'(t) = 1 + \operatorname{csch}^2 t = \coth^2 t$

24. $g(x) = \operatorname{sech}^2 3x$

$g'(x) = -2\operatorname{sech}(3x)\operatorname{sech}(3x)\tanh(3x)(3)$

$= -6\operatorname{sech}^2 3x \tanh 3x$

26. $f(x) = e^{\sinh x}$

$f'(x) = (\cosh x)(e^{\sinh x})$

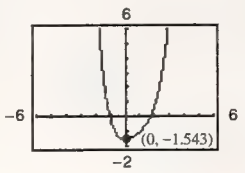
28. $y = \operatorname{sech}(x + 1)$

$y' = -\operatorname{sech}(x + 1)\tanh(x + 1)$

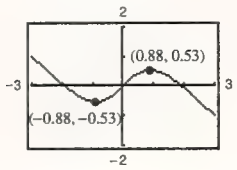
30. $f(x) = x\sinh(x - 1) - \cosh(x - 1)$

$f'(x) = x\cosh(x - 1) + \sinh(x - 1) - \sinh(x - 1) = x\cosh(x - 1)$

$f'(x) = 0$ for $x = 0$. By the First Derivative Test, $(0, -\cosh(-1)) \approx (0, -1.543)$ is a relative minimum.



32. $h(x) = 2\tanh x - x$



Relative maximum: $(0.88, 0.53)$

Relative minimum: $(-0.88, -0.53)$

34. $y = a\cosh x$

$y' = a\sinh x$

$y'' = a\cosh x$

Therefore, $y'' - y = 0$.

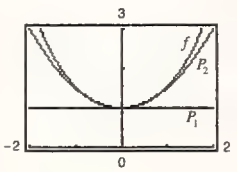
36. $f(x) = \cosh x \qquad f(1) = \cosh(0) \approx 1$

$f'(x) = \sinh x \qquad f'(1) = \sinh(0) \approx 0$

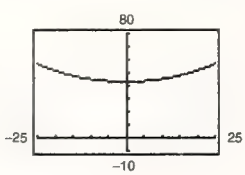
$f''(x) = \cosh x \qquad f''(1) = \cosh(0) \approx 1$

$P_1(x) = f(0) + f'(0)(x - 0) = 1$

$P_2(x) = 1 + \frac{1}{2}x^2$



38. (a) $y = 18 + 25\cosh\dfrac{x}{25}, -25 \le x \le 25$



(b) At $x = \pm 25$, $y = 18 + 25\cosh(1) \approx 56.577$.

At $x = 0$, $y = 18 + 25 = 43$.

(c) $y' = \sinh\dfrac{x}{25}$. At $x = 25$, $y' = \sinh(1) \approx 1.175$

40. Let $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}dx$.

$$\int \frac{\cosh\sqrt{x}}{\sqrt{x}}dx = 2\int \cosh\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)dx = 2\sinh\sqrt{x} + C$$

42. Let $u = \cosh x$, $du = \sinh x\,dx$.

$$\int \frac{\sinh}{1 + \sinh^2 x}dx = \int \frac{\sinh x}{\cosh^2 x}dx = \frac{-1}{\cosh x} + C$$

$$= -\operatorname{sech} x + C$$

44. Let $u = 2x - 1$, $du = 2\,dx$.

$$\int \operatorname{sech}^2(2x - 1)\,dx = \frac{1}{2}\int \operatorname{sech}^2(2x - 1)(2)\,dx$$

$$= \frac{1}{2}\tanh(2x - 1) + C$$

46. Let $u = \operatorname{sech} x$, $du = -\operatorname{sech} x \tanh x\,dx$.

$$\int \operatorname{sech}^3 x \tanh x\,dx = -\int \operatorname{sech}^2 x(-\operatorname{sech} x \tanh x)\,dx$$

$$= -\frac{1}{3}\operatorname{sech}^3 x + C$$

48. $\displaystyle\int \cosh^2 x\,dx = \int \frac{1+\cosh 2x}{2}\,dx$

$\displaystyle= \frac{1}{2}\left[x + \frac{\sinh 2x}{2}\right] + C$

$\displaystyle= \frac{1}{2}x + \frac{1}{4}\sinh 2x + C$

50. $\displaystyle\int_0^4 \frac{1}{\sqrt{25-x^2}}\,dx = \left[\arcsin\frac{x}{5}\right]_0^4 = \arcsin\frac{4}{5}$

52. $\displaystyle\int \frac{2}{x\sqrt{1+4x^2}}\,dx = 2\int \frac{1}{(2x)\sqrt{1+(2x)^2}}(2)\,dx = -2\ln\left(\frac{1+\sqrt{1+4x^2}}{|2x|}\right) + C$

54. Let $u = \sinh x$, $du = \cosh x\,dx$.

$\displaystyle\int \frac{\cosh x}{\sqrt{9-\sinh^2 x}}\,dx = \arcsin\left(\frac{\sinh x}{3}\right) + C$

$\displaystyle= \arcsin\left(\frac{e^x - e^{-x}}{6}\right) + C$

56. $y = \tanh^{-1}\left(\frac{x}{2}\right)$

$\displaystyle y' = \frac{1}{1-(x/2)^2}\left(\frac{1}{2}\right) = \frac{2}{4-x^2}$

58. $y = \operatorname{sech}^{-1}(\cos 2x),\ 0 < x < \frac{\pi}{4}$

$\displaystyle y' = \frac{-1}{\cos 2x\sqrt{1-\cos^2 2x}}(-2\sin 2x) = \frac{2\sin 2x}{\cos 2x|\sin 2x|} = \frac{2}{\cos 2x} = 2\sec 2x,$

since $\sin 2x \geq 0$ for $0 < x < \pi/4$.

60. $y = (\operatorname{csch}^{-1} x)^2$

$\displaystyle y' = 2\operatorname{csch}^{-1} x\left(\frac{-1}{|x|\sqrt{1+x^2}}\right) = \frac{-2\operatorname{csch}^{-1} x}{|x|\sqrt{1+x^2}}$

62. $\displaystyle y = x\tanh^{-1} x + \ln\sqrt{1-x^2} = x\tanh^{-1} x + \frac{1}{2}\ln(1-x^2)$

$\displaystyle y' = x\left(\frac{1}{1-x^2}\right) + \tanh^{-1} x + \frac{-x}{1-x^2} = \tanh^{-1} x$

64. See page 401, Theorem 5.22.

66. Equation of tangent line through $P = (x_0, y_0)$:

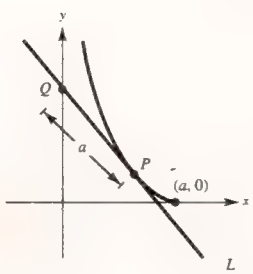
$\displaystyle y - a\operatorname{sech}^{-1}\frac{x_0}{a} + \sqrt{a^2 - x_0^2} = -\frac{\sqrt{a^2-x_0^2}}{x_0}(x - x_0)$

When $x = 0$,

$\displaystyle y = a\operatorname{sech}^{-1}\frac{x_0}{a} - \sqrt{a^2-x_0^2} + \sqrt{a^2-x_0^2} = a\operatorname{sech}^{-1}\frac{x_0}{a}.$

Hence, Q is the point $[0, a\operatorname{sech}^{-1}(x_0/a)]$.

Distance from P to Q: $d = \sqrt{x_0^2 + \left(-\sqrt{a^2-x_0^2}\right)^2} = a$



68. $\displaystyle\int \frac{x}{9-x^4}\,dx = -\frac{1}{2}\int \frac{-2x}{9-(x^2)^2}\,dx = -\frac{1}{2}\left(\frac{1}{6}\right)\ln\left|\frac{3-x^2}{3+x^2}\right| + C$

$\displaystyle= -\frac{1}{12}\ln\left|\frac{3-x^2}{3+x^2}\right| + C$

70. Let $u = x^{3/2}$, $du = \frac{3}{2}\sqrt{x}\,dx$.

$$\int \frac{\sqrt{x}}{\sqrt{1+x^3}}\,dx = \frac{2}{3}\int \frac{1}{\sqrt{1+(x^{3/2})^2}}\left(\frac{3}{2}\sqrt{x}\right)dx = \frac{2}{3}\sinh^{-1}(x^{3/2}) + C = \frac{2}{3}\ln\left(x^{3/2} + \sqrt{1+x^3}\right) + C$$

72. $$\int \frac{dx}{(x+2)\sqrt{x^2+4x+8}} = \int \frac{dx}{(x+2)\sqrt{(x+2)^2+4}}$$
$$= -\frac{1}{2}\ln\left(\frac{2+\sqrt{(x+2)^2+4}}{|x+2|}\right) + C$$

74. $$\int \frac{1}{(x+1)\sqrt{2x^2+4x+8}}\,dx = \int \frac{1}{(x+1)\sqrt{2(x+1)^2+6}}\,dx$$
$$= \frac{1}{\sqrt{2}}\int \frac{1}{(x+1)\sqrt{(x+1)^2+(\sqrt{3})^2}}\,dx = -\frac{1}{\sqrt{6}}\ln\left(\frac{\sqrt{3}+\sqrt{(x+1)^2+3}}{x+1}\right) + C$$

76. Let $u = 2(x-1)$, $du = 2\,dx$.

$$y = \int \frac{1}{(x-1)\sqrt{-4x^2+8x-1}}\,dx = \int \frac{2}{2(x-1)\sqrt{(\sqrt{3})^2-[2(x-1)]^2}}\,dx = -\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}+\sqrt{-4x^2+8x-1}}{2(x-1)}\right| + C$$

78. $$y = \int \frac{1-2x}{4x-x^2}\,dx = \int \frac{4-2x}{4x-x^2}\,dx + 3\int \frac{1}{(x-2)^2-4}\,dx$$
$$= \ln|4x-x^2| + \frac{3}{4}\ln\left|\frac{(x-2)-2}{(x-2)+2}\right| + C = \ln|4x-x^2| + \frac{3}{4}\ln\left|\frac{x-4}{x}\right| + C$$

80. $$A = \int_0^2 \tanh 2x\,dx$$
$$= \int_0^2 \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\,dx$$
$$= \frac{1}{2}\int_0^2 \frac{1}{e^{2x}+e^{-2x}}(2)(e^{2x}-e^{-2x})\,dx$$
$$= \left[\frac{1}{2}\ln(e^{2x}+e^{-2x})\right]_0^2$$
$$= \frac{1}{2}\ln(e^4+e^{-4}) - \frac{1}{2}\ln 2$$
$$= \ln\sqrt{\frac{e^4+e^{-4}}{2}} \approx 1.654$$

82. $$A = \int_3^5 \frac{6}{\sqrt{x^2-4}}\,dx$$
$$= \left[6\ln\left(x+\sqrt{x^2-4}\right)\right]_3^5$$
$$= 6\ln\left(5+\sqrt{21}\right) - 6\ln\left(3+\sqrt{5}\right)$$
$$= 6\ln\left(\frac{5+\sqrt{21}}{3+\sqrt{5}}\right) \approx 3.626$$

84. (a) $v(t) = -32t$

(b) $$s(t) = \int v(t)\,dt = \int (-32t)\,dt = -16t^2 + C$$
$$s(0) = -16(0)^2 + C = 400 \Rightarrow C = 400$$
$$s(t) = -16t^2 + 400$$

—CONTINUED—

84. —CONTINUED—

(c) $$\frac{dv}{dt} = -32 + kv^2$$

$$\int \frac{dv}{kv^2 - 32} = \int dt$$

$$\int \frac{dv}{32 - kv^2} = -\int dt$$

Let $u = \sqrt{k}\,v$, then $du = \sqrt{k}\,dv$.

$$\frac{1}{\sqrt{k}} \cdot \frac{1}{2\sqrt{32}} \ln\left|\frac{\sqrt{32} + \sqrt{k}\,v}{\sqrt{32} - \sqrt{k}\,v}\right| = -t + C$$

Since $v(0) = 0$, $C = 0$.

$$\ln\left|\frac{\sqrt{32} + \sqrt{k}\,v}{\sqrt{32} - \sqrt{k}\,v}\right| = -2\sqrt{32k}\,t$$

$$\frac{\sqrt{32} + \sqrt{k}\,v}{\sqrt{32} - \sqrt{k}\,v} = e^{-2\sqrt{32k}\,t}$$

$$\sqrt{32} + \sqrt{k}\,v = e^{-2\sqrt{32k}\,t}\left(\sqrt{32} - \sqrt{k}\,v\right)$$

$$v\left(\sqrt{k} + \sqrt{k}\,e^{-2\sqrt{32k}\,t}\right) = \sqrt{32}\left(e^{-2\sqrt{32k}\,t} - 1\right)$$

$$v = \frac{\sqrt{32}\left(e^{-2\sqrt{32k}\,t} - 1\right)}{\sqrt{k}\left(e^{-2\sqrt{32k}\,t} + 1\right)} \cdot \frac{e^{\sqrt{32k}\,t}}{e^{\sqrt{32k}\,t}}$$

$$= \frac{\sqrt{32}}{\sqrt{k}}\left[\frac{-\left(e^{\sqrt{32k}\,t} - e^{-\sqrt{32k}\,t}\right)}{e^{\sqrt{32k}\,t} + e^{-\sqrt{32k}\,t}}\right]$$

$$= -\frac{\sqrt{32}}{\sqrt{k}} \tanh\left(\sqrt{32k}\,t\right)$$

(d) $$\lim_{t\to\infty}\left[-\frac{\sqrt{32}}{\sqrt{k}} \tanh\left(\sqrt{32k}\,t\right)\right] = -\frac{\sqrt{32}}{\sqrt{k}}.$$

The velocity is bounded by $-\sqrt{32}/\sqrt{k}$.

(e) Since $\int \tanh(ct)\,dt = (1/c)\ln\cosh(ct)$ (which can be verified by differentiation), then

$$s(t) = \int -\frac{\sqrt{32}}{\sqrt{k}} \tanh\left(\sqrt{32k}\,t\right) dt$$

$$= -\frac{\sqrt{32}}{\sqrt{k}} \frac{1}{\sqrt{32k}} \ln\left[\cosh\left(\sqrt{32k}\,t\right)\right] + C$$

$$= -\frac{1}{k} \ln\left[\cosh\left(\sqrt{32k}\,t\right)\right] + C.$$

When $t = 0$,

$$s(0) = C$$

$$= 400 \Rightarrow 400 - (1/k)\ln\left[\cosh\left(\sqrt{32k}\,t\right)\right].$$

When $k = 0.01$,

$$s_2(t) = 400 - 100\ln\left(\cosh\sqrt{0.32}\,t\right)$$

$$s_1(t) = -16t^2 + 400.$$

$s_1(t) = 0$ when $t = 5$ seconds.

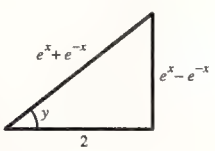
$s_2(t) = 0$ when $t \approx 8.3$ seconds

When air resistance is not neglected, it takes approximately 3.3 more seconds to reach the ground.

86. Let $y = \arcsin(\tanh x)$. Then,

$$\sin y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ and}$$

$$\tan y = \frac{e^x - e^{-x}}{2} = \sinh x.$$



Thus, $y = \arctan(\sinh x)$. Therefore,

$$\arctan(\sinh x) = \arcsin(\tanh x).$$

88.

$$y = \operatorname{sech}^{-1} x$$

$$\operatorname{sech} y = x$$

$$-(\operatorname{sech} y)(\tanh y)y' = 1$$

$$y' = \frac{-1}{(\operatorname{sech} y)(\tanh y)} = \frac{-1}{(\operatorname{sech} y)\sqrt{1 - \operatorname{sech}^2 y}} = \frac{-1}{x\sqrt{1 - x^2}}$$

90.

$$y = \sinh^{-1} x$$

$$\sinh y = x$$

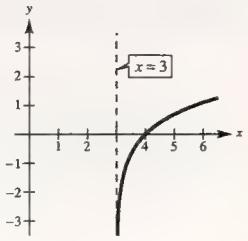
$$(\cosh y)y' = 1$$

$$y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{\sinh^2 y + 1}} = \frac{1}{\sqrt{x^2 + 1}}$$

Review Exercises for Chapter 5

2. $f(x) = \ln(x - 3)$

Horizontal shift 3 units to the right
Vertical asymptote: $x = 3$



4. $\ln[(x^2 + 1)(x - 1)] = \ln(x^2 + 1) + \ln(x - 1)$

6. $3[\ln x - 2\ln(x^2 + 1)] + 2\ln 5 = 3\ln x - 6\ln(x^2 + 1) + \ln 5^2 = \ln x^3 - \ln(x^2 + 1)^6 + \ln 25 = \ln\left[\dfrac{25x^3}{(x^2 + 1)^6}\right]$

8. $\ln x + \ln(x - 3) = 0$

$\ln x(x - 3) = 0$

$x(x - 3) = e^0$

$x^2 - 3x - 1 = 0$

$x = \dfrac{3 \pm \sqrt{13}}{2}$

$x = \dfrac{3 + \sqrt{13}}{2}$ only since $\dfrac{3 - \sqrt{13}}{2} < 0.$

10. $h(x) = \ln\dfrac{x(x - 1)}{x - 2} = \ln x + \ln(x - 1) - \ln(x - 2)$

$h'(x) = \dfrac{1}{x} + \dfrac{1}{x - 1} - \dfrac{1}{x - 2} = \dfrac{x^2 - 4x + 2}{x^3 - 3x^2 + 2x}$

12. $f(x) = \ln[x(x^2 - 2)^{2/3}] = \ln x + \dfrac{2}{3}\ln(x^2 - 2)$

$f'(x) = \dfrac{1}{x} + \dfrac{2}{3}\left(\dfrac{2x}{x^2 - 2}\right) = \dfrac{7x^2 - 6}{3x^3 - 6x}$

14. $y = \dfrac{1}{b^2}[a + bx - a\ln(a + bx)]$

$\dfrac{dy}{dx} = \dfrac{1}{b^2}\left(b - \dfrac{ab}{a + bx}\right) = \dfrac{x}{a + bx}$

16. $y = -\dfrac{1}{ax} + \dfrac{b}{a^2}\ln\dfrac{a + bx}{x}$

$= -\dfrac{1}{ax} + \dfrac{b}{a^2}[\ln(a + bx) - \ln x]$

$\dfrac{dy}{dx} = -\dfrac{1}{a}\left(-\dfrac{1}{x^2}\right) + \dfrac{b}{a^2}\left[\dfrac{b}{a + bx} - \dfrac{1}{x}\right]$

$= \dfrac{1}{ax^2} + \dfrac{b}{a^2}\left[\dfrac{-a}{x(a + bx)}\right] = \dfrac{1}{ax^2} - \dfrac{b}{ax(a + bx)}$

$= \dfrac{(a + bx) - bx}{ax^2(a + bx)} = \dfrac{1}{x^2(a + bx)}$

18. $u = x^2 - 1, du = 2x\,dx$

$\displaystyle\int \frac{x}{x^2 - 1}\,dx = \frac{1}{2}\int \frac{2x}{x^2 - 1}\,dx = \frac{1}{2}\ln|x^2 - 1| + C$

20. $u = \ln x, du = \dfrac{1}{x}\,dx$

$\displaystyle\int \frac{\ln\sqrt{x}}{x}\,dx = \frac{1}{2}\int (\ln x)\left(\frac{1}{x}\right)dx = \frac{1}{4}(\ln x)^2 + C$

22. $\displaystyle\int_1^e \frac{\ln x}{x}\,dx = \int_1^e (\ln x)^1\left(\frac{1}{x}\right)dx = \left[\frac{1}{2}(\ln x)^2\right]_1^e = \frac{1}{2}$

24. $\displaystyle\int_0^{\pi/4} \tan\left(\frac{\pi}{4} - x\right)dx = \left[\ln\left|\cos\left(\frac{\pi}{4} - x\right)\right|\right]_0^{\pi/4}$

$= 0 - \ln\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{1}{2}\ln 2$

26. (a) $f(x) = 5x - 7$

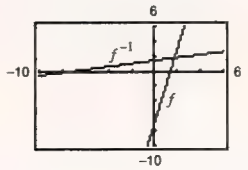
$y = 5x - 7$

$\dfrac{y + 7}{5} = x$

$\dfrac{x + 7}{5} = y$

$f^{-1}(x) = \dfrac{x + 7}{5}$

(b)



(c) $f^{-1}(f(x)) = f^{-1}(5x - 7) = \dfrac{(5x - 7) + 7}{5} = x$

$f(f^{-1}(x)) = f\left(\dfrac{x + 7}{5}\right) = 5\left(\dfrac{x + 7}{5}\right) - 7 = x$

28. (a) $f(x) = x^3 + 2$

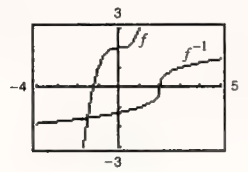
$y = x^3 + 2$

$\sqrt[3]{y - 2} = x$

$\sqrt[3]{x - 2} = y$

$f^{-1}(x) = \sqrt[3]{x - 2}$

(b)



(c) $f^{-1}(f(x)) = f^{-1}(x^3 + 2) = \sqrt[3]{(x^3 + 2) - 2} = x$

$f(f^{-1}(x)) = f\left(\sqrt[3]{x - 2}\right) = \left(\sqrt[3]{x - 2}\right)^3 + 2 = x$

30. (a) $f(x) = x^2 - 5, x \geq 0$

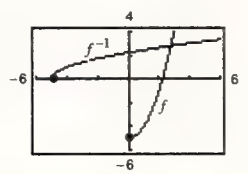
$y = x^2 - 5$

$\sqrt{y + 5} = x$

$\sqrt{x + 5} = y$

$f^{-1}(x) = \sqrt{x + 5}$

(b)



(c) $f^{-1}(f(x)) = f^{-1}(x^2 - 5) = \sqrt{(x^2 - 5) + 5} = x$ for $x \geq 0$.

$f(f^{-1}(x)) = f\left(\sqrt{x + 5}\right) = \left(\sqrt{x + 5}\right)^2 - 5 = x$

32. $f(x) = x\sqrt{x - 3}$

$f(4) = 4$

$f'(x) = \sqrt{x - 3} + \dfrac{1}{2}x(x - 3)^{-1/2}$

$f'(4) = 1 + 2 = 3$

$(f^{-1})'(4) = \dfrac{1}{f'(4)} = \dfrac{1}{3}$

34. $f(x) = \ln x$

$f^{-1}(x) = e^x$

$(f^{-1})'(x) = e^x$

$(f^{-1})'(0) = e^0 = 1$

36. (a) $f(x) = e^{1-x}$

$y = e^{1-x}$

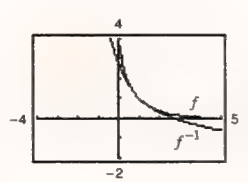
$\ln y = 1 - x$

$x = 1 - \ln y$

$y = 1 - \ln x$

$f^{-1}(x) = 1 - \ln x$

(b)

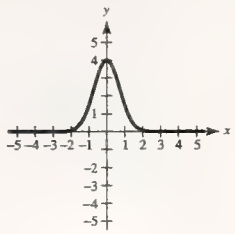


(c) $f^{-1}(f(x)) = f^{-1}\left(e^{1-x}\right) = 1 - \ln(e^{1-x})$

$= 1 - (1 - x) = x$

$f(f^{-1}(x)) = f(1 - \ln x) = e^{1-(1-\ln x)} = e^{\ln x} = x$

38. $y = 4e^{-x^2}$



40. $g(x) = \ln\left(\dfrac{e^x}{1+e^x}\right)$

$= \ln e^x - \ln(1+e^x) = x - \ln(1+e^x)$

$g'(x) = 1 - \dfrac{e^x}{1+e^x} = \dfrac{1}{1+e^x}$

42. $h(z) = e^{-z^2/2}$

$h'(z) = -ze^{-z^2/2}$

44. $y = 3e^{-3/t}$

$y' = 3e^{-3/t}(3t^{-2}) = \dfrac{9e^{-3/t}}{t^2}$

46. $f(\theta) = \dfrac{1}{2}e^{\sin 2\theta}$

$f'(\theta) = \cos 2\theta\, e^{\sin 2\theta}$

48. $\cos x^2 = xe^y$

$-2x \sin x^2 = xe^y \dfrac{dy}{dx} + e^y$

$\dfrac{dy}{dx} = -\dfrac{2x \sin x^2 + e^y}{xe^y}$

50. Let $u = \dfrac{1}{x}$, $du = \dfrac{-1}{x^2}\,dx$.

$\displaystyle\int \frac{e^{1/x}}{x^2}\,dx = -\int e^{1/x}\left(-\frac{1}{x^2}\right)dx = -e^{1/x} + C$

52. Let $u = e^{2x} + e^{-2x}$, $du = (2e^{2x} - e^{-2x})\,dx$.

$\displaystyle\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx = \frac{1}{2}\int \frac{2e^{2x} - 2e^{-2x}}{e^{2x} + e^{-2x}}\,dx$

$= \dfrac{1}{2}\ln(e^{2x} + e^{-2x}) + C$

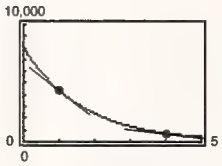
54. Let $u = x^3 + 1$, $du = 3x^2\,dx$.

$\displaystyle\int x^2 e^{x^3+1}\,dx = \frac{1}{3}\int e^{x^3+1}(3x^2)\,dx = \frac{1}{3}e^{x^3+1} + C$

56. $\displaystyle\int \frac{e^{2x}}{e^{2x}+1}\,dx = \frac{1}{2}\int \frac{1}{e^{2x}+1}2e^{2x}\,dx$

$= \dfrac{1}{2}\ln(e^{2x} + 1) + C$

58. (a), (c)



(b) $V = 8000e^{-0.6t}$, $0 \le t \le 5$

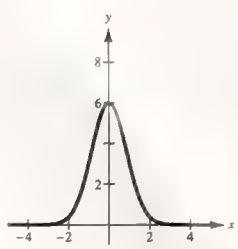
$V'(t) = -4800e^{-0.6t}$

$V'(1) = -2634.3$ dollars/year

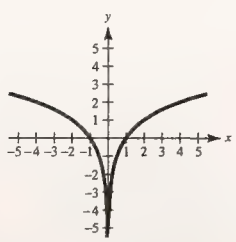
$V'(4) = -435.4$ dollars/year

60. Area $= \displaystyle\int_0^2 2e^{-x}\,dx = \left[-2e^{-x}\right]_0^2 = -2e^{-2} + 2 = 2 - \frac{2}{e^2} \approx 1.729$

62. $g(x) = 6(2^{-x^2})$



64. $y = \log_4 x^2$



66. $f(x) = 4^x e^x$

$f'(x) = 4^x e^x + (\ln 4)4^x e^x = 4^x e^x(1 + \ln 4)$

68. $y = x(4^{-x})$

$y' = 4^{-x} - x \cdot 4^{-x} \ln 4$

70. $h(x) = \log_5 \dfrac{x}{x-1} = \log_5 x - \log_5(x-1)$

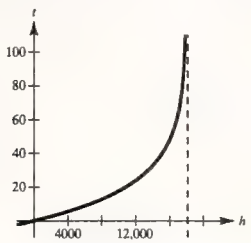
$h'(x) = \dfrac{1}{\ln 5}\left[\dfrac{1}{x} - \dfrac{1}{x-1}\right] = \dfrac{1}{\ln 5}\left[\dfrac{-1}{x(x-1)}\right]$

72. $\displaystyle\int \frac{2^{-1/t}}{t^2}\,dt = \frac{1}{\ln 2}2^{-1/t} + C$

74. $t = 50 \log_{10}\left(\dfrac{18{,}000}{18{,}000 - h}\right)$

(a) Domain: $0 \le h < 18{,}000$

(b)



Vertical asymptote: $h = 18{,}000$

(c) $t = 50 \log_{10}\left(\dfrac{18{,}000}{18{,}000 - h}\right)$

$10^{t/50} = \dfrac{18{,}000}{18{,}000 - h}$

$18{,}000 - h = 18{,}000(10^{-t/50})$

$h = 18{,}000(1 - 10^{-t/50})$

As $h \to 18{,}000$, $t \to \infty$.

(d) $t = 50 \log_{10} 18{,}000 - 50 \log_{10}(18{,}000 - h)$

$\dfrac{dt}{dh} = \dfrac{50}{(\ln 10)(18{,}000 - h)}$

$\dfrac{d^2t}{dh^2} = \dfrac{50}{(\ln 10)(18{,}000 - h)^2}$

No critical numbers

As t increases, the rate of change of the altitude is increasing.

76. $2P = Pe^{10r}$

$2 = e^{10r}$

$\ln 2 = 10r$

$r = \dfrac{\ln 2}{10} \approx 6.93\%$

78. $y = 5\left(\dfrac{1}{2}\right)^{t/1620}$

$y(600) = 5\left(\dfrac{1}{2}\right)^{600/1620} \approx 3.868$ grams

80. (a) $\dfrac{dy}{ds} = -0.012y, s > 50$

$\dfrac{-1}{0.012}\displaystyle\int \frac{dy}{y} = \int ds$

$\dfrac{-1}{0.012}\ln y = s + C_1$

$y = Ce^{-0.012s}$

When $s = 50$, $y = 28 = Ce^{-0.012(50)} \Rightarrow C = 28e^{0.6}$

$y = 28e^{0.6 - 0.012s}, s > 50$

(b)

Speed(s)	50	55	60	65	70
Miles per Gallon (y)	28	26.4	24.8	23.4	22.0

82. $\dfrac{dy}{dx} = \dfrac{e^{-2x}}{1 + e^{-2x}}$

$\displaystyle\int dy = \int \frac{e^{-2x}}{1 + e^{-2x}}\,dx = -\frac{1}{2}\int \frac{-2e^{-2x}}{1 + e^{-2x}}\,dx$

$y = -\dfrac{1}{2}\ln(1 + e^{-2x}) + C$

84. $y' - e^y \sin x = 0$

$$\frac{dy}{dx} = e^y \sin x$$

$$\int e^{-y}\,dy = \int \sin x\,dx$$

$$-e^{-y} = -\cos x + C_1$$

$$e^y = \frac{1}{\cos x + C} \qquad (C = -C_1)$$

$$y = \ln\left|\frac{1}{\cos x + C}\right| = -\ln|\cos x + C|$$

86. $\dfrac{dy}{dx} = \dfrac{3(x+y)}{x}$ (homogeneous differential equation)

$$3(x + y)\,dx - x\,dy = 0$$

Let $y = vx$, $dy = x\,dv + v\,dx$.

$$3(x + vx)\,dx - x(x\,dv + v\,dx) = 0$$

$$(3x + 2vx)\,dx - x^2\,dv = 0$$

$$(3 + 2v)\,dx = x\,dv$$

$$\int \frac{1}{x}\,dx = \int \frac{1}{3 + 2v}\,dv$$

$$\ln|x| = \frac{1}{2}\ln|3 + 2v| + C_1 = \ln(3 + 2v)^{1/2} + \ln C_2$$

$$x = C_2(3 + 2v)^{1/2}$$

$$x^2 = C(3 + 2v) = C\left(3 + 2\left(\frac{y}{x}\right)\right)$$

$$x^3 = C(3x + 2y) = 3Cx + 2Cy$$

$$y = \frac{x^3 - 3Cx}{2C}$$

88. $\dfrac{dv}{dt} = kv - 9.8$

(a) $$\int \frac{dv}{kv - 9.8} = \int dt$$

$$\frac{1}{k}\ln|kv - 9.8| = t + C_1$$

$$\ln|kv - 9.8| = kt + C_2$$

$$kv - 9.8 = e^{kt + C_2} = C_3 e^{kt}$$

$$v = \frac{1}{k}\left[9.8 + C_3 e^{kt}\right]$$

At $t = 0$, $\quad v_0 = \dfrac{1}{k}(9.8 + C_3) \Rightarrow C_3 = kv_0 - 9.8$

$$v = \frac{1}{k}[9.8 + (kv_0 - 9.8)e^{kt}]$$

Note that $k < 0$ since the object is moving downward.

(b) $$\lim_{t\to\infty} v(t) = \frac{9.8}{k}$$

(c) $$s(t) = \int \frac{1}{k}[9.8 + (kv_0 - 9.8)e^{kt}]\,dt$$

$$= \frac{1}{k}\left[9.8t + \frac{1}{k}(kv_0 - 9.8)e^{kt}\right] + C$$

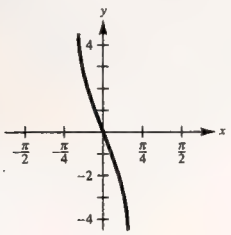
$$= \frac{9.8t}{k} + \frac{1}{k^2}(kv_0 - 9.8)e^{kt} + C$$

$$s(0) = \frac{1}{k^2}(kv_0 - 9.8) + C \Rightarrow C = s_0 - \frac{1}{k^2}(kv_0 - 9.8)$$

$$s(t) = \frac{9.8t}{k} + \frac{1}{k^2}(kv_0 - 9.8)e^{kt} + s_0 - \frac{1}{k^2}(kv_0 - 9.8)$$

$$= \frac{9.8t}{k} + \frac{1}{k^2}(kv_0 - 9.8)(e^{kt} - 1) + s_0$$

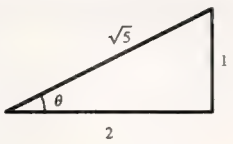
90. $h(x) = -3 \arcsin(2x)$



92. (a) Let $\theta = \operatorname{arccot} 2$

$\cot\theta = 2$

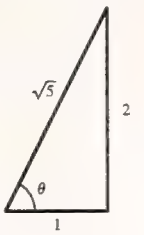
$\tan(\operatorname{arccot} 2) = \tan\theta = \frac{1}{2}.$



(b) Let $\theta = \operatorname{arcsec}\sqrt{5}$

$\sec\theta = \sqrt{5}$

$\cos(\operatorname{arcsec}\sqrt{5}) = \cos\theta = \frac{1}{\sqrt{5}}.$



94. $y = \arctan(x^2 - 1)$

$$y' = \frac{2x}{1 + (x^2 - 1)^2} = \frac{2x}{x^4 - 2x^2 + 2}$$

96. $y = \frac{1}{2}\arctan e^{2x}$

$$y' = \frac{1}{2}\left(\frac{1}{1 + e^{4x}}\right)(2e^{2x}) = \frac{e^{2x}}{1 + e^{4x}}$$

98. $y = \sqrt{x^2 - 4} - 2\operatorname{arcsec}\frac{x}{2}, 2 < x < 4$

$$y' = \frac{x}{\sqrt{x^2 - 4}} - \frac{1}{(|x|/2)\sqrt{(x/2)^2 - 1}} = \frac{x}{\sqrt{x^2 - 4}} - \frac{4}{|x|\sqrt{x^2 - 4}} = \frac{x^2 - 4}{|x|\sqrt{x^2 - 4}} = \frac{\sqrt{x^2 - 4}}{x}$$

100. Let $u = 5x$, $du = 5\,dx$.

$$\int \frac{1}{3 + 25x^2}\,dx = \frac{1}{5}\int \frac{1}{(\sqrt{3})^2 + (5x)^2}(5)\,dx = \frac{1}{5\sqrt{3}}\arctan\frac{5x}{\sqrt{3}} + C$$

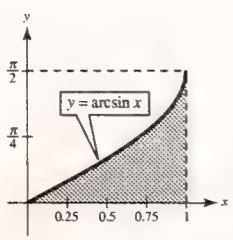
102. $\displaystyle\int \frac{1}{16 + x^2}\,dx = \frac{1}{4}\arctan\frac{x}{4} + C$

104. $\displaystyle\int \frac{4 - x}{\sqrt{4 - x^2}}\,dx = 4\int \frac{1}{\sqrt{4 - x^2}}\,dx + \frac{1}{2}\int (4 - x^2)^{-1/2}(-2x)\,dx = 4\arcsin\frac{x}{2} + \sqrt{4 - x^2} + C$

106. Let $u = \arcsin x$, $du = \dfrac{1}{\sqrt{1 - x^2}}\,dx$.

$$\int \frac{\arcsin x}{\sqrt{1 - x^2}}\,dx = \frac{1}{2}(\arcsin x)^2 + C$$

108.



Since the area of region A is $1\left(\displaystyle\int_0^{\pi/2} \sin y\,dy\right)$,

the shaded area is $\displaystyle\int_0^1 \arcsin x\,dx = \frac{\pi}{2} - 1 \approx 0.571.$

110. $y = x\tanh^{-1} 2x$

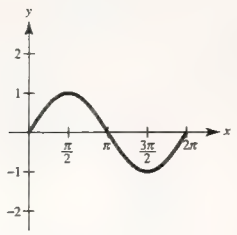
$$y' = x\left(\frac{2}{1 - 4x^2}\right) + \tanh^{-1} 2x = \frac{2x}{1 - 4x^2} + \tanh^{-1} 2x$$

112. Let $u = x^3$, $du = 3x^2\,dx$.

$$\int x^2(\operatorname{sech} x^3)^2\,dx = \frac{1}{3}\int (\operatorname{sech} x^3)^2(3x^2)\,dx = \frac{1}{3}\tanh x^3 + C$$

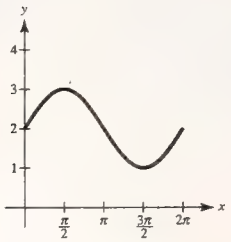
Problem Solving for Chapter 5

2. (a)



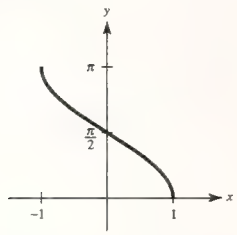
$$\int_0^{\pi} \sin x\,dx = -\int_{\pi}^{2\pi} \sin x\,dx \implies \int_0^{2\pi} \sin x\,dx = 0$$

(b)



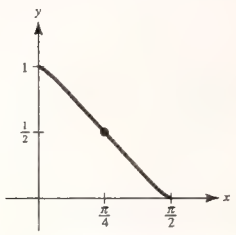
$$\int_0^{2\pi} (\sin x + 2)\,dx = 2(2\pi) = 4\pi$$

(c)



$$\int_{-1}^{1} \arccos x\,dx = 2\left(\frac{\pi}{2}\right) = \pi$$

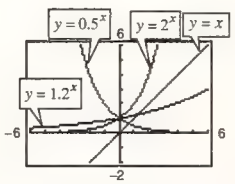
(d)



$y = \dfrac{1}{1 + (\tan x)\sqrt{2}}$ is symmetric with respect to the point $\left(\frac{\pi}{4}, \frac{1}{2}\right)$. $\displaystyle\int_0^{\pi/2} \frac{1}{1 + (\tan x)\sqrt{2}}\,dx = \frac{\pi}{2}\left(\frac{1}{2}\right) = \frac{\pi}{4}$

4. $y = 0.5^x$ and $y = 1.2^x$ intersect $y = x$.

$y = 2^x$ does not intersect $y = x$.



Suppose $y = x$ is tangent to $y = a^x$ at (x, y).

$a^x = x \implies a = x^{1/x}$.

$y' = a^x \ln a = 1 \implies x \ln x^{1/x} = 1 \implies \ln x = 1 \implies x = e, a = e^{1/e}$

For $0 < a \le e^{1/e} \approx 1.445$, the curve $y = a^x$ intersects $y = x$.

6. (a) $y = f(x) = \arcsin x$

$\sin y = x$

$$\text{Area } A = \int_{\pi/6}^{\pi/4} \sin y \cdot dy = -\cos y\Big]_{\pi/6}^{\pi/4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} = \frac{\sqrt{3} - \sqrt{2}}{2} \approx 0.1589$$

$$\text{Area } B = \left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right) = \frac{\pi}{12} \approx 0.2618$$

(b) $\displaystyle\int_{1/2}^{\sqrt{2}/2} \arcsin x\,dx = \text{Area}(C) = \left(\frac{\pi}{4}\right)\left(\frac{\sqrt{2}}{2}\right) - A - B$

$$= \frac{\pi\sqrt{2}}{8} - \frac{\sqrt{3} - \sqrt{2}}{2} - \frac{\pi}{12}$$

$$= \pi\left(\frac{\sqrt{2}}{8} - \frac{1}{12}\right) + \frac{\sqrt{2} - \sqrt{3}}{2} \approx 0.1346$$

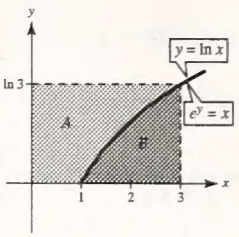
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6. —CONTINUED—

(c) $\text{Area } A = \int_0^{\ln 3} e^y\,dy$

$= e^y\Big]_0^{\ln 3} = 3 - 1 = 2$

$\text{Area } B = \int_1^3 \ln x\,dx = 3(\ln 3) - A = 3\ln 3 - 2 = \ln 27 - 2 \approx 1.2958$



(d) $\tan y = x$

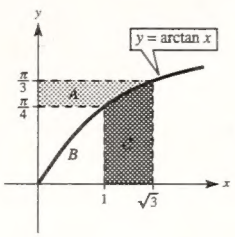
$\text{Area } A = \int_{\pi/4}^{\pi/3} \tan y\,dy$

$= -\ln|\cos y|\Big]_{\pi/4}^{\pi/3}$

$= -\ln\frac{1}{2} + \ln\frac{\sqrt{2}}{2} = \ln\sqrt{2} = \frac{1}{2}\ln 2$

$\text{Area } C = \int_1^{\sqrt{3}} \arctan x\,dx = \left(\frac{\pi}{3}\right)(\sqrt{3}) - \frac{1}{2}\ln 2 - \left(\frac{\pi}{4}\right)(1)$

$= \frac{\pi}{12}(4\sqrt{3} - 3) - \frac{1}{2}\ln 2 \approx 0.6818$



8. $y = e^x$

$y' = e^x$

$y - b = e^a(x - a)$

$y = e^a x - ae^a + b$ Tangent line

If $y = 0$,

$e^a x = ae^a - b$

$bx = ab - b \quad (b = e^a)$

$x = a - 1$

$c = a - 1$

Thus, $a - c = a - (a - 1) = 1$.

10. Let $u = \tan x$, $du = \sec^2 x\,dx$

$\text{Area} = \int_0^{\pi/4} \frac{1}{\sin^2 x + 4\cos^2 x}\,dx = \int_0^{\pi/4} \frac{\sec^2 x}{\tan^2 x + 4}\,dx$

$= \int_0^1 \frac{du}{u^2 + 4}$

$= \left[\frac{1}{2}\arctan\left(\frac{u}{2}\right)\right]_0^1$

$= \frac{1}{2}\arctan\left(\frac{1}{2}\right)$

12. (a) $\frac{dy}{dt} = y(1 - y),\ y(0) = \frac{1}{4}$

$\int\left(\frac{1}{y} + \frac{1}{1 - y}\right)dy = \int dt$

$\ln|y| - \ln|1 - y| = t + C$

$\ln\left|\frac{y}{1 - y}\right| = t + C$

$\frac{y}{1 - y} = e^{t+C} = C_1 e^t$

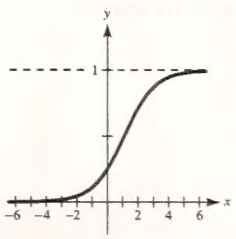
$y = C_1 e^t - yC_1 e^t$

$y = \frac{C_1 e^t}{1 + C_1 e^t} = \frac{1}{1 + C_2 e^{-t}}$

$y(0) = \frac{1}{4} = \frac{1}{1 + C_2} \Rightarrow C_2 = 3$

Hence, $y = \frac{1}{1 + 3e^{-t}}$.

(b)



$\frac{dy}{dt} = y(1 - y) = y - y^2$

$\frac{d^2y}{dt^2} = y'' = y' - 2yy' \Rightarrow y'' = 0 \text{ for } y = \frac{1}{2}$

$\frac{d^2y}{dt^2} > 0 \text{ if } 0 < y < \frac{1}{2} \text{ and } \frac{d^2y}{dt^2} < 0 \text{ if } \frac{1}{2} < y < 1.$

Thus, the rate of growth is maximum at $y = \frac{1}{2}$, the point of inflection.

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12. —CONTINUED—

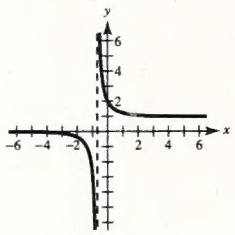
(c) $y' = y(1 - y), y(0) - 2$

As before, $y = \dfrac{1}{1 + C_2 e^{-t}}$

$$y(0) = 2 = \frac{1}{1 + C_2} \Rightarrow C_2 = -\frac{1}{2}$$

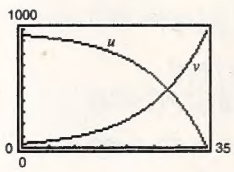
Thus, $y = \dfrac{1}{1 - \frac{1}{2}e^{-t}} = \dfrac{2}{2 - e^{-t}}.$

The graph is different:



14. (a) $u = 985.93 - \left(985.93 - \dfrac{(120{,}000)(0.095)}{12}\right)\left(1 + \dfrac{0.095}{12}\right)^{12t}$

$$v = \left(985.93 - \frac{(120{,}000)(0.095)}{12}\right)\left(1 + \frac{0.095}{12}\right)^{12t}$$



(b) The larger part goes for interest. The curves intersect when $t \approx 27.7$ years.

(c) The slopes are negatives of each other. Analytically,

$$u = 985.93 - v \Rightarrow \frac{du}{dt} = -\frac{dv}{dt}$$

$u'(15) = -v'(15) = -14.06.$

(d) $t = 12.7$ years

Again, the larger part goes for interest.



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